Bayesian Designs for Progressively Type-I Censored Simple Step-Stress Accelerated Life Tests Under Cost Constraint and Order-Restriction Based on a Three-Parameter Gamma Prior

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Abstract
In this work, we investigate order-restricted Bayesian cost constrained design optimization for progressively Type-I censored simple step-stress accelerated life tests with exponential lifetimes under continuous inspections. Previously we showed that using a three-parameter gamma distribution as a conditional prior ensures order restriction for parameter estimation and that the conjugate-like structure provides computational simplicity. Adding on to our Bayesian design work, we explore incorporating a cost constraint to various criteria based on Shannon information gain and the posterior variance-covariance matrix. We derive the formula for expected termination time and expected total cost and propose estimation procedures for each. We conclude with results and a comparison of the efficiencies for the constrained vs. unconstrained tests from an application of these methods to an extension of our previous simulation study.

Key Words: accelerated life tests, Bayesian analysis, design of experiments, cost constrained, progressive Type-I censoring, step-stress loading

1. Introduction

Accelerated life testing (ALT) is known to be a useful technique to determine the life distribution of products and devices that have high reliability, e.g.) when running a test at normal operating conditions would be too time consuming to perform and/or would yield little useful information. The term accelerated generally refers to running a life test at higher levels of stress. This could be higher temperature, voltage, pressure, etc. Once performed, another model is used to relate the life characteristics back to the use stress. These tests can be executed in various ways, monitoring the failures continuously or on intervals, imposing k-levels of stress in a constant, step, ramp or random fashion, and by possibly censoring the data, such as with progressive Type-I, Type-II or hybrid censoring. Deciding which settings to use can be an experiment in itself, and these choices may be motivated by other factors including cost.

When designing an experiment with a cost constraint in mind, one must set a prespecified budget. This budget should account for costs associated with setup, operation, inspection and the test units. Therefore, other necessary choices to consider include the sample size and design criterion. By setting the number of available units, the maximum duration of time that a test can run can be identified. With this, design criteria choices become relevant.

Regarding design criteria, most Frequentist research involves solving some function of the Fisher information matrix. Using Bayesian statistics, analogous criteria can be defined using the posterior variance-covariance matrix. In the work to follow, we will utilized Bayesian statistics and in addition to criteria based on the variance-covariance matrix, we will also detail another information-theoretic criterion based on Shannon information gain.

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This work will focus on adding a cost constraint to progressively Type-I censored simple ($k = 2$) step-stress accelerated life tests (SSALTs) under continuous inspections assuming that the lifetimes are exponential and that a cumulative exposure model holds. For more details related to this type of censoring, we recommend Balakrishnan and Cramer (2014). For step-stress testing and the cumulative exposure model, we recommend reviewing Nelson (1980) or Xu and Fei (2012).

Fundamental and intriguing work for constrained design optimization problems is seen in the literature from various authors. Han (2015) compares constant-stress and step-stress ALT cost and time constrained designs for k-levels under exponential failure time distributions and Type-I censoring. Lim (2015) examines constant-stress accelerated degradation testing using a gamma process model. Xiang et al. (2017) use an adaptive plan from Bayesian methods to approach cost constrained constant-stress ALT design problems for log-location-scale failure time distributions. Hakamipour (2019) explores a Rayleigh lifetime distribution for progressively Type-I censored data from SSALTs. Han (2020) derives the expected termination times of progressively Type-I censored step-stress accelerated life tests under continuous and interval inspections. Hakamipour (2021) compares constant-stress and step-stress ALTs under a Rayleigh lifetime distribution, and lastly Han (2021) provides cost function derivations and considers non-uniform step durations for SSALTs.

To contribute to this growing collection, we introduce an attractive Bayesian approach to the cost-constrained design optimization problem. The selected prior distribution is based off of related work and is particularly valuable for order-restricted inference, see Wiedner and Han (2020). This article is organized as follows. In Section 2 we provide the model for simple SSALTs. In Section 3 we discuss the expected termination time, and in Section 4 we discuss the expected total cost. In Section 5 we examine the specific design criteria under consideration. In Section 6 we review the simulation results for this Bayesian optimal design. And lastly, in Section 7 we conclude the paper and note our plans for future work.

2. Model Description

Assuming that the lifetime of a test unit follows an exponential distribution, the rate parametrized probability distribution function (PDF) and cumulative distribution function (CDF) at stress level x_i are respectively given by

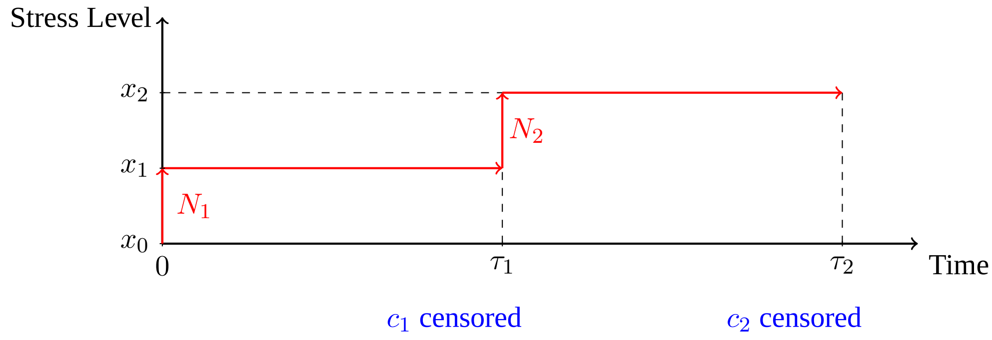
$$f_i(t) = \lambda_i \exp(-\lambda_i t) \quad \text{and}$$
$$F_i(t) = 1 - S_i(t) = 1 - \exp(-\lambda_i t) \tag{1}$$

for $t \geq 0$.

In the following, N_i will represent the number of units entering stress level x_i, and n_i will represent the number of units that failed at stress level x_i. These stress levels are run on the intervals $[\tau_{(i-1)}, \tau_i)$, also denoted as Δ_i. Using continuous monitoring, $y_{i,l}$ will represent the l-th ordered failure time of the n_i failed units with $l = 1, 2, \ldots, n_i$. c_i^* is a fixed number of units to be censored at the end of stress level x_i. The actual number of censored units at time τ_i will be denoted as c_i, which is equal to $\min\{c_i^*, N_i - n_i\}$.

Therefore for a simple step-stress ALT under progressive Type-I censoring, $N_1 \equiv n$ units enter at stress level x_1, which is run until τ_1. At this time, c_1 of the surviving units are randomly removed from the test, leaving $N_2 = n - n_1 - c_1$ units to enter the new stress level x_2. At time τ_2 the test is stopped and the remaining units $c_2 = N_2 - n_2 = n - n_1 - n_2 - c_1$ are right censored. This is depicted in Figure 1.

Figure 1: Simple SSALT with Progressive Type-I Censoring



We can assume a model to relate the effects observed at the different stress levels. Here we will choose the cumulative exposure model. The PDF and CDF of a test unit for a simple step-stress ALT are respectively then

$$f(t) = \begin{cases} f_1(t), & 0 \leq t < \tau_1 \\ S_1(\tau_1) f_2(t - \tau_1), & \tau_1 \leq t < \infty \end{cases} \tag{2}$$

$$F(t) = \begin{cases} 1 - S_1(t), & 0 \leq t < \tau_1 \\ 1 - S_1(\tau_1) S_2(t - \tau_1), & \tau_1 \leq t < \infty. \end{cases} \tag{3}$$

The joint distribution of the failure counts, $\boldsymbol{n}$, and times, $\boldsymbol{y}$, is

$$f_J(\boldsymbol{y}, \boldsymbol{n}) = \left[\prod_{i=1}^{2} \frac{N_i!}{(N_i - n_i)!} \right] \left[\prod_{i=1}^{2} \lambda_i^{n_i} \right] \exp\left(- \sum_{i=1}^{2} \lambda_i U_i \right) \tag{4}$$

where $U_1 = \sum_{l=1}^{n_1} y_{1,l} + (n - n_1)\tau_1$ and $U_2 = \sum_{l=1}^{n_2} (y_{2,l} - \tau_1) + (n - n_1 - n_2 - c_1)(\tau_2 - \tau_1)$. This model has been used and described in other works including Gouno et al. (2004).

Using a Bayesian framework we propose an order restricted conjugate-like prior, $\pi(\lambda_1, \lambda_2)$, that is a joint distribution of 3-parameter gamma ($viz.$, Erlang) distributions as in Equation 5. Utilizing the shift parameter in this distribution allows us to ensure that the rate parameter increases as the stress level increases.

$$\begin{aligned} \pi(\lambda_1) &= \frac{\gamma_1^{\alpha_1}}{(\alpha_1 - 1)!} (\lambda_1)^{\alpha_1 - 1} \exp(-\gamma_1 \lambda_1) \\ \pi(\lambda_2 | \lambda_1) &= \frac{\gamma_2^{\alpha_2}}{(\alpha_2 - 1)!} (\lambda_2 - \lambda_1)^{\alpha_2 - 1} \exp(-\gamma_2 (\lambda_2 - \lambda_1)) \\ \alpha_i &\in \{1, 2, 3, \ldots\} \text{ and } \gamma_i > 0 \qquad i = 1, 2 \end{aligned} \tag{5}$$

The exact joint posterior distribution, $\pi(\lambda_1, \lambda_2 | \boldsymbol{y})$, was computed in Wiedner and Han (2020).

3. Termination Time

For a simple step-stress accelerated life test under progressive type-I censoring, the termination time of a test can be expressed as

$$T_1 = \begin{cases} \min\{Y_{1,N_1}, \Delta_1\}, & \text{if } N_2 = 0 \\ \Delta_1 + \min\{Y_{2,N_2} - \tau_1, \Delta_2\}, & \text{if } N_2 > 0 \end{cases} \tag{6}$$

where Y_{i,N_i} is the largest order statistic from a sample of size N_i starting at time τ_{i-1}. As previously mentioned, Han (2020) has derived the expected termination time for the Frequentist setting, which is

$$\begin{aligned} E[T_1] &= \frac{1}{\lambda_1}\sum_{l=1}^{N_1}\binom{N_1}{l}\frac{(-1)^{(l+1)}}{l}F_1(l\Delta_1)+F_1(\Delta_1)^{N_1} \\ &\times \mathbb{1}_{(N_2>0)}\sum_{N_2=1}^{N_1-c_1^*}\binom{N_1}{N_2+c_1^*}\left[\frac{S_1(\Delta_1)}{F_1(\Delta_1)}\right]^{N_2+c_1^*}\frac{1}{\lambda_2}\sum_{l=1}^{N_2}\binom{N_2}{l}\frac{(-1)^{(l+1)}}{l}F_2(l\Delta_2) \\ &= \frac{1}{\lambda_1}\sum_{l=1}^{N_1}\binom{N_1}{l}\frac{(-1)^{(l+1)}}{l}F_1(l\Delta_1)+\frac{1}{\lambda_2}\sum_{m=0}^{n-c_1^*-1}\sum_{j=0}^{m}\sum_{l=1}^{n-m-c_1^*}\binom{n}{m}\binom{m}{j}\binom{n-m-c_1^*}{l} \\ &\times \frac{(-1)^{(j+l+1)}}{l}S_1(\Delta_1)^{n-m+j}F_2(l\Delta_2). \end{aligned} \tag{7}$$

Here $\mathbb{1}$ represents the indicator function. Recall c_1^* was the fixed number of units to be censored at the end of stress level x_1. In the special case where $c_1^* = 0$, i.e.) Type-I censoring, the expected termination time was given as

$$E[T_1] = \sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{(l+1)}}{l}\left[\frac{F_1(l\Delta_1)}{\lambda_1}+\frac{S_1(l\Delta_1)F_2(l\Delta_2)}{\lambda_2}\right]. \tag{8}$$

In this Bayesian framework, λ_1 and λ_2 are random, so to compute the expected termination time, we need to take an additional expectation with respect to these parameters. Specifically, $E_{Bayes}[T_1] = E_{\boldsymbol{\lambda}}[E[T_1|\lambda_1, \lambda_2]]$. This result is complex and involves mixtures of independent gamma random variables with unequal scales. To simplify, we can approximate this expectation. Two approaches were considered. One was to simply use Monte Carlo simulation, and the other was to use a moment-matching method which lets us use a single gamma variable in place of a mixture, see Covo and Elalouf (2014). Equation 9 is the result for the case of Type-I censoring. In this formula, γ_{i*} and α_{i*}, $i = 1, 2$, are the parameters identified for two different single gammas and τ is the total test duration.

$$\begin{aligned} E_{\boldsymbol{\lambda}}[E[T_1|\lambda_1,\lambda_2]] &= \sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{(l+1)}}{l}\left\{\frac{\gamma_1}{\alpha_1-1}\left[1-\left(1+\frac{l\Delta_1}{\gamma_1}\right)^{1-\alpha_1}\right]\right. \\ &+ \left.\frac{\gamma_{1*}^{[l]}}{\alpha_{1*}^{[l]}-1}\left(1+\frac{l\Delta_1}{\gamma_1}\right)^{-\alpha_1} - \frac{\gamma_{2*}^{[l]}}{\alpha_{2*}^{[l]}-1}\left(1+\frac{l\tau}{\gamma_1}\right)^{-\alpha_1}\left(1+\frac{l\Delta_2}{\gamma_2}\right)^{-\alpha_2}\right\} \end{aligned} \tag{9}$$

4. Total Cost

For a simple step-stress accelerated life test under progressive type-I censoring, the total cost of a test is given by

$$\begin{aligned} C_T &= C_{set} + nC_{unit} + C_{op}^{[1]} + C_{ins}(U_1+U_2) \\ &+ (n_1+n_2)C_{fail} + (n-n_1-n_2)C_{unfail}. \end{aligned} \tag{10}$$

Here C_{set} denotes the set up cost for the experiment, and C_{unit} is the cost of each unit. If $y_{1,N_1} > \tau_1$, then the total operating cost, $C_{op}^{[1]}$, is $C_{op}(x_1)\Delta_1 + C_{op}(x_2)\min\{y_{2,N_2} -$

$\Delta_1, \Delta_2\}$, otherwise it will simply be $C_{op}(x_1)\min\{y_{1,N_1}, \Delta_1\}$. $C_{op}(x_i)$ is the operating cost at the corresponding stress level. C_{ins} is the cost of inspection, C_{fail} is the cost associated with a failed unit and C_{unfail} is the cost associated with an unfailed unit.

With this we should ensure that the total cost does not exceed the experimental budget, C_B. However, the total cost is random, so to conservatively run the test, we should set the largest possible cost, $\max\{C_T\}$, to be less than the budget.

Our previous work for progressive Type-I censoring used a predefined proportion of units to censor at each stress level. Specifically, $0 \leq \pi_1^* < 1$ and by design $\pi_2^* \equiv 1$. To follow these choices, we should determine the actual number of units censored, which we can identify as $c_i = \Gamma((N_i - n_i)\pi_i^*)$, where Γ is some discretizing function. We can say $c_i \approx ((N_i - n_i)\pi_i^*)$, so to estimate $\max\{C_T\}$ we use

$$\begin{aligned}\max\{C_T\} &= C_{set} + nC_{unit} + C_{op}(x_1)(\Delta_1) + C_{op}(x_2)(\Delta_2) \\ &+ nC_{ins}(\Delta_1 + (1-\pi_1^*)\Delta_2) + n(1-\pi_1^*)C_{fail} + n\pi_1^* C_{unfail}. \end{aligned} \tag{11}$$

It may be of interested to know the expected total cost, which is computed as

$$\begin{aligned}E[C_T] &= C_{set} + nC_{unit} + E[C_{op}^{[1]}] + C_{ins}(E[U_1] + E[U_2]) \\ &+ C_{fail}(E[n_1] + E[n_2]) + C_{unfail}(n - E[n_1] - E[n_2]). \end{aligned} \tag{12}$$

These expectations are obtained as seen in Han (2021) and for our simple SSALT problem, they are

$$\begin{aligned}E[C_{op}^{[1]}] &= \frac{C_{op}(x_1)}{\lambda_1}\sum_{l=1}^{N_1}\binom{N_1}{l}\frac{(-1)^{(l+1)}}{l}F_1(l\Delta_1) + F_1(\Delta_1)^{N_1} \\ &\times \mathbb{1}_{(N_2>0)}\sum_{l=1}^{N_1}\binom{N_1}{l}\left[\frac{S_1(\Delta_1)}{F_1(\Delta_1)}\right]^l \frac{C_{op}(x_2)}{\lambda_2}\sum_{l=1}^{N_2}\binom{N_2}{l}\frac{(-1)^{(l+1)}}{l}F_2(l\Delta_2) \\ &= \frac{C_{op}(x_1)}{\lambda_1}\sum_{l=1}^{N_1}\binom{N_1}{l}\frac{(-1)^{(l+1)}}{l}F_1(l\Delta_1) + \frac{C_{op}(x_2)}{\lambda_2} \\ &\times \sum_{m=0}^{n-c_1^*-1}\sum_{j=0}^{m}\sum_{l=1}^{n-m-c_1^*}\binom{n}{m}\binom{m}{j}\binom{n-m-c_1^*}{l}\frac{(-1)^{(j+l+1)}}{l}S_1(\Delta_1)^{n-m+j}F_2(l\Delta_2)\end{aligned} \tag{13}$$

$E[U_i] = E[n_i]/\lambda_i$ and $E[n_i] = E[N_i]F_i(\Delta_i) = nF_i(\Delta_i)\prod_{j=1}^{i-1}S_j(\Delta_j)(1-\pi_j^*)$. Again $\mathbb{1}$ represents the indicator function. We can express the expected total cost compactly as

$$\begin{aligned}E[C_T] &= C_{set} + n(C_{unit} + C_{unfail}) + E[C_{op}^{[1]}] \\ &+ n\sum_{i=1}^{2}\left(\frac{C_{ins}}{\lambda_i} + C_{fail} - C_{unfail}\right)F_i(\Delta_i)\prod_{j=1}^{i-1}S_j(\Delta_j)(1-\pi_j^*). \end{aligned} \tag{14}$$

In the case of Type-I censoring, the expected total cost is given as

$$
\begin{aligned}
E[C_T] &= C_{set} + n(C_{unit} + C_{fail}F(\tau_2) + C_{unfail}S(\tau_2)) + nC_{ins}(\lambda_1^{-1}F_1(\Delta_1) \\
&+ \lambda_2^{-1}S_1(\Delta_1)F_2(\Delta_2)) + \frac{C_{op}(x_1)}{\lambda_1}\sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{(l+1)}}{l}F_1(l\Delta_1) \\
&+ \frac{C_{op}(x_2)}{\lambda_2}\sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{(l+1)}}{l}S_1(l\Delta_1)F_2(l\Delta_2). \qquad (15)
\end{aligned}
$$

These results were derived for the Frequentist setting, and to utilize our Bayesian framework, we again must account for the randomness of λ_1 and λ_2. Similar to the expected termination time, we need $E_{\text{Bayes}}[C_T] = E_{\boldsymbol{\lambda}}[E[C_T|\lambda_1, \lambda_2]]$. For the study to follow in Section 6, to obtain the expected total cost, we again relied on Monte Carlo simulations and similarly computed an alternative moment-matching method estimate. Equation 16 shows this approximated expectation for the Type-I censoring case.

$$
\begin{aligned}
E_{\boldsymbol{\lambda}}[E[C_T|\lambda_1,\lambda_2]] &= C_{set} + nC_{unit} + nC_{fail} \\
&+ n(C_{unfail} - C_{fail})\left(1+\frac{\tau}{\gamma_1}\right)^{-\alpha_1}\left(1+\frac{\Delta_2}{\gamma_2}\right)^{-\alpha_2} \\
&+ nC_{ins}\left\{\frac{\gamma_1}{\alpha_1-1}\left[1-\left(1+\frac{\Delta_1}{\gamma_1}\right)\right]^{1-\alpha_1}\right. \\
&+ \left.\frac{\gamma_{1*}^{[1]}}{\alpha_{1*}^{[1]}-1}\left(1+\frac{\Delta_1}{\gamma_1}\right)^{-\alpha_1} - \frac{\gamma_{2*}^{[1]}}{\alpha_{2*}^{[1]}-1}\left(1+\frac{\tau}{\gamma_1}\right)^{-\alpha_1}\left(1+\frac{\Delta_2}{\gamma_2}\right)^{-\alpha_2}\right\} \\
&+ C_{op}(x_1)\sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{(l+1)}}{l}\frac{\gamma_1}{\alpha_1-1}\left[1-\left(1+\frac{l\Delta_1}{\gamma_1}\right)\right]^{1-\alpha_1} \\
&+ C_{op}(x_2)\sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{(l+1)}}{l}\left\{\frac{\gamma_{1*}^{[l]}}{\alpha_{1*}^{[l]}-1}\left(1+\frac{l\Delta_1}{\gamma_1}\right)^{-\alpha_1}\right. \\
&- \left.\frac{\gamma_{2*}^{[l]}}{\alpha_{2*}^{[l]}-1}\left(1+\frac{l\tau}{\gamma_1}\right)^{-\alpha_1}\left(1+\frac{l\Delta_2}{\gamma_2}\right)^{-\alpha_2}\right\} \qquad (16)
\end{aligned}
$$

5. Design Criteria

Bayesian methods are known to be a good alternatives to use for design optimization because they reduce the dependence on assumptions that are needed in the Frequentist setting as well as the asymptotic results that are usually used. The utility functions that we consider for design optimization include an information theoretic criterion, the H-optimal design, and designs based off of the posterior variance-covariance matrix, the D, C, A, E and M-optimal designs.

The H-optimal design maximizes

$$U_H(\zeta) = E\big[\log \pi(\boldsymbol{\Lambda}|\boldsymbol{T})\big] - E\big[\log \pi(\boldsymbol{\Lambda})\big].$$

This specific function is used to measure the uncertainty about $\boldsymbol{\Lambda}$ in terms of Shannon entropy. Specifically, this is defined as the prior-posterior differential gain in the Shannon information content. We have this expressed above as the expectation of the log of the prior distribution subtracted from the expectation of the log of the posterior distribution.

The D-optimal design maximizes

$$U_D(\zeta) \quad = \quad \frac{1}{2}\log\det\big(\boldsymbol{\Sigma}_0\big) - \frac{1}{2}\int_{\boldsymbol{t}} \log\det\big(\boldsymbol{\Sigma}(\boldsymbol{t};\zeta)\big) f_{\boldsymbol{T}}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\gamma})\, d\boldsymbol{t}.$$

$\boldsymbol{\Sigma}_0$ is the variance-covariance matrix of the prior distribution, and $\boldsymbol{\Sigma}(\boldsymbol{t};\zeta)$ is the variance-covariance matrix of the posterior distribution. The square root of the determinant of the posterior variance-covariance matrix is correlatable to the overall volume of the joint credible region, which gives us the basis of the formula given here for the D-optimal design.

The C-optimal design maximizes

$$U_C(\zeta) \quad = \quad \frac{1}{2}\log\big(\boldsymbol{c}^\top\boldsymbol{\Sigma}_0\boldsymbol{c}\big) - \frac{1}{2}\int_{\boldsymbol{t}} \log\big(\boldsymbol{c}^\top\boldsymbol{\Sigma}(\boldsymbol{t};\zeta)\boldsymbol{c}\big) f_{\boldsymbol{T}}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\gamma})\, d\boldsymbol{t}.$$

This design allows us to estimate a quantity of interest with maximum precision and minimum variability. For our specific work, what we consider is the variance of our estimator at normal usage conditions.

The A-optimal design maximizes

$$U_A(\zeta) \quad = \quad \frac{1}{2}\log\mathrm{tr}\big(\boldsymbol{\Sigma}_0\big) - \frac{1}{2}\int_{\boldsymbol{t}} \log\mathrm{tr}\big(\boldsymbol{\Sigma}(\boldsymbol{t};\zeta)\big) f_{\boldsymbol{T}}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\gamma})\, d\boldsymbol{t}.$$

This looks similar to D, but instead of the determinant we now are using the trace. The trace gives us insight into the total marginal posterior variance of $\boldsymbol{\Lambda}$.

The E-optimal design maximizes

$$U_E(\zeta) \quad = \quad \frac{1}{2}\log\psi_{\max}\big(\boldsymbol{\Sigma}_0\big) - \frac{1}{2}\int_{\boldsymbol{t}} \log\psi_{\max}\big(\boldsymbol{\Sigma}(\boldsymbol{t};\zeta)\big) f_{\boldsymbol{T}}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\gamma})\, d\boldsymbol{t}.$$

This optimal design effectively minimizes the maximum variance of all possible normalized linear combinations of parameter estimates; essentially, this is minimizing the maximum eigenvalue.

The M-optimal design maximizes

$$U_M(\zeta) \quad = \quad \frac{1}{2}\log\max_j\left\{\boldsymbol{e}_j^T\boldsymbol{\Sigma}_0\boldsymbol{e}_j\right\} - \frac{1}{2}\int_{\boldsymbol{t}} \log\max_j\left\{\boldsymbol{e}_j^T\boldsymbol{\Sigma}(\boldsymbol{T};\zeta)\boldsymbol{e}_j\right\} f_{\boldsymbol{T}}(\boldsymbol{t};\boldsymbol{\alpha},\boldsymbol{\gamma})\, d\boldsymbol{t}.$$

This design is used to minimize the maximum variance in the variance-covariance matrix.

The simulation algorithm we used in the results to follow was adapted from Hong et al. (2014) as seen in Algorithm 1.

Algorithm 1 Stochastic Algorithm

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1. Simulate m samples of $\boldsymbol{\lambda}$ from the prior in Equation 5.
2. For each $\boldsymbol{\lambda_i}$, simulate n random samples of $\boldsymbol{t_i}$ from the likelihood.
3. For each $\boldsymbol{t_i}$, compute the posterior variance-covariance matrix Σ_i for $\boldsymbol{\lambda_i}$ (or $\boldsymbol{\beta_i}$ based on a linear link).
4. Compute the value of a respective utility function $U(\Delta)$ by obtaining the mean of the simulated measures, $\sum_{i=1}^m g(\Sigma_i)/m$ with an appropriate transformation $g(\cdot)$.
5. Identify the value of Δ which maximizes $U(\Delta)$.

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6. Simulation Results

The results to follow are based off of $m = 1000$ simulations with $n = 24$. Given the progressive Type-I censoring scheme, choices for the proportion of surviving units to censor after the first level were chosen as $\pi_1^* = 0\%$, $\pi_1^* = 10\%$ and $\pi_1^* = 20\%$. Only equal step durations were considered for this study. For c_1^*, we use the 1000 simulated values for λ_1 and estimate $c_1^* \approx E[(N_1 - n_1)\pi_1^*]$ as the rounded mean of $(nS_1(\Delta_1)\pi_1^*)$.

For the cost constraint, we used the following settings: $C_{set} = 10$, $C_{unit} = 1$, $C_{fail} = 0.5$, $C_{unfail} = 0.2$, $C_{ins} = 0.01$, $C_{op}(x_1) = 0.1$, $C_{op}(x_2) = 1.1$ and $C_B = 48$. With these settings, the maximum allowed test durations are 2.380952, 2.415459 and 2.450980 respectively for the censoring proportions of 0%, 10% and 20%. The optimal test durations for C, D and H-optimality are all longer than the maximum allowed due to the constraint. In tables 1 and 2 we show the unconstrained as well as the constrained results for designs considered based on the variance-covariance matrix for $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$.

In tables 1 and 2 we include the optimal test duration as well as the constrained duration along with the corresponding utility function values. We additionally report the approximated expected termination times, approximated expected total costs and maximum total costs. There were two approximations used, one using Monte Carlo simulations and the other using the moment-matching method. These approaches result in fairly close values for all utility / censoring proportion combinations. For the expected termination times, the moment-matching results are all larger for Type-I censoring. Except for the unconstrained C-optimal design, and one setting for the M-optimal design, the simulated results are larger for the other two censoring proportions. For the expected total costs, the moment-matching results are all larger for Type-I censoring. There isn't a particular pattern observed when it comes to the expected cost and the other censoring proportions. We previously noted that our optimal test durations were somewhat erratic, which was attributed to the stochastic nature of the simulation, finding local maximums. Here we see that the expected termination times appear to follow the patterns for the test durations. It's a little harder to make statements about the expected total costs, since it's a function of several components. But we do observe some erratic patterns here as well.

In tables 3 and 4 we include the efficiency calculations. The efficiency is computed as the ratio of the utility function values. When unconstrained and constrained results are the same, the efficiency is equal to one, and when imposing the constraint has an impact, the efficiency decreases. By introducing the cost constraint, we see that the efficiency of optimal designs C, D and H are reduced, with that for C being the most pronounced. This observation related to the C-optimal design was also observed in the Frequentist work of Han (2015).

Table 1: Design Optimization Based on the Variance-Covariance Matrix of β_0 and β_1.

$U_i(\zeta)$		Unconstrained $C_B = \infty$					Constrained $C_B = 48$				
i	π^*	Δ^*	U^*	$E[T_1]$	$E[C_T]$	$\max\{C_T\}$	Δ^*	U^*	$E[T_1]$	$E[C_T]$	$\max\{C_T\}$
A	0%	1.4983	0.6741	1.4090/1.4132	46.1333/46.1417	47.2586	1.4983	0.6741	1.4090/1.4132	46.1333/46.1417	47.2586
	10%	1.3268	0.6495	1.2643/1.2628	45.5827/45.5828	47.0986	1.3268	0.6495	1.2643/1.2628	45.5827/45.5828	47.0986
	20%	1.4499	0.6180	1.3505/1.3413	45.4766/45.4709	47.1831	1.4499	0.6180	1.3505/1.3413	45.4766/45.4709	47.1831
C	0%	7.8447	1.3018	3.7263/3.7315	47.2681/47.2730	52.5895	2.3810	1.1873	1.9944/2.0054	46.8678/46.8916	48.0000
	10%	7.0694	1.2990	3.5520/3.5751	47.1792/47.2082	51.8535	2.4155	1.1865	1.9964/1.9734	46.6353/46.6227	48.0000
	20%	7.9181	1.2943	3.6969/3.7454	47.0703/47.1289	52.4611	2.4510	1.1869	1.9876/1.9368	46.3944/46.3521	48.0000
D	0%	2.4230	1.8367	2.0179/2.0291	46.8878/46.9121	48.0353	2.3810	1.8359	1.9944/2.0054	46.8678/46.8916	48.0000
	10%	3.1312	1.8145	2.3486/2.3081	46.9031/46.8717	48.5927	2.4155	1.8067	1.9964/1.9734	46.6353/46.6227	48.0000
	20%	3.2743	1.7835	2.3747/2.3669	46.7123/46.7178	48.6718	2.4510	1.7764	1.9876/1.9368	46.3944/46.3521	48.0000
E	0%	1.2300	0.5857	1.1890/1.1918	45.7095/45.7117	47.0332	1.2300	0.5857	1.1890/1.1918	45.7095/45.7117	47.0332
	10%	1.2414	0.5568	1.1931/1.1926	45.4322/45.4318	47.0279	1.2414	0.5568	1.1931/1.1926	45.4322/45.4318	47.0279
	20%	1.2658	0.5237	1.2054/1.1853	45.1763/45.1556	47.0329	1.2658	0.5237	1.2054/1.1853	45.1763/45.1556	47.0329
M	0%	1.2177	0.5984	1.1783/1.1811	45.6863/45.6882	47.0228	1.2177	0.5984	1.1783/1.1811	45.6863/45.6882	47.0228
	10%	1.0972	0.5656	1.0683/1.0689	45.1399/45.1380	46.9085	1.0972	0.5656	1.0683/1.0689	45.1399/45.1380	46.9085
	20%	1.1321	0.5329	1.0931/1.0801	44.9142/44.8990	46.9238	1.1321	0.5329	1.0931/1.0801	44.9142/44.8990	46.9238

Table 2: Design Optimization Based on the Variance-Covariance Matrix of λ_1 and λ_2.

$U_i(\zeta)$		Unconstrained $C_B = \infty$					Constrained $C_B = 48$				
i	π^*	Δ^*	U^*	$E[T_1]$	$E[C_T]$	$\max\{C_T\}$	Δ^*	U^*	$E[T_1]$	$E[C_T]$	$\max\{C_T\}$
A	0%	1.2494	0.8370	1.2056/1.2085	45.7452/45.7479	47.0495	1.2494	0.8370	1.2056/1.2085	45.7452/45.7479	47.0495
	10%	1.3470	0.8054	1.2807/1.2791	45.6159/45.6162	47.1153	1.3470	0.8054	1.2807/1.2791	45.6159/45.6162	47.1153
	20%	1.3858	0.7715	1.3012/1.2740	45.3791/45.3526	47.1308	1.3858	0.7715	1.3012/1.2740	45.3791/45.3526	47.1308
D	0%	2.4230	1.8367	2.0179/2.0291	46.8878/46.9121	48.0353	2.3810	1.8359	1.9944/2.0054	46.8678/46.8916	48.0000
	10%	3.1312	1.8145	2.3486/2.3081	46.9031/46.8717	48.5927	2.4155	1.8067	1.9964/1.9734	46.6353/46.6227	48.0000
	20%	3.2743	1.7835	2.3747/2.3669	46.7123/46.7178	48.6718	2.4510	1.7764	1.9876/1.9368	46.3944/46.3521	48.0000
E	0%	1.2440	0.8316	1.2009/1.2038	45.7353/45.7378	47.0449	1.2440	0.8316	1.2009/1.2038	45.7353/45.7378	47.0449
	10%	1.2403	0.7966	1.1922/1.1917	45.4302/45.4298	47.0269	1.2403	0.7966	1.1922/1.1917	45.4302/45.4298	47.0269
	20%	1.2496	0.7599	1.1920/1.1729	45.1467/45.1268	47.0197	1.2496	0.7599	1.1920/1.1729	45.1467/45.1268	47.0197
H	0%	3.1832	1.5928	2.3913/2.4076	47.1287/47.1586	48.6739	2.3810	1.5830	1.9944/2.0054	46.8678/46.8916	48.0000
	10%	3.0097	1.5705	2.2935/2.2562	46.8691/46.8413	48.4920	2.4155	1.5603	1.9964/1.9734	46.6353/46.6227	48.0000
	20%	3.7734	1.5436	2.5732/2.5567	46.8230/46.8181	49.0791	2.4510	1.5307	1.9876/1.9368	46.3944/46.3521	48.0000
M	0%	1.2440	0.7841	1.2009/1.2038	45.7352/45.7378	47.0449	1.2440	0.7841	1.2009/1.2038	45.7352/45.7378	47.0449
	10%	1.2414	0.7488	1.1931/1.1926	45.4323/45.4319	47.0279	1.2414	0.7488	1.1931/1.1926	45.4323/45.4319	47.0279
	20%	1.1201	0.7111	1.0827/1.0703	44.8886/44.8737	46.9140	1.1201	0.7111	1.0827/1.0703	44.8886/44.8737	46.9140

Table 3: Design Optimization Efficiencies Based on the Variance-Covariance Matrix of β_0 and β_1.

$U_i(\zeta)$		U^*		
i	π^*	Unconstrained	Constrained	Efficiency
A	0%	0.6741	0.6741	1.0000
	10%	0.6495	0.6495	1.0000
	20%	0.6180	0.6180	1.0000
C	0%	1.3018	1.1873	0.9120
	10%	1.2990	1.1865	0.9134
	20%	1.2943	1.1869	0.9170
D	0%	1.8367	1.8359	0.9995
	10%	1.8145	1.8067	0.9957
	20%	1.7835	1.7764	0.9960
E	0%	0.5857	0.5857	1.0000
	10%	0.5568	0.5568	1.0000
	20%	0.5237	0.5237	1.0000
M	0%	0.5984	0.5984	1.0000
	10%	0.5656	0.5656	1.0000
	20%	0.5329	0.5329	1.0000

Table 4: Design Optimization Efficiencies Based on the Variance-Covariance Matrix of λ_1 and λ_2.

$U_i(\zeta)$		U^*		
i	π^*	Unconstrained	Constrained	Efficiency
A	0%	0.8370	0.8370	1.0000
	10%	0.8054	0.8054	1.0000
	20%	0.7715	0.7715	1.0000
D	0%	1.8367	1.8359	0.9995
	10%	1.8145	1.8067	0.9957
	20%	1.7835	1.7764	0.9960
E	0%	0.8316	0.8316	1.0000
	10%	0.7966	0.7966	1.0000
	20%	0.7599	0.7599	1.0000
H	0%	1.5928	1.5830	0.9938
	10%	1.5705	1.5603	0.9935
	20%	1.5436	1.5307	0.9916
M	0%	0.7841	0.7841	1.0000
	10%	0.7488	0.7488	1.0000
	20%	0.7111	0.7111	1.0000

7. Conclusion

Using a 3-parameter gamma distribution as a conditional prior, in this work we have performed order-restricted Bayesian cost constrained design optimization for progressively

Type-I censored simple step-stress accelerated life tests with exponential lifetimes under continuous inspections. Under this framework, we examined the formula for the expected termination time and expected total cost and proposed estimation procedures for each. From the simulation results, we saw that the two estimation procedures yield similar results.

Our future work will include the cost constrained results under constant stress so that we can compare efficiencies. Other considerations are to explore other constraints, explore the general k-level accelerated life test, to consider other censoring schemes and to explore cost savings.

REFERENCES

Balakrishnan, N. and Cramer, E. (2014). *The art of progressive censoring: applications to reliability and quality.* New York, NY: Birkhäuser.

Covo, S. and Elalouf, A. (2014), "A novel single-gamma approximation to the sum of independent gamma variables, and a generalization to in
nitely divisible distributions," *Electronic Journal of Statistics*, 8, 894–926.

Gouno, E., Sen, A. and Balakrishnan, N. (2004), "Optimal step-stress test under progressive Type-I censoring," *IEEE Transactions on Reliability*, 53, 388–393.

Hakamipour, N. (2019), "Time and cost constrained optimal designs of multiple step stress tests under progressive censoring," *International Journal of Quality & Reliability Management*, 36, 1721–1733.

Hakamipour, N. (2021), "Comparison between constant-stress and step-stress accelerated life tests under a cost constraint for progressive type I censoring," *Sequential Analysis*, 40, 17–31.

Han, D. (2015), "Time and cost constrained optimal designs of constant-stress and step-stress accelerated life tests," *Reliability Engineering and System Safety*, 140, 1–14.

Han, D. and Kundu, D. (2015), "Inference for a step-stress model with competing risks for failure from the generalized exponential distribution under Type-I censoring," *IEEE Transactions on Reliability*, 64, 31–43.

Han, D. (2020), "Expected termination times of progressively Type-I censored step-stress accelerated life tests under continuous and interval inspections," *Statistica Neerlandica*, 74, 112–124.

Han, D. and Bai, T. (2021), "Parameter estimation using EM algorithm for lifetimes from step-stress and constant-stress accelerated life tests with interval monitoring," *IEEE Transactions on Reliability*, 70, 49–64.

Han, D. (2021), "Time and cost constrained design of a simple step-stress accelerated life test under progressive Type-I censoring," *Quality Engineering*, 33, 156–171.

Hong, Y., King, C., Zhang, Y. and Meeker, W.Q. (2014), "Bayesian life test planning for the log-location-scale family of distributions," *Statistics Preprints*, 131, 1–25.

Lim, H. (2015), "Optimum accelerated degradation tests for the gamma degradation process case under the constraint of total cost," *Entropy*, 17, 2556–2572.

Lindley, D. V. (1956), "On a Measure of the Information Provided by an Experiment," *The Annals of Mathematical Statistics*. 27(4): 986–1005.

Nelson, W. (1980), "Accelerated life testing step-stress models and data analyses," *IEEE Transactions on Reliability*. 29(2): 103–108.

Wiedner, C. and Han, D. (2020), "Order-restricted Bayesian inference and optimal designs for the simple step-stress accelerated life tests under progressive Type-I censoring based on three-parameter gamma prior," *In JSM Proceedings, Quality and Productivity Section. Alexandria, VA: American Statistical Association*, 647–660.

Xiang, S., Yang, J. and Shen, L. (2017), "Designing adaptive accelerated life tests using Bayesian methods," *Qual Reliab Engng Int.*, 33, 2269–2279.

Xu, H. and Fei, H. (2012), "Models comparison for step-stress accelerated life testing," *Communications in Statistics - Theory and Methods*, 41, 3878–3887.