The Asymmetric Hyperbolic Generalized Autoregressive Conditional Heteroscedastic (A-HYGARCH) model

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Abstract
The Hyperbolic Generalized Autoregressive Conditional Heteroscedastic (HYGARCH) model proposed by James Davidson in 2004, nests both the GARCH and fractionally integrated GARCH models. It is an extensively used long memory process to model the long-range dependence in volatility. The HYGARCH model, however treats both positive and negative shocks the same.To address this shortcoming, we propose a parsimonious asymmetric HYGARCH(A-HYGARCH) model to capture both long memory as well as the asymmetric response to positive and negative shocks. Small sample properties of the parameter estimates is studied using Monte-Carlo simulation and the utility of the proposed model is illustrated using a real-life data set.

Key Words: Long memory, Hyperbolic GARCH model, Volatility models, Asymmetry, Threshold models;

1. Introduction

Understanding the behavior of volatility in financial assets is important for option pricing, risk management, and many other financial activities. The Autoregressive Conditional Heteroskedastic (ARCH) model introduced by Engle (1982), the Generalized Autoregressive Conditional Heteroskedastic (GARCH) model by Bollerslev (1986) and variants of these formulations are extensively used for forecasting stock market volatility. Many of these models could capture key properties such as volatility clustering, but they fail to handle the long-term persistence of volatility found in many financial time series. This led to introduction of long memory extensions of the GARCH family such as Fractionally Integrated GARCH (FIGARCH) by Baillie (1996) and Hyperbolic GARCH (HYGARCH) by Davidson (2004). The HYGARCH model is widely used due to its versatility, as its conditional variance is a nested model with a short-memory component (GARCH) & a long-memory component(FIGARCH). In addition, most variants of ARCH & GARCH models are nested in the HYGARCH framework.

One key feature observed in financial market volatility is the presence of an asymmetric response, where positive and negative shocks of equal magnitude do not generate equal responses. That is, larger increases in volatility are observed when the shocks are negative compared to positive shocks of the same magnitude. The HYGARCH model, unfortunately, does not take this asymmetry into consideration. There are many extensions of the HYGARCH model proposed to address various shortcomings. Kwan, Li and Li (2011) proposed a threshold HYGARCH model which allows a different HYGARCH structure for each regime. Li, Li and Li (2015) proposed a new hyperbolic model with finite variance and a form nearly as simple as the FIGARCH model.Ferdous & Saeid introduced a HYGARCH model with a smooth transition structure for the conditional variance (2017) and a model with a Markov switching smooth-transition structure (2018).Shi & Yang(2018) introduced A-HYEGARCH model to estimate the long memory of high frequency time series

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with potential structural breaks.

None of these models, however, directly address the shortcoming of the symmetric response to positive and negative shocks. In this paper we propose a parsimonious model, the A-β-HYGARCH model to capture both the long memory as well as the asymmetric response in volatility. The parameters of this model are estimated using the quasi maximum likelihood estimation method. Small sample properties of the parameter estimates are studied using Monte-Carlo simulations.The effectiveness of the model and its performance in an empirical setting are illustrated using two time series of USD-SKW and USD-EUR exchange rates.
The paper is organized as follows. Section 2 introduces the proposed A-β-HYGARCH model. In section 3, the methodology for parameter estimation is developed. Section 4 presents the simulation results. In Section 5, we apply the proposed model to two daily exchange rate time series. Section 6 concludes this paper.

2. A-β-HYGARCH model

2.1 HYGARCH model

Although the FIGARCH model is developed to handle the long memory behavior of financial time series, there are some issues with the model such as not having a well-defined unconditional variance and not being covariance stationary. Thus, Davidson (2004) proposed the HYGARCH (hyperbolic GARCH) model to overcome these issues. The HYGARCH(p,d,q) model is defined as follows.

Let

$$\epsilon_t = \sigma_t z_t; \qquad z_t \overset{iid}{\sim} (0,1),$$

for all $t = 1, ..., T$. The HYGARCH models σ_t with

$$\sigma_t^2 = \omega + \lambda(L)\epsilon_t^2,$$

where

$$\omega = \frac{\delta}{1-\beta(1)},$$

$$0 \leq d \leq 1, \eta \geq 0,$$

$$\lambda(L) = \left[1 - \frac{\varphi(L)}{\beta(L)} ft(1 + \eta\left((1-L)^d - 1\right)\right] = \sum_{i=1}^{\infty} \lambda_i L^i,$$

$$\beta(L) = 1 - \sum_{i=1}^{q} \beta_i L^i \quad \& \quad \varphi(L) = 1 - \sum_{i=1}^{max(p,q)} \varphi_i L^i.$$

The HYGARCH model reduces to the GARCH,IGARCH and FIGARCH versions depending on the values of η and d. The HYGARCH conditional variance can be expressed as a convex combination of short-run GARCH component and a long-run FIGARCH component as follows:

$$\sigma_t^2 = \boldsymbol{\eta}\left[\omega + \left(1 - \frac{\varphi(L)}{\beta(L)}\right)(1-L)^d \epsilon_t^2\right] + (\mathbf{1} - \boldsymbol{\eta})\left[\omega + \left(1 - \frac{\varphi(L)}{\beta(L)}\right)\epsilon_t^2\right].$$

Similar to any ARCH type model, there are restrictive conditions to ensure conditional variance of HYGARCH model remain positive for all t. Conrad (2010) derived non negativity conditions for the HYGARCH model and the reader is referred to that publication for details.

2.2 A-β-HYGARCH model

The HYGARCH model is capable of modeling time series with highly persistent volatility and addresses some shortcomings found in the FIGARCH model. It, however, ignores the asymmetric response in volatility to positive and negative shocks. Hence, we propose the following parsimonious model capable of handling both long memory and asymmetry in volatility shown by financial time series.

Take

$$\epsilon_t = \sigma_t z_t,$$

with z_t as above and let

$$P_t = \sum_{i=1}^{n} \epsilon_{t-i},$$

be the net returns over the past n days ($n = 1$ for this study) and the conditional variance of A-β-HYGARCH model is defined as follows:

$$\sigma_t^2 = \begin{cases} \omega + \lambda_1(L)\epsilon_t^2 & : P_t \geq 0 \\ \omega + \lambda_2(L)\epsilon_t^2 & : P_t < 0. \end{cases}$$

The HYGARCH weights for the A-β-HYGARCH model is calculated as follows

$$\lambda_1(L) = \left[1 - \frac{\varphi(L)}{\beta_1(L)}\left(1 + \eta\left((1-L)^d - 1\right)\right)\right] = \sum_{i=1}^{\infty} \lambda_{1_i} L^i,$$

$$\lambda_2(L) = \left[1 - \frac{\varphi(L)}{\beta_2(L)}\left(1 + \eta\left((1-L)^d - 1\right)\right)\right] = \sum_{i=1}^{\infty} \lambda_{2_i} L^i.$$

Here we define two sets of β parameters which will eventually result in two sets of HYGARCH weights and one of the weights sets will be used in calculating conditional variance depending on the value of $P_t = \sum_{i=1}^{n} \epsilon_{t-i}$, hence resulting in an asymmetric response to positive and negative returns.

2.3 A-β-HYGARCH(1,d,1) model weights

A-β-HYGARCH model is defined as in Section 2.2 and following results were derived using the same arguments and assumptions employed by Conrad(2010) for the HYGARCH model:

$$\lambda_j(L) = \left[1 - \frac{1-\varphi L}{1-\beta_j L}\left(1 + \eta\left((1-L)^d - 1\right)\right)\right] = \sum_{i=1}^{\infty} \lambda_{j_i} L^i,$$

$$\lambda_{j_1} = \eta d + \varphi - \beta_j,$$

$$\lambda_{j_k} = \beta_j \lambda_{k-1} + \eta((k-1-d/k) - \varphi) * (-g_{k-1}),$$

where

$$j = 1,2 \quad \& \quad g_k = (k-1-d/k) * (g_{k-1}).$$

3. Parameter Estimation

To estimate the parameters of A-β-HYGARCH$(1, d, 1)$ & A-φ-HYGARCH$(1, d, 1)$ models, we chose the method of Quasi Maximum Likelihood Estimation (QMLE), with assumption that $z_t \overset{iid}{\sim} \mathcal{N}(0, 1)$. We want to maximize

$$\ln\left(L(\boldsymbol{\theta}; \{\epsilon_t\}_{t=1}^{T})\right) = -\frac{1}{2}\left[T\ln(2\pi) + \sum_{i=1}^{T}\left(\ln(\sigma_t^2) + \epsilon_t^2\sigma_t^{-2}\right)\right],$$

where $\boldsymbol{\theta_\beta} = (\omega, \eta, d, \varphi, \boldsymbol{\beta_1}, \boldsymbol{\beta_2})'$ & $\boldsymbol{\theta_\varphi} = (\omega, \eta, d, \beta, \boldsymbol{\varphi_1}, \boldsymbol{\varphi_2})'$ are the parameters of σ_t^2 of A-β-HYGARCH model and A-φ-HYGARCH model respectively. The optimization was done following the restrictions obtained by using arguments similar to those employed by Conrad (2010).

4. Simulation Study

In this section we provide details of a Monte Carlo simulation study conducted to investigate the performance of the A-β-HYGARCH model. Time series of length 15,000 are generated and the first 15,000-T observations are discarded depending on the required time series length T to mitigate effects due to initial conditions. In the simulation experiment, two sample lengths T = 2,500 and 5,000 observations are considered. A total of 500 simulation runs were made for each sample size. The simulation results are used to estimate any estimation bias that is present and finite sample performance of the Gaussian QMLE method.

The calculated bias of the estimates and the standard error (SE) values of the parameters estimates for the A-β-HYGARCH model is summarized in Table 1 and Table 2.
From Table 1 and Table 2 it can be seen that both bias and standard error (SE) are within an acceptable range and decrease as the sample size increases. This is true for both selected parameter sets we studied.

Table 1: Simulation results for A-β HYGARCH model: Parameter Set 1

		$T = 2500$			$T = 5000$		
Parameter	Real Value	Estimated	Bias	SE	Estimated	Bias	SE
ω	0.0800	0.0900	0.0100	0.0006	0.0856	0.0056	0.0004
η	0.9500	0.9397	0.0103	0.0013	0.9452	0.0048	0.0008
d	0.8500	0.8630	0.0130	0.0042	0.8599	0.0099	0.0034
φ	0.200	0.1848	0.0152	0.0027	0.1901	0.0099	0.0023
$\beta 1$	0.8700	0.8606	0.0094	0.0023	0.8662	0.0038	0.0017
$\beta 2$	0.7200	0.7091	0.0109	0.0033	0.7177	0.0023	0.0024

Table 2: Simulation results for A-β HYGARCH model: Parameter Set 2

		$T = 2500$			$T = 5000$		
Parameter	Real Value	Estimated	Bias	SE	Estimated	Bias	SE
ω	0.1000	0.1557	0.0557	0.0018	0.1319	0.0319	0.0011
η	0.9500	0.9236	0.0264	0.0015	0.9329	0.0171	0.0011
d	0.5500	0.6667	0.1167	0.0045	0.6198	0.0698	0.0032
φ	0.200	0.1598	0.0402	0.0034	0.1798	0.0202	0.0028
$\beta 1$	0.5500	0.6021	0.0521	0.0049	0.5820	0.0320	0.0037
$\beta 2$	0.3500	0.3963	0.0463	0.0061	0.3826	0.0326	0.0043

5. Empirical Study

In this section, we apply the proposed A-β-HYGARCH model to two financial time series: daily exchange rates of Korean Won against the US dollar and Euro against the US dollar. Then HYGARCH (accounts for long memory) , FIGARCH (accounts for long memory) and EGARCH (accounts for asymmetry) models are applied to the same time series to compare the performance against the proposed model. The time spans for the two data sets are as follows:

- The daily exchange rates of Korean Won against the US dollar are from 12/9/2003 to 12/1/2010, and there are 1822 observations in total.
- The daily exchange rates of Euro against the US dollar are from 9/22/2006 to2/6/2012, and there are 1402 observations in total.

It is noteworthy to point out that the selected financial time series span the global financial crisis from 2007 to 2009. Four models A-β-HYGARCH(1,d,1) , HYGARCH(1,d,1), FIGARCH(1,d,1) and EGARCH(1,1) are then applied to $100 * log(\frac{X_t}{X_{t-1}})$ of the daily exchange rates X_t. The results pertaining to the comparison of the empirical performance between the four models are presented in Table 3 and Table 4.

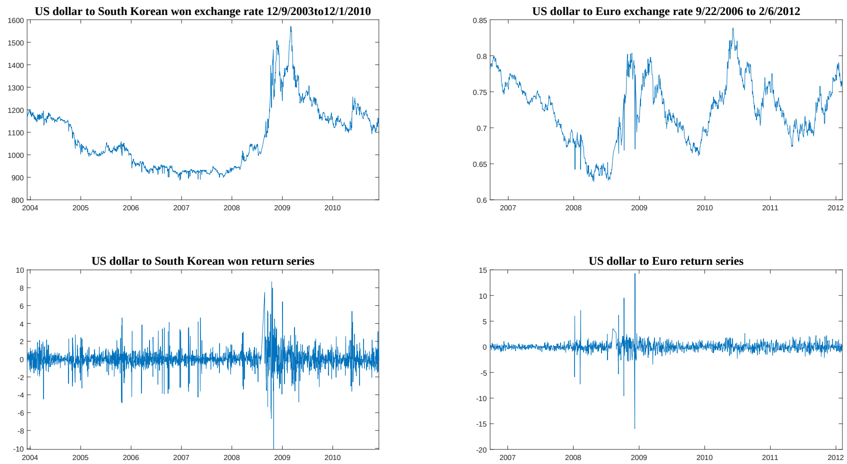

Figure 1: USD-SKW and USD-EUR daily exchange rates

Table 3: Summary statistics for the two daily exchange rates return series.

	USD-SKW	USD-EUR
Min	-10.1195	-15.9632
Max	8.6818	14.3324
Mean	-0.0010	-0.0018
S.D	1.1668	1.0905
Skew	0.2188	-0.5726
Kurt.	14.0831	69.7957

Table 4: Empirical Study of US dollar to South Korean won exchange rate 12/9/2003 to 12/1/2010

	A-β-HYGARCH	HYGARCH	FIGARCH	EGARCH
AIC	4931.9962	4965.1112	4976.2601	4998.6769
BIC	4919.9962	4955.1112	4968.2601	4990.6769
ω	0.2772	0.3120	0.1797	-
η	0.9532	0.8421	-	-
d	0.8472	0.6135	0.2616	-
φ	0.1589	0.2368	0.2248	-
β	-	0.5414	0.2689	-
β_1	0.8747	-	-	-
β_2	0.6709	-	-	-

From the results presented in Table 3 and Table 4 it is observed that A-β HYGARCH model outperforms HYGARCH, FIGARCH and EGARCH models in terms of AIC and BIC values.

Table 5: Empirical Study of US dollar to Euro exchange rate 9/22/2006 to 2/6/2012

	A-β-HYGARCH	HYGARCH	FIGARCH	EGARCH
AIC	3277.8719	3298.6782	3297.4318	3360.6816
BIC	3265.8719	3288.6782	3289.4318	3352.6816
ω	0.0606	0.02871	0.0026	-
η	1.0036	0.9945	-	-
d	0.8604	0.8504	0.8353	-
φ	0.2028	0.2139	0.2195	-
β	-	0.9336	0.9319	-
β_1	0.9448	-	-	-
β_2	0.9008	-	-	-

6. Conclusion

This paper proposes an extension to the HYGARCH model, nameley the Asymmetric-β-HYGARCH, which can capture long memory as well as the asymmetric response to positive and negative shocks.The quasi-maximum likelihood procedure is used to estimate the parameters of this model. In light of the Monte-Carlo simulation results, it is evident that Maximum Likelihood Estimation provides quite accurate estimates, specially for large sample sizes.The usefulness of the A-β HYGARCH model in practical situations is evaluated with empirical evidence.The examples on exchange rates (United States Dollar to South Korean won and United States Dollar to Euro) showed that the proposed A-β HYGARCH model performs better than HYGARCH, FIGARCH and EGARCH models. Hence, the A-β HYGARCH model has the potential to be a powerful toll for modeling both long memory and asymmetric behavior of volatility series.

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