# Deep Neural Network Classifier for Multi-Dimensional Functional Data

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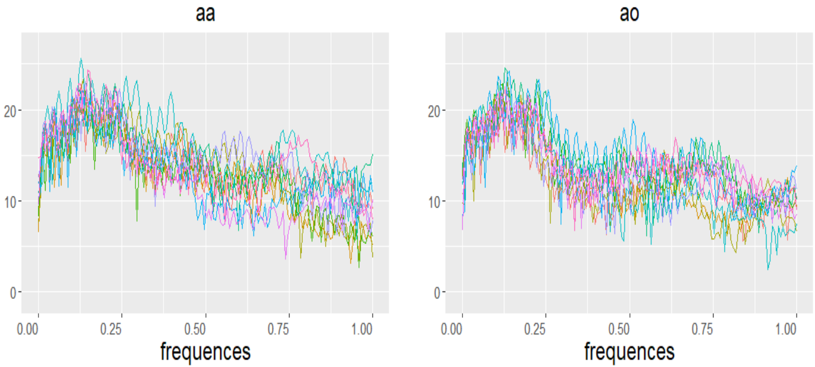
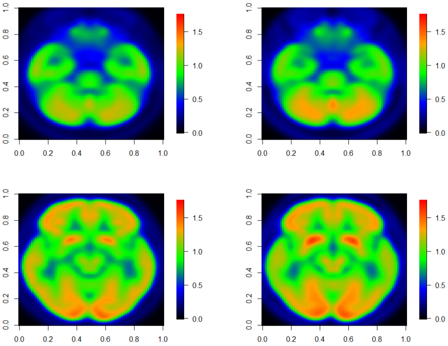
# Motivating example: speech data




aa $\Longleftrightarrow$ dark
ao $\Longleftrightarrow$ water

# Motivating example: Alzheimer disease

**EMCI** **AD**




EMCI : early mild cognitive impairment
AD : Alzheimer disease

# Literature Review

## Functional data classification (one-dimensional)

- k-nearest neighbor classifiers [Biau et al., 2005, Biau et al., 2010].

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- Functional discriminant analysis [Delaigle and Hall, 2012].

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- logistic regression [Araki et al., 2009].
- Functional discriminant analysis [Delaigle and Hall, 2012].
- Bayesian classifier [Dai et al., 2017].

Which classifier(s) can achieve optimality?

# Framework for functional data classification

- Class label $Y = \{1, -1\}$
- $X(\boldsymbol{s}) = \sum_{j=1}^{\infty} \xi_j \psi_j(\boldsymbol{s}) \Longleftrightarrow (\xi_1, \xi_2, \dots)^\intercal$
- Density for functional observations $h = (h_1, h_{-1}) \in \mathcal{H}$
  - Domain for $h$ has infinite dimension
  - Likelihood ratio $h_1/h_{-1}$ has composition structures of $q$ layers
  - The $u$-th composition layer:
    - $\beta_u$-Hölder functions
    - Intrinsic dimension bounded by $c_u < \infty$
- Noise condition with $\alpha$

$$\mathrm{P}\left(\left|\frac{h_1 - h_{-1}}{h_1 + h_{-1}}\right| \leq x\right) \leq Cx^{\alpha}, \qquad \forall x > 0. \tag{1}$$

# Statistical optimality for binary classification

## Misclassification risk for $\widehat{G}$

$$R_h(\widehat{G}_n) := \mathbb{E}_h[\mathbb{I}\{\widehat{G}_n(X) \neq Y\} | \{X_i, Y_i\}_{i=1}^n]$$

## Excess misclassification risk (EMR)

$$\mathcal{E}_h(\widehat{G}) := \mathbb{E}[R_h(\widehat{G}_n) - R_h(G^*)], \quad G^* \textit{ is the Bayesian classifier}$$

## Minimax excess misclassification risk (MEMR)

$$\max_{h \in \mathcal{H}} \mathcal{E}_h(\widetilde{G}) \asymp \inf_{\widehat{G}} \max_{h \in \mathcal{H}} \mathcal{E}_h(\widehat{G})$$

$\widetilde{G}$ is statistically optimal!

# Optimality for Functional Deep Neural Network (FDNN) Classifier

## MEMR lower bound

$$\inf_{\widehat{G}} \sup_{h \in \mathcal{H}} E\left[R_h(\widehat{G}) - R_h(G^*)\right] \gtrsim \left(\frac{1}{n}\right)^{S_0}.$$

## EMR for FDNN classifier

$$\sup_{h \in \mathcal{H}} E\left[R_h(\widehat{G}^{FDNN}) - R_h(G^*)\right] \lesssim \left(\frac{\log^3 n}{n}\right)^{S_0},$$

$S_0 = \min_{u=0,\ldots,q} \frac{\beta_u^*(\alpha+1)}{\beta_u^*(\alpha+2)+c_u}$, $\beta_u^* = \beta_u \prod_{w=u+1}^{q} \beta_w \wedge 1$.

$c_u \uparrow \Longleftrightarrow$ easier to classify (Bayesian) $\Longleftrightarrow$ slower convergence

$c_u \downarrow \Longleftrightarrow$ harder to classify (Bayesian) $\Longleftrightarrow$ faster convergence

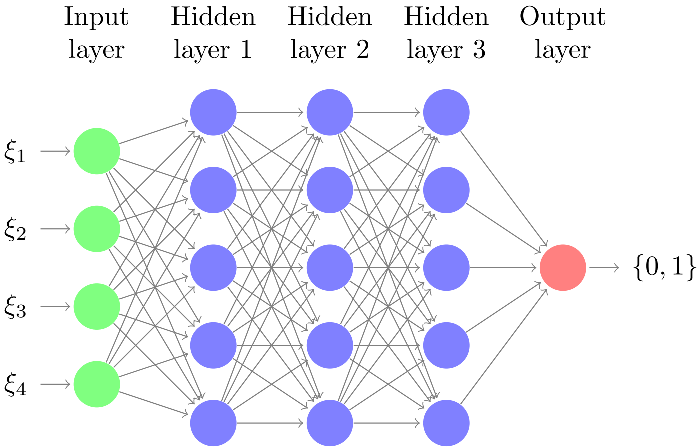
# Data driven architecture

## Hyperparameters for DNN

- Number of inputs: $\left(n\log^{-3} n\right)^{S_0/\rho} \lesssim J \lesssim (n\log^{-3} n)^{S_1}$;
- Depth: $L \asymp \log n$;
- Width: $\max_{1\leq \ell \leq L} p_\ell \asymp (n\log^{-3} n)^{S_1}$;
- Maximal weight: $B \asymp (n\log^{-3} n)^{S_2}$,

$$S_1 = \max_{0\leq u\leq q} \frac{c_u}{\beta_u^*(\alpha+2)+c_u}, S_2 = \min_{0\leq u\leq q} \frac{1}{\beta_u^*(\alpha+2)+c_u}, \rho > 0.$$

# Deep neural network structures ($J = 4$)



# Functional deep neural networks (FDNN) classifier

## Deep neural networks

Deep neural networks with ReLU active function and hinge loss

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## Projection scores as inputs

- Use the first $J$ projection scores $\{\xi_i^{(j)}\}_{j=1}^J$ as inputs of DNN
- Classification rule: $\widehat{G}^{FDNN}(Z) = \mathbb{I}\left(\widehat{f}_\phi(\boldsymbol{z}) < 0\right)$

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## Choice of ingredients

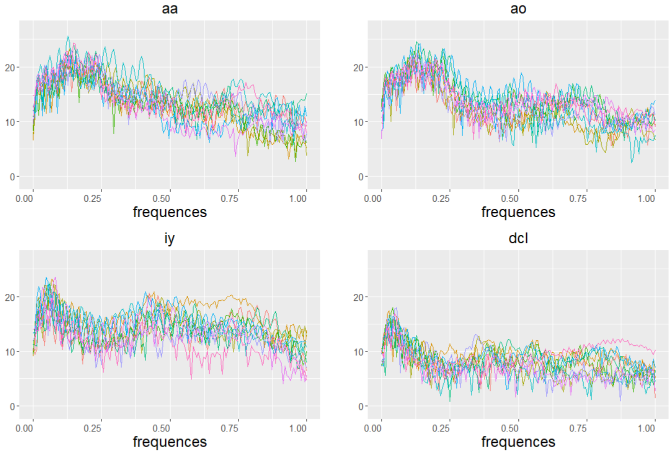
- Data driven network structure
- Flexible choice of basis functions

# Network hyperparameter selection

## Data-splitting method

- Step 1. Randomly divide the whole sample $\left(\{\xi_i^{(j)}\}_{j=1}^{J}, Y_i\right)$'s into two subsets indexed by $\mathcal{I}_{train}$ and $\mathcal{I}_{test}$, respectively, with about $|\mathcal{I}_{train}| = 0.8n$ and $|\mathcal{I}_{test}| = 0.2n$.
- Step 2. For each $(L, J, \boldsymbol{p}, B)$, we train a DNN based on subset $\mathcal{I}_{train}$, and then calculate the testing error based on subset $\mathcal{I}_{test}$ as
$$\mathrm{err}(L, J, \boldsymbol{p}, B) = \frac{1}{|\mathcal{I}_{test}|} \sum_{i \in \mathcal{I}_{test}} I(\widehat{f}_{L,J,\boldsymbol{p},B}(\widehat{\xi}_i^J) Y_i < 0).$$
- Step 3. Choose $(L, J, \boldsymbol{p}, B)$, possibly from a preselected set, to minimize $\mathrm{err}(L, J, \boldsymbol{p}, B)$.

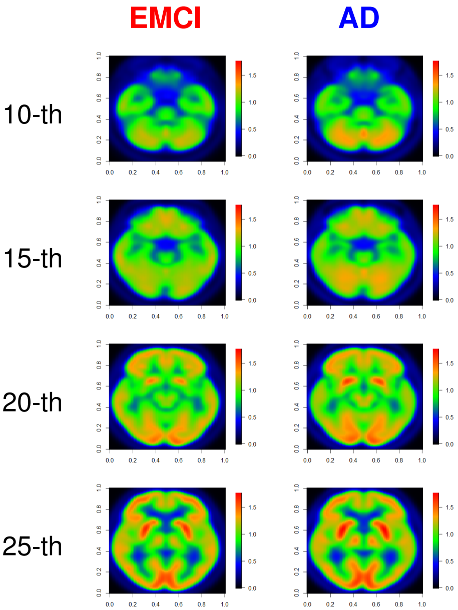
# Speech data



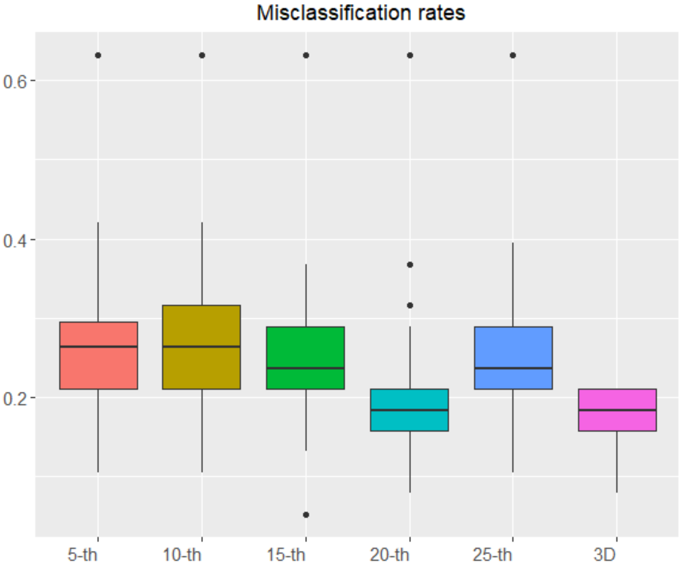
## Speech data misclassification rates

| Phonemes | FDNN | QD | NB |
|---|---|---|---|
| "aa" vs "ao" | 20.744 | 25.402 | 25.378 |
| "aa" vs "iy" | 0.193 | 0.288 | 0.273 |
| "ao" vs "iy" | 0.183 | 0.578 | 0.232 |
| "ao" vs "dcl" | 0.229 | 0.391 | 0.472 |

# ADNI data



# ADNI data misclassification rates



## Takeaways

**Motivation:** Whether and which of existing functional classifiers are statistically optimal? How to construct an optimal functional classifier with better performances?

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**Novelty:** Establish the first minimax theory for multi-dimensional functional data classification problem under non-Gaussian assumption.

**Importance:** Understand how the “best" functional classifier looks like, as well as provide a guidance to find the “best" classifiers.

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