Revisiting Convexity-Preserving Signal Recovery with the Linearly Involved GMC Penalty

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Introduction

Optimization problem for sparse signal and image recovery:

$$\min_x F(x) = \frac{1}{2}\|y - Ax\|_2^2 + \mu\phi(x)$$

where $y \in R^m$ is the response, $A \in R^{m\times n}$ is a matrix, and $\phi: R^n \to R$ is the penalty term.

❖ *Convex penalty*

- Lasso, Group Lasso
- underestimate large magnitude components

❖ *Nonconvex penalty*

- MCP, SCAD
- better signal recovery but nonconvex optimization, computation difficulties

❖ ***Convexity-preserving nonconvex penalty***

- generalized minimax concave (GMC) penalty

$$\phi_B(x) = \|x\|_1 - \min_{v\in R^n}\{\|v\|_1 + \frac{1}{2}\|B(x - v)\|_2^2\}$$

- convex optimization with nonconvex penalty

The **linearly involved GMC model** [1]:

$$\min_x F_L(x) = \frac{1}{2}\|y - Ax\|_2^2 + \mu\phi_B \circ L(x) \quad (1)$$

where $L \in R^{l\times n}$ and $B \in R^{m\times l}$. [1] states that F_L maintains convex when B satisfies the convexity-preserving condition

$$A^TA - \mu L^TB^TBL \succcurlyeq O_n \quad (2)$$

❑ ***Contributions:***

- New method to set B satisfying condition (2)
- A fast algorithm to solve model (1)
- Theoretical properties of the solution path

Algorithms for matrix B

Find a matrix $Z = \mu B^TB$ satisfying (2), which is equivalent to find a pair of symmetric matrices Z and D satisfying the following three conditions:

$$\begin{cases} Z \succcurlyeq O_l \\ L^TZL = D \\ A^TA - D \succcurlyeq O_n \end{cases} \quad (3)$$

❖ CQ algorithm

Let $C = \{Z \in R^{l\times l} | Z \succcurlyeq O_l\}$ and $D = \{D \in R^{n\times n} | D$ is symmetric and $\lambda_n(A^TA - D) \geq \sigma_\theta\}$ where $\sigma_\theta = (1-\theta)\lambda_n(A^TA) \geq 0$ and $\theta \in (0, 1)$. Here λ_n indicates the smallest eigenvalue.

A solution to (3) solves the following split feasibility problem:

Find $Z \in C$ with $L^TZL = D$,

which can be solved by CQ algorithm.

❖ ADMM algorithm

Define three sets in space $\Theta = R^{l\times l} \times R^{n\times n}$:

$$G = \{(Z, D) \in \Theta | L^TZL = D\}$$
$$C' = \{(Z, D) \in \Theta | Z \in C\}$$
$$Q' = \{(Z, D) \in \Theta | D \in Q\}$$

which are all nonempty, closed and convex.

Then finding Z satisfying (3) is finding a point in the intersection of C', G and Q', which can be solved by ADMM.

Algorithm for model (1)

Model (1) can be written as a saddle-point problem of the form:

$$\min_{x\in R^n} \max_{v \in R^l} f(x) + v^TPx - g(v)$$

where f, g are convex functions and $P = ZL$.

The Primal-Dual Hybrid Gradient (PDHG) a powerful tool for solving this problem.

- **Note:** Using matrix Z, instead of B, avoids redundant computations.

Properties of solution path

Theorem 1: Suppose $A^TA - \mu B^TL^TLB \succ O_n$, then the solution $x^*(\mu)$ to (1) exists, is unique, and is continuous in μ.

Theorem 2: Let $\tilde{x}$ be a solution to

$$\min_x \frac{1}{2}\|y - Ax\|_2^2 \text{ s.t. } Lx = 0$$

Then F_L in (1) is minimized by $\tilde{x}$ for all $\mu \geq \mu_0$, where $\mu_0 = \|A^T(A\tilde{x} - y)\|_2 / \sigma_{min}(L)$.

Numerical experiment

True image, $S \in R^{20\times 20}$, is a matrix in a shape of triangle with entries 1 inside of the triangle and −1 outside. The observation $y \in R^{80}$ is generated by $y_i = \text{vec}(X_i)^T\text{vec}(S) + \epsilon_i$, where entries of X_i are drawn from $N(0,1)$ and ϵ_i are white Gaussian noise with standard deviation σ. We compare the performance of model (1) and 2-D TV.

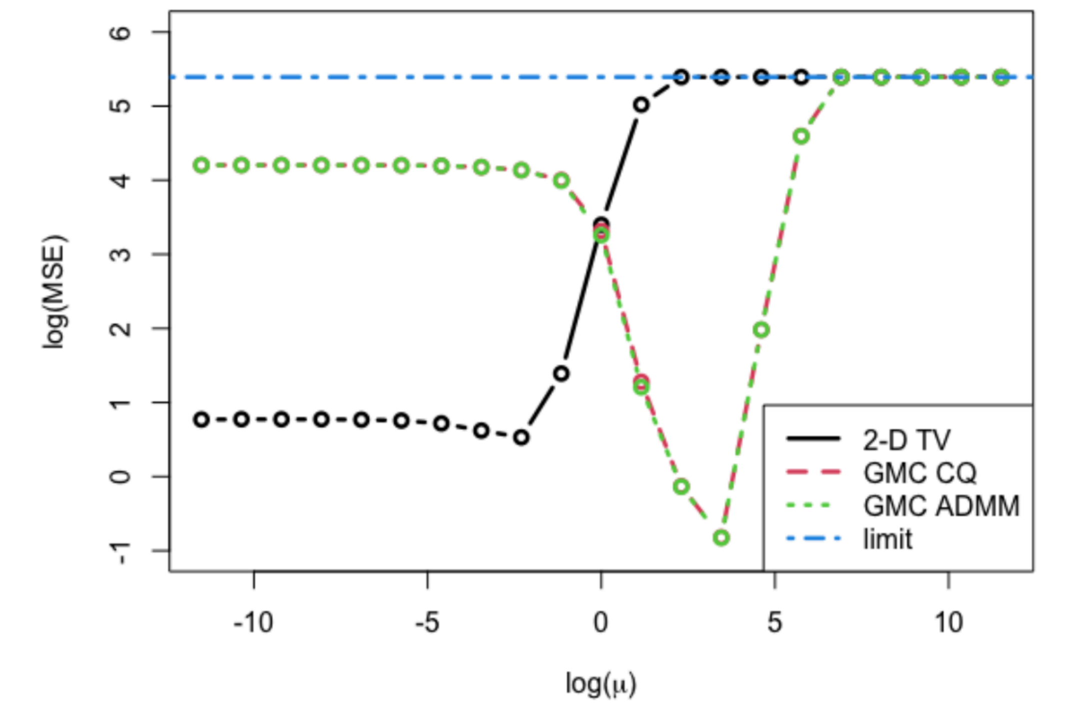

Figure 1: log(MSE) as a function of log(μ). The limit of log(MSE) is obtained by the $\tilde{x}$ in Theorem 2.

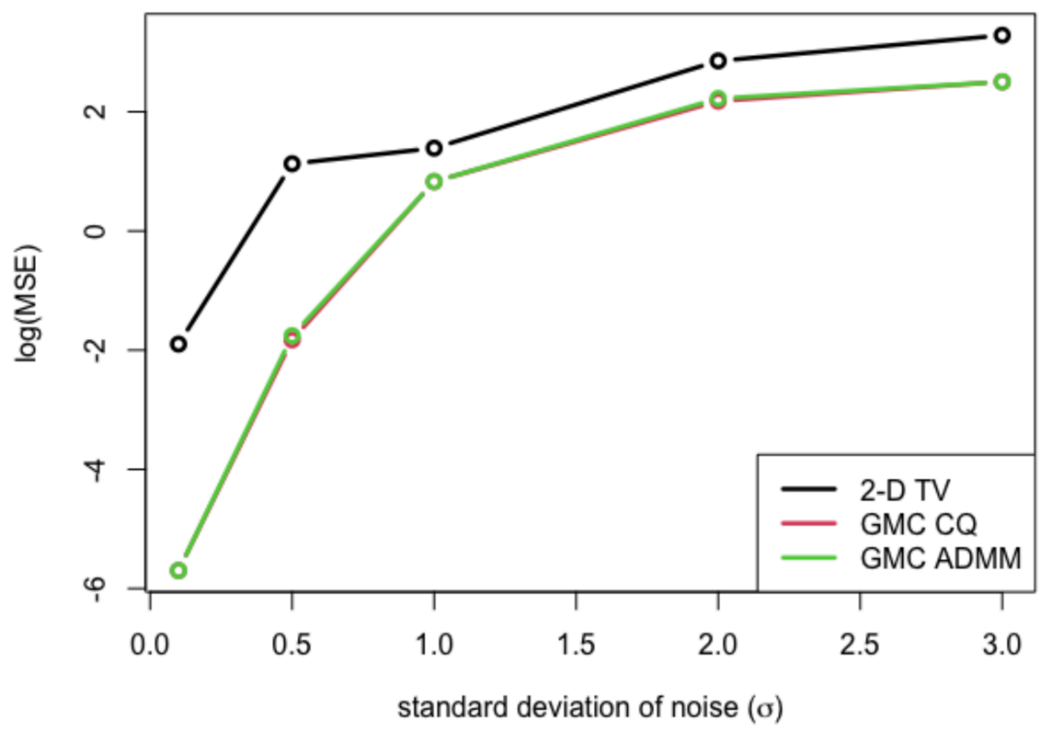

Figure 2: log(MSE) under different levels of noise.

Conclusion

Both CQ and ADMM algorithms work well to produce Z matrix, and the linearly involved GMC penalty has a better performance on the estimate accuracy comparing with the standard TV under different noise levels.

Bibliography

1. Abe, J. et.al, 2019. Convexity-edge-preserving signal recovery with linearly involved generalized minimax concave penalty function.