The Hyperbolic Conditional Autoregressive Range (HYCARR) Model

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Abstract

This paper proposes the Hyperbolic Conditional Autoregressive Range (HYCARR) model to analyze the long-memory or long-term dependencies of the price range of a financial asset. The Conditional Autoregressive Range (CARR) model explains the current conditional mean of a price ranges as a function of past conditional mean price range values and past prices range data. However, the Auto covariance Function (ACF) of the price range series exhibits statistically significance correlation up to far end, which indicates the presence of long-memory properties in the finance data. In this paper long-memory properties in the price range data are examined. The standard CARR process accounts for short-memory properties in the conditional price range data. The long-memory behavior in the return-based models were well explained in the Integer GARCH (IGARCH), Fractionally IGARCH (FIGARCH) and Hyperbolic GARCH (HYGARCH) studies. This paper mainly focuses on discussing the gap knowledge exists in the long-memory properties in the range-based models. Further, the paper examines the non-negativity conditions of the HYCARR model for the conditional mean range term are derived. The Maximum Likelihood Estimation (MLE) technique is discussed to estimate the proposed model. The simulation study is carried out to estimate the finite sample performance of the proposed HYCARR model. The empirical study of the HYCARR model is illustrated by usingS&P500 index data.

Key Words: Long-memory, Range model, CARR process, Hyperbolic Function.

1. Introduction

Modelling the volatility is indispensable in better understanding the dynamics of financial markets. Financial volatility of asset prices has been discussed extensively in the financial and econometric literature over past few decades. Engle (1982) proposed the Autoregressive Conditional Heteroscedasticity (ARCH) model to address the complexities of time-varying volatility and volatility clustering in the financial time series. In the ARCH formulation, the conditional volatility is modeled as a function of past returns. Bollerslev (1986) proposed the Generalized Autoregressive Conditional Heteroscedasticity (GARCH), which remains one of the most popular volatility models up to date. The GARCH model is an extension of ARCH formulation, and it models the conditional volatility as a function of lagged squared returns, as well as past conditional variances. Due to the fact that ARCH and GARCH models are earned the popularity among researchers as a useful tool to explain the real world phenomena and successful method to forecast future volatilities, several versions of ARCH /GARCH family models were proposed such as Exponential GARCH (EGARCH) by Nelson (1991), Threshold ARCH (TARCH) and Threshold GARCH (TGARCH) by Zakoian (1994), GJR-GARCH by Glosten, Jagannathan and Runkle (1993), Quadratic GARCH (QGARCH) by Sentana (1995) Etc. Since ARCH/GARCH family models aforementioned focus on modeling price returns, they can be identified as examples of return-based volatility models. Engle and Russel (1998) proposed the Autoregressive Conditional Duration or ACD to model the time series of durational data.

The price range of an asset is an alternative measure of the financial volatility. Parkinson (1980) concluded that the range-based method is far superior to the standard methods based on returns. Several researchers (see. Beckers (1983), Kunitomo (1992), and Alizadeh, Brandt, and Diebold (2002)) pointed out that range-based volatility estimators are highly efficient when compared to the classical volatility proxies that are based on log absolute returns or squared returns. This opened the floor for the range-based volatility models as a new class of volatility models that uses range as a measure of price volatility.

Chou (2005) invented a range-based volatility model called the Conditional Autoregressive Range or the CARR model. The CARR model is primarily an ACD process. It is employed to explain the price volatility of an asset by considering range of the log prices for a given fixed time interval while the ACD process is used to model the time intervals between events with positive observations. The CARR model is similar to the standard volatility models such as the GARCH model. One distinct difference between the two models is that the GARCH model uses rate of return as its volatility measure while the CARR model uses range as its volatility measure. The CARR model proposed by Chou is a simple yet efficient tool for analyzing the volatility clustering property when compared to the GARCH models. For example, Chou (2005) showed that the effectiveness of volatility estimates produced by the CARR models is higher than the estimates of standard return-based models such as GARCH models. Due to the growing interest and development of the range-based financial time series, researchers became invested in analyzing the CARR family models namely, Exponential CARR, Weibull CARR, CARR-X (Chou 2005), Asymmetric CARR (Chou 2006), Threshold ACARR (Ratnayake 2021) etc.

In time series literature long-memory process or long-range dependent process is defined based on its autocovariance function. According to the Palma (2007), if the autocovariances of the stationary process are not absolutely summable then the process has long-memory. Another method to detect the existence of a long-memory is by checking whether the stationary time series process exhibits statistically significant dependence between far away observations. Therefore, the long-memory process is displaying a very slow decay to zero, or hyperbolic decay autocorrelation function. Many financial time series such as price return, price range, and transaction duration series exhibit long-memory properties meaning that autocorrelations of volatilities remain non-zero for very large lags. However, the above-mentioned standard ARCH/GARCH models, CARR model, and ACD model have exponentially decaying coefficients and absolutely summable exponentially decaying autocorrelation function (see. Bollerslev, (1986), Engle and Russel (1998), and Chou (2005)) and hence these models are unable to capture the persistence that presents in the volatility series.

Multitudes of time series models were proposed to analyze the long-memory properties in return-based model. The Integrated GARCH (IGARCH) by Engle and Bollerslev (1986), where the lag coefficients sum to unity. Similar to the ARIMA model, IGARCH process is a long-memory process with d=1, where d is the long-memory parameter. Clearly it is too restrictive to set d= 0 (standard GARCH) and d=1 (IGARCH). To overcome this shortcoming, fractionally integrated ($0<d<1$) return based models were proposed. The term ‘hyperbolic memory’ is therefore preferred to distinguish it from the ‘geometric memory’ cases such as GARCH and IGARCH. Following the idea of the fractional integrated autoregressive moving average (ARFIMA) model, Baillie, Bollerslev and

Mikkeslen (1996) extended the common GARCH model to the Fractional Integrated GARCH (FIGARCH) model. The FIGARCH model allows analyzing the slow hyperbolic decay present in squared error series using a fractional differencing parameter. Further, the FIGARCH model formulated conditional error variance as an infinite series of weighted past error variances. The paper also developed the QMLE for the FIGARCH parameters and empirically showed that proposed model works well with time series exhibiting long-range dependencies. Conrad and Haag (2006) proposed the non-negativity conditions for the FIGARCH model. The significance of the imposing non-negativity constrains were illustrated using exchange rate data (Japanese yen versus U.S. dollar. Davidson (2004) proposed the Hyperbolic GARCH (HYGARCH) model that can be viewed as a two-component GARCH specification with one component being GARCH and the other being FIGARCH. The HYGARCH shares with the GARCH model the desirable property of covariance stationarity. It also obeys the hyperbolically decaying impulse response coefficients as the FIGARCH. Davidson also derived the moment properties for the HYGARCH model. However, the FIGARCH process always has infinite variance, and the HYGARCH model has a more complicated form. Alternative model to HYGARCH model was proposed by Muyi Li, Wai Keung Li, and Guodong Li (2014) and named it as a new hyperbolic GARCH (HGARCH) model. The HGARCH model is parsimonious as the FIGARCH, and it addresses the issue of the infinite variance of the FIGARCH processes by allowing the existence of a finite variance as in HYGARCH model. The above processes discuss the long-range dependencies in a return-based setting. Several models were proposed to discuss the long-range dependencies present in the durational data. For an example, Karanasos (2003) proposed the long-memory ACD (LMACD) process and derived its moment properties. However, to the best of our knowledge, there is no study that discussed a range-base long-memory model. This paper aims to fill this gap.

In this study we propose the long-memory range based conditional heteroscedastic model to pick up the persistence in the price range data and name the model as Hyperbolic Conditional Autoregressive Range (HYCARR) model. This is a range base alternative of HYGARCH which we discussed in the previous section.

The remainder of the paper organized as follows: Section 2 reviews the standard CARR prosses properties and expresses the conditional mean range term as an infinite series of past range like to ARCH (∞) in return-based setting. Section 3 introduces the Integrated CARR (ICARR) model and Fractionally ICARR (FICARR) model. In section 4, we propose Hyperbolic CARR (HYCARR) and HYCARR model of order (1, d,1) model is introduced. Non-negativity conditions for the proposed model are presented in section 5. Finite sample performance of the proposed HYCARR model is discussed in the simulation study in Section 6. Section 7 presents the empirical study based on S&P 500 data followed by concluding results in section 8.

2. The CARR Model

Let us assume a daily time series and say that t denotes any trading day, then the maximum and minimum logarithmic price of an asset for a given trading day are denoted by P_t^H and P_t^L respectively. Then the price range of an asset is defined as: $R_t = P_t^H - P_t^L$.

The Conditional Autoregressive Range (CARR) model of order p and q is expressed as:

$$R_t = \lambda_t \varepsilon_t,$$

$$E(R_t|\mathcal{F}_{t-1}) = \lambda_t = \gamma + \sum_{i=1}^{p} \alpha_i R_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j}. \quad (1)$$

Here, $\lambda_t$is the conditional expectation of the price range $R_t$given $\mathcal{F}_{t-1}$ where $\mathcal{F}_{t-1}$is the information up to time $t-1$. Moreover, ε_t is the independent and identically distributed (*i.i.d.*) residual series which follows a probability density function $f_\varepsilon(.)$ (i.e., $\varepsilon_t \sim f_\varepsilon(.)$) with unit mean, such that $E(\varepsilon_t) = 1$.

Let L be the lag operator and it is defined as $LY_t = Y_{t-1}$, then the above equation (1) can be rewritten as follows:

$$\lambda_t = \gamma + \sum_{i=1}^{p} \alpha_i L^i R_t + \sum_{j=1}^{q} \beta_j L^j \lambda_t,$$

$$\lambda_t = \gamma + \alpha(L) R_t + \beta(L) \lambda_t,$$

$$[1 - \beta(L)]\lambda_t = \gamma + \alpha(L) R_t,$$

$$B(L)\lambda_t = \gamma + \alpha(L) R_t. \quad (2)$$

Here, $B(L) = \left(1 - \sum_{j=1}^{q} \beta_j L^j\right) = 1 - \beta_1 L^1 - \beta_2 L^2 - \cdots \beta_q L^q$, and $\alpha(L) = \sum_{i=1}^{p} \alpha_i L^i$, are the lag polynomials.

Let us introduce a stochastic difference component η_t, such that $\eta_t = R_t - \lambda_t$, with$E(\eta_t) = 0$, and $cov(\eta_t, \eta_u) = 0, t \neq u$.Alternatively, the price range series R_t can be expressed as an Autoregressive Moving Average (ARMA) process.

$$R_t = \gamma + \sum_{i=1}^{s} \theta_i R_{t-i} + \eta_t + \sum_{j=1}^{q} \phi_j \eta_{t-j},$$

Here, $s = max(p, q)$ and $\theta_i = \begin{cases} \alpha_i + \beta_i & : i \leq \min(p, q) \\ \alpha_i & : q < i \leq p \\ \beta_i & : p < i \leq q \end{cases}$ and $\phi_j = (-\beta_j) : 1 \leq j \leq q$. Therefore, the CARR process of order *p* and *q* can be represented as ARMA process of order *s* and *q*.

Let $\Theta(L)$ and $\Phi(L)$be the lag polynomials such that $\Theta(L) = 1 - \theta_1 L^1 - \theta_2 L^2 - \cdots - \theta_s L^s$, and $\Phi(L) = 1 + \phi_1 L^1 + \phi_2 L^2 + \cdots + \phi_q L^q = 1 - \sum_{j=1}^{q} \beta_j L^j = B(L)$. The following derivation can then be used to set the stationarity, inversibility and non-reducibility conditions for the CARR process.

$$\Theta(L) R_t = \gamma + \Phi(L) \eta_t.$$

Replacing $\eta_t = R_t - \lambda_t$,

$$\Theta(L) R_t = \gamma + \Phi(L)[R_t - \lambda_t],$$

$$\Phi(L)\lambda_t = \gamma + [\Phi(L) - \Theta(L)] R_t,$$

$$B(L)\lambda_t = \gamma + [B(L) - \Theta(L)] R_t.$$

If the roots of the lag polynomial $\Theta(z) = 1 - \theta_1 z^1 - \theta_2 z^2 - \cdots - \theta_s z^s$, lie outside the unit circle, then the CARR process is stationary. Also, if the roots of lag polynomial $B(z) = 1 - \beta_1 z^1 - \beta_2 z^2 - \cdots - \beta_q z^q$, lie outside the unit circle and if $\Theta(z) \neq B(z)$then the CARR process is invertible and non-reducible. If the CARR process satisfies the above conditions, then the conditional expectation of the price range (λ_t), can be represented as an infinite series of past realization of price range series such that:

$$\lambda_t = \frac{\gamma}{B(1)} + \Psi(L)R_t = \frac{\gamma}{B(1)} + \sum_{i=1}^{\infty} \psi_i^{CA} R_{t-i},$$

with

$$\Psi^{CA}(L) = \frac{B(L) - \Theta(L)}{B(L)} = 1 - \frac{\Theta(L)}{B(L)} = \frac{\alpha(L)}{B(L)} = \sum_{i=1}^{\infty} \psi_i^{CA} L^i.$$

The CARR model has an exponentially decaying Auto Covariance Function (ACF) for the price range data and therefore unable to capture the persistence in the conditional mean range of such time series. In this situation, the conditional mean is formulated by assigning higher weights to the most recent price range information and hence it is categorized as short-memory process. However, in practice it is possible to expect a time series of range data that exhibits a slow decaying ACF function. Then the regular CARR model does not fit for such a data set. To overcome this shortcoming three long-memory range based conditional heteroscedastic models are proposed in this study namely Integrated CARR (ICARR) model, Fractionally Integrated CARR (FICARR) model and Hyperbolic CARR model (HYCARR). The ICARR and FICARR models are introduced in section 3, but this study heavily invested on the HYCARR process.

3. The Integrated CARR (ICARR) model and The Fractionally ICARR (FICARR) model.

The Integrated GARCH (IGARCH) model for price return series was formulated by Nelson (1990) and the important development of IGARCH is that the sum of the persistence parameters equals to 1. Following the IGARCH formulation for return series, the Integrated CARR (ICARR) model for price range series can be derived. The ICARR process of *p* and *q* can be formed by setting the lag coefficients, $\sum_{i=1}^{\infty} \psi_i^{CA} = 1$. Moreover, it can also be written in the form,

$$\Psi^I(L) = 1 - \tfrac{\Theta(L)}{B(L)}(1 - L) = \sum_{i=1}^{\infty} \psi_i^I L^i, \tag{3}$$

where, $\Theta(L)$define accordingly. The resulting ICARR (*p*, *q*) model is:

$$\lambda_t = \frac{\gamma}{B(1)} + \left[1 - \frac{\Theta(L)}{B(L)}(1 - L)\right] R_t = \frac{\gamma}{B(1)} + \sum_{i=1}^{\infty} \psi_i^I R_{t-i}.$$

The FIGARCH model introduced by Baillie, Bollerslev, and Mikkelsen (1996) bought the concept of fractional differencing to the GARCH family models. Due to this addition, FIGARCH model distinguishes short-memory and long-memory in the return series data. The derivation for the Fractionally Integrated CARR (FICARR) model of order*s*, *d*, and *q* for range data, follows similar to its return-based counterpart FIGARCH (*s*, *d*, *q*) model.

The FICARR model replaces the simple difference in ICARR model given in (3) by using a fractional difference term$d$,

$$\Psi^{FI}(L) = 1 - \frac{\Theta(L)}{B(L)}(1-L)^d,$$

where, $d \in (0,1)$such that, $\lim_{d \to 0} \Psi^{FI}(L) = \Psi^{CA}(L)$ and $\lim_{d \to 1} \Psi^{FI}(L) = \Psi^{I}(L)$. The parameter d becomes the bridge between CARR and ICARR processes.

By using the hypergeometric function $H(.)$, notations the fractional differencing operator $(1-L)^d$ can be expressed as given bellow.

$$(1-L)^d = H(-d,1;1;L) = \sum_{j=0}^{\infty} \pi_j L^j. \tag{4}$$

Here, $\pi_j = \delta_j \pi_{j-1} = \prod_{i=1}^{j} \delta_i$ with $\delta_j = \frac{j-1-d}{j}$ for $j = 1,2,\ldots$ and $\pi_0 = 1, \delta_1 = -d$.Following the representation(4), the fractional difference operator $(1-L)^d$ and it is as follows:$(1-L)^d = \sum_{n=0}^{\infty} \frac{(-d)_n L^n}{n!}$, with $(-d)_n = \prod_{j=1}^{n}(j-1-d)$, and define $(-d)_0 = 1$.

As a summary, we can state that both CARR and ICARR processes have a geometric convergence rate while the FICARR process has a hyperbolic convergence rate. Further, both ICARR and FICARR are long-memory processes in which the conditional mean range can be expressed as an infinite series of past range values.

4. The Hyperbolic CARR (HYCARR) Model

Let R_t be the price range of an asset for a given time interval t and λ_t is the conditional mean range of price given information set up to time $t-1$, such that $\lambda_t = E(R_t|\mathcal{F}_{t-1})$, and the residual series $\varepsilon_t$is independently and identically distributed (*i.i.d.*) which follows a probability density function $f_\varepsilon(.)$ (i.e., $\varepsilon_t \sim f_\varepsilon(.)$) with unit mean, such that $E(\varepsilon_t) = 1$. Then the HYGARCH model of order *s*, *d*, and *q*,

$$R_t = \lambda_t \varepsilon_t,$$

$$\lambda_t = \frac{\gamma}{B(1)} + \left\{1 - \frac{\Theta(L)}{B(L)}\left[1 + \eta\left((1-L)^d - 1\right)\right]\right\} R_t = \frac{\gamma}{B(1)} + \sum_{i=1}^{\infty} \psi_i^{HY} L^i R_t. \tag{5}$$

where, $s = max(p,q)$, $\Theta(L) = 1 - \sum_{i=1}^{s} \theta_i L^i$,and $B(L) = 1 - \sum_{i=1}^{q} \beta_i L^i$. Here, $d \geq 0$, and $\eta \geqslant 0$.

$$\Psi^{HY}(L) = \left\{1 - \frac{\Theta(L)}{B(L)}\left[1 + \eta\left((1-L)^d - 1\right)\right]\right\} = \sum_{i=1}^{\infty} \psi_i^{HY} L^i.$$

The HYCARR coefficients $\Psi^{HY}(L)$ can also be decomposed as:

$$\Psi^{HY}(L) = (1-\eta)\Psi^{CA}(L) + \eta \Psi^{FI}(L). \tag{6}$$

Here, $\lim_{\eta \to 0} \Psi^{HY}(L) = \Psi^{CA}(L)$ and $\lim_{\eta \to 1} \Psi^{HY}(L) = \Psi^{FI}(L)$.This suggests that the HYCARR model is nested with the CARR component and the FICARR component depends on the parameter η.

Another way to look at the HYCARR coefficients ($\Psi^{HY}(L)$) is by changing the differencing parameter$d$. In the case where $d = 1$,that is$\Psi^{HY}(L) = \left\{1 - \frac{\Theta(L)}{B(L)}(1 - \eta L)\right\}$, η parameter determines whether the HYCARR model falls into CARR model ($\eta < 1$)or ICARR model ($\eta = 1$).

4.1. The HYCARR (1, *d*,1) Model

The HYCARR process of order *s*, *d*, and *q* is presented in the equation (5) and by setting *s*=1, and *q*=1, the HYCARR (1, *d*,1) can be derived. Here, $\Theta(L) = 1 - \theta L$ and $B(L) = 1 - \beta L$. The coefficients of the proposed HYCARR (1, *d*,1) model can be expressed as given bellow:

$$\Psi^{HY}(L) = \left\{1 - \frac{1-\theta L}{1-\beta L}[1 + \eta((1-L)^1 - 1)]\right\} = \sum_{i=1}^{\infty} \psi_i^{HY} L^i. \quad (7)$$

Then the coefficients are,

$$\psi_1^{HY} = \theta + \eta d - \beta, \quad (8)$$

$$\psi_2^{HY} = \beta(\theta + \eta d - \beta) + \eta d\left(\frac{1-d}{2} - \theta\right), \quad (9)$$

$$\psi_3^{HY} = \beta\left[\beta(\theta + \eta d - \beta) + \eta d\left(\frac{1-d}{2} - \theta\right)\right] + \eta d\left(\frac{1-d}{2}\right)\left(\frac{2-d}{3} - \theta\right), \quad (10)$$

$$\vdots$$

$$\psi_k^{HY} = \beta\psi_{k-1}^{HY} - \eta\left(\frac{k-1-d}{k} - \theta\right)\pi_{k-1} = \beta\psi_{k-1}^{HY} - \eta\pi_k + \eta\theta\pi_{k-1}, \forall k \geq 2, (11)$$

or alternatively,

$$\psi_k^{HY} = \beta^2\psi_{k-2}^{HY} - \eta[\beta(\delta_{k-1} - \theta) + (\delta_k - \theta)\delta_{k-1}]\pi_{k-2}, \forall k \geq 3. (12)$$

Here, $\pi_k = \delta_k\pi_{k-1} = \prod_{i=1}^{k}\delta_i$ with $\delta_k = \frac{k-1-d}{k}$ for $k = 1,2,\ldots$ and $\pi_0 = 1, \delta_1 = -d$. Therefore, it can be showed that the coefficients of the HYCARR (1, *d*,1) model follows a recursive representation, and which is like the HYGARCH (1, *d*,1) formulation derived by Conrad (2010).

The coefficients of the HYCARR (1, *d*, 1) model can also be used to establish the necessary and sufficient conditions for the non- negativity of the conditional mean range term in the HYCARR model.

4.2. The Non-negativity conditions for the HYCARR (1, *d*, 1) model.

Since, the HYCARR process models the non-negative range data it is important to guarantee that the conditional mean of the price range data to be non-negative. In this section we derive the constraints which are the necessary and sufficient conditions to ensure the non-negativity of the conditional mean range in HYCARR (1, *d*,1) process.

The non-negativity conditions of the conditional variance term in the return based GARCH model was first discussed by Nelson and Cao (1992), Baillie, Bollerslev, and Mikkelsen (1996), Bollerslev and Mikkelsen (1996), Chung (1999) and Conrad and Haag (2006) for the FIGARCH model. Conrad (2010) derived the non-negativity conditions for the HYGARCH model. However, there is only a limited number of research work discussing the non-negativity of range-based models. For an example Chou (2005) discussed the non-negativity of the conditional mean term in the CARR process, in another study Chou (2006) presented it for the Asymmetric CARR(ACARR) process, Xie (2017) talked about the non-negativity in Feedback ACARR (FACARR) model and Ratnayake and Samaranayake (2020) derived necessary and sufficient conditions for the Generalized FACARR process.

Motivated by the initial work presented by Conrad (2010) for the HYGARCH model, we present the necessary and sufficient conditions to ensure the non-negativity of the conditional mean range term of the HYCARR process is presented. According to the equation (5), the conditional mean range (λ_t) of the HYCARR model is expressed as an infinite sum of weighted past realization of range values. If the HYCARR coefficients (ψ_i^{HY}) of the above infinite series are non-negative ($\psi_i^{HY} \geq 0$), then this ensures the non-negativity conditional mean range term. To guarantee this inequality constraints need to be placed on the model parameters.

First, we look into the case where $0 < \eta < 1$, a sufficient condition for having non-negative HYCARR coefficients ($\psi_i^{HY} \geq 0$), is that all the weights of the CARR (ψ_i^{CA}) and the FICARR (ψ_i^{FI}) are non-negative.The non-negativity of the CARR can be guaranteed by following the condition proposed by Chou (2004) and the FICARR can be derived by mimicking the non-negativity conditions of FIGARCH by Conrad and Hagg (2006). This is a more restrictive sufficient condition. Moreover, it also ignores the scenario where$\eta > 1$, in which CARR coefficients (ψ_i^{CA}) on the equation (6) adding negative weights to the expression and make doubts about the previous sufficient condition. This is because adding a negative CARR weight (ψ_i^{CA}) to the infinite series given in (6) can still be ended up with the non-negative HYCARR coefficients (ψ_i^{HY}) depending on FICARR weights (ψ_i^{FI}). In the next section we focus on deriving the necessary and sufficient non-negativity conditions for the HYCARR (1, *d*, 1) coefficients (ψ_i^{HY}) regardless of the non-negativity of the ψ_i^{CA} and ψ_i^{FI} coefficients.

Recall the HYCARR (1, *d*, 1) model coefficients (ψ_i^{HY}) can be derived recursively and it is presented in equations (8)-(12).

Theorem 1. The conditional mean of HYCARR (1, *d*,1) is non-negative a.s. *iff*

Case 1:$0 < \beta < 1$

Either $\psi_1^{HY} \geq 0$ and $\theta \leq \delta_2$ or for $k > 2$ with $\delta_{k-1} < \theta \leq \delta_k$ it holds that $\psi_{k-1}^{HY} \geq 0$.

Case 2:$-1 < \beta < 0$

Either $\psi_1^{HY}, \psi_2^{HY} \geq 0$ and $|\beta| \leq \delta_2$ or for $k > 3$ with $f_{k-2} < |\beta| \leq f_{k-1}$ it holds that $\psi_1^{HY}, \dots, \psi_2^{HY} \geq 0$.

Here,η acts as a scaling parameter which does not influence the main argument of the theorem 1. The proof of the theorem 1 follows similar arguments as the proof of HYGARCH (1, *d* ,1) in Conrad (2010) and considered the case where parameter $\eta \in (-1,0)$ for the HYGARCH (1, *d*, 1) model. Full detail of the theorem 1 is presented in the full paper based on the HYCARR model.

Based on the findings of Conrad (2010) for HYGARCH (1, *d*, 1) process and modifying these to the proposed HYCARR (1, *d*,1) process, we can say that if β is positive (i.e., Case 1) it suffices to check two conditions and three conditions ifβ is negative (i.e., Case 2) to ensure the non-negativity of conditional mean range λ_t.

For an example let consider Case 1 of theorem 1, where $0 < \beta < 1$, and arbitrary pick $k = 3$.This equivalent toψ_1^{HY},$\psi_2^{HY} \geq 0$, and $\theta < \delta_3$ andprovides the following results,

$$\beta - \eta d \leq \theta \leq \frac{2-d}{3} \text{ and } \eta d\left(\theta - \frac{1-d}{2}\right) \leq \beta(\theta - \beta + \eta d). \quad (13)$$

These inequality constraints are coinciding with the modified version of sufficient conditions for the HYGARCH (1, *d*, 1*)* derived by Conrad (2010). However, these sufficient conditions exclude wide range of necessary and sufficient conditions given by Theorem 1.

5. Parameter Estimation, In-Sample Performance, and Out of Sample Forecasting

In this section, parameter estimation and forecasting methods for proposed HYCARR model are presented. For the illustrative purposes, the Exponential HYCARR (EHYCARR) models is introduced and model parameters for the EHYCARR is discussed.

5.1. Parameter Estimation of Exponential HYCARR (EHYCARR) Model

In this subsection, the EHYCARR model is introduced. Then conditional log likelihood function is derived, and finally Maximum Likelihood Estimation (MLE) method is employed to estimate the model parameters.

In the case where residual series ε_t , is independently and identically distributed as an exponential distribution with unit mean such that, $\varepsilon_t \sim i.i.d. \exp(1)$ with $E(\varepsilon_t) = 1$, then the resulting HYCARR (*s*, *d*, *q*) model is named as Exponential HYCARR (EHYCARR) process with order s, *d*, and *q* and written as EHYCARR (*s*, *d*, *q)*.

The conditional probability density function of $R_t|\mathcal{F}_{t-1} \sim \exp(\lambda_t(\mathbf{\Omega}))$and it can be expressed as:

$$f(R_t|\mathcal{F}_{t-1}) = \frac{1}{\lambda_t(\mathbf{\Omega})} exp\left(-\frac{R_t}{\lambda_t(\mathbf{\Omega})}\right).$$

Here, $\lambda_t(\mathbf{\Omega}) = \frac{\gamma}{B(1)} + \sum_{i=1}^{\infty} \psi_i^{HY} L^i R_t$ and $\mathbf{\Omega} = (\boldsymbol{\beta}, \boldsymbol{\theta}, \gamma, \eta, d)' \subset \mathbb{R}^{q+s+3}$ is the parameter vector, such that $\boldsymbol{\beta} = \left(\beta_1, \dots, \beta_q\right)' \in \mathbb{R}^q$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_s)' \in \mathbb{R}^s$.

Then the conditional log likelihood function of EHYCARR (*s*, *d*, *q*) is derived as follows:

$$LLF(\mathbf{\Omega}|\mathcal{F}_{t-1}) = -\sum_{\forall t}\left(ln\lambda_t(\mathbf{\Omega}) + \frac{R_t}{\lambda_t(\mathbf{\Omega})}\right). \quad (14)$$

Note that the EHYCARR process, ψ_i^{HY} is a function of $\mathbf{\Omega}$and $\lambda_t(\mathbf{\Omega})$ is an infinite series of weighted average of past range values (R_t). In practice some pre sample values must be assigned to initiate the recursion for the conditional mean function. Here, the pre sample values of R_t for $t \leq 0$,by the unconditional mean range μ_1 (Similar argument was made by Davidson (2004) for FIGARCH process). Then the resulting conditional mean function and conditional log likelihood function are $\widetilde{\lambda_t}(\mathbf{\Omega})and\widetilde{LLF}(\mathbf{\Omega}|\mathcal{F}_{t-1})$ respectively. Final approach here is to maximize the log likelihood function conditional on these pre sample values using MLE technique and resulting vector $(\widehat{\mathbf{\Omega}})$ can be defined as:

$$\widehat{\mathbf{\Omega}} = \arg max\, \widetilde{LLF}(\mathbf{\Omega}|\mathcal{F}_{t-1}). \quad (15)$$

5.2. In-Sample Performance

After estimating the HYCARR model parameters $(\widehat{\mathbf{\Omega}})$, then the price range values can be predicted as $\widehat{R}_t \approx \lambda_t(\widehat{\mathbf{\Omega}})$. Next, in-sample performance of the proposed long-memory HYCARR model is measured, and it is compared with short-memory CARR model by using the Root Mean Square Error (RMSE) and the Mean Absolute Error (MAE) values.

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(R_t - \hat{R}_t)^2}{N}} \text{ and } MADE = \frac{\sum_{i=1}^{N}|R_t - \hat{R}_t|}{N}. \tag{16}$$

Here, R_t is the price range at time *t*, $\hat{R}_t$ be the predicted price range at time *t* and in-sample size is denoted by N.

5.3. Out-of-Sample Forecasting

Under out of sampling forecast, we use the rolling window approach to forecast out-of-sample value. In the rolling window approach, first we divided the entire sample period (sample size =*T*) into two periods namely in-sample period (in-sample size=*N*<*T*) and out-of-sample period (out-of-sample size =*T*-*N*). The first one-step-ahead out-of-sample forecasting is carried out using the all the *N* in-sample data. The method is given bellow:

Let define, $R_{1:N}(1)$be the one step ahead forecast (based on in-sample data i.e., 1st observation to Nth observation)of $R_{N+1}$where$R_{N+1} = \lambda_{N+1}\varepsilon_{N+1}$ and $E(R_{N+1}|\mathcal{F}_N) = \lambda_{N+1}$. Therefore, one step ahead forecast value $R_{1:N}(1)$ can be estimated by using the $\lambda_{N+1}(\widehat{\boldsymbol{\Omega}})$:

$$\lambda_{N+1}(\widehat{\boldsymbol{\Omega}}) = \begin{cases} \hat{\gamma}^* + \sum_{i=1}^{k}\hat{\psi}_i^{HY}R_{N+1-i} & : k \leq N \\ \hat{\gamma}^* + \sum_{i=1}^{N}\hat{\psi}_i^{HY}R_{N+1-i} + \hat{\mu}\sum_{i=N+1}^{k}\hat{\psi}_i^{HY} & : k > N \end{cases}. \tag{17}$$

Here $\hat{\mu} = \frac{\sum_{t=1}^{N}\lambda_t}{N}$, and *k* is the number of previous lags that we use to calculate the weighted sum of range series. After calculating the forecasted value for the $(N+1)^{th}$ observation (i.e., $f_{N+1} = R_{1:N}(1)$), the sample window is moved to (2: *N*+1) to forecast $(N+2)^{th}$ observation (i.e., f_{N+2}). Next, we considered the window of (2: *N*+1) as the new in-sample data and recalculated the model parameters based on this new data. After the estimation, the estimated parameters were applied to the one step ahead forecasting method in equation (20) to calculate $R_{2:N+1}(1)$ which the forecasted value is for f_{N+2}. This process is repeated until all the future values are estimated in the out-of-sample data. Moreover, to check the forecasting accuracy of the proposed HYCARR model with CARR models DM test was used (See. Diebold & Marino, 1995).

6. Simulation Study

In this section, we conduct a simulation study to examine the finite sample performance of the Maximum Likelihood Estimation (MLE) method discussed in the previous section. In this simulation study two time series lengths $n = 2000$, and $n = 4000$ are considered and each time series are $M = 500$times replicated. As discussed in Conrad (2010), to avoid the initial value issue first 6000 observations are discarded. Therefore, for each time series $n + 6000$ observations are generated. For the demonstration purpose, the EHYCARR process is considered to generate the data.

First, we generate range data by using EHYCARR (1, *d*,1) process as follows:

$$R_t = \lambda_t\varepsilon_t, \text{ and } \lambda_t = \gamma^* + \sum_{i=1}^{k}\psi_i^{HY}R_{t-i}.$$

In this simulation study,ε_t is *i.i.d.* random error term which follows an exponential distribution with unit mean and the resulting model is EHYCARR process.

In the EHYCARR process conditional mean range λ_t, is modeled as a weighted sum of infinite series of past range values but in practice this is an impossible to reach. Therefore, reasonable number of past lags are considered to model the $\lambda_t$term. This number of (truncation) lags is denoted by k in this study k is set to 1000. Weighted coefficients ψ_i^{HY} are calculated recursively as mentioned in the equations (12) and (13) (or alternatively (14)). As mentioned in the earlier section when $t \leq 0$, replace R_t by the unconditional mean range value μ_1 and after generating $n + 6000$ observations first 6000 values are dropped to avoid the pre sample complications. The conditional log likelihood function (17) is maximized by using '*nloptr*' package in R software to estimate the model parameters. The accuracy of the estimates is evaluated using the Mean Absolute Error Deviation (MAED) which is formulated as$MAED(\Omega) = \sum_{i=1}^{M} \frac{|\Omega - \widehat{\Omega}_i|}{M}$, where Ω is the parameter of interest and $\widehat{\Omega}_i$ is the estimated parameter value at i^{th} iteration. The simulation result for EHYCARR (1, *d*,1) is presented in Table 1.

According to the simulation results given in table 1, MLE procedure estimate the model parameters with quite accurate. In the case where η close to 1, estimated values have lower MADE when compared to the mid-range values of η. For higher time series length parameters are estimated with higher accuracy.

Table 1: Means of MLE estimates and MADE (within parentheses) for EHYCARR model with order (1, *d*,1)

	$\boldsymbol{\gamma}$	$\boldsymbol{\theta}$	$\boldsymbol{\beta}$	$\boldsymbol{d}$	$\boldsymbol{\eta}$
True parameter	**0.01**	**0.20**	**0.45**	**0.50**	**0.95**
$n = 2000$	0.0138 (0.0055)	0.1642 (0.0799)	0.4885 (0.1249)	0.5900 (0.1257)	0.9318 (0.0375)
$n = 4000$	0.0118 (0.0033)	01759 (0.0580)	0.4592 (0.0903)	0.5405 (0.0783)	0.9400 (0.0275)
True Parameter	**0.01**	**0.40**	**0.30**	**0.10**	**0.15**
$n = 2000$	0.0101 (0.0067)	0.3513 (0.1407)	0.2810 (0.1460)	0.1224 (0.1412)	0.2095 (0.1883)
$n = 4000$	0.0100 (0.0006)	0.3670 (0.1218)	0.2924 (0.1211)	0.1330 (0.1436)	0.2223 (0.1861)
True Parameter	**0.01**	**0.30**	**0.40**	**0.80**	**0.35**
$n = 2000$	0.0098 (0.0006)	0.2574 (0.0931)	0.4546 (0.1898)	0.8308 (0.1608)	0.4373 (0.2337)
$n = 4000$	0.0099 (0.0005)	0.2607 (0.0936)	0.4622 (0.1819)	0.8264 (0.1516)	0.4413 (0.2240)
True Parameter	**0.10**	**0.20**	**0.30**	**0.60**	**0.75**
$n = 2000$	0.1074 (0.0173)	0.1662 (0.1124)	0.3726 (0.1953)	0.7081 (0.1802)	0.7820 (0.0594)
$n = 4000$	0.1049 (0.0128)	0.1704 (0.1106)	0.3443 (0.1734)	0.6751 (0.1287)	0.7707 (0.0434)

7. Empirical Study

In this section we illustrate the usefulness of the proposed HYCARR model in real world scenario through an empirical application based on the S&P 500 stock index. First the S&P 500 sample data is divided in to two sample periods namely in-sample and out-of-sample periods and then in-sample data is used to estimate the model parameters and predictions for the HYCARR process. The RMSE and the MAE values are calculated, and we compare the in-sample performance of long-memory HYCARR with short-memory CARR models. Finally, the out-of-sample data is used to gauge the forecast performance of the HYCARR model and then compare it with the CARR process.

7.1. Data Set

As mentioned above, the S&P 500 stock indices are used to gauge the performance of the proposed HYCARR model and compare it with the CARR models. The sample periods for the S&P 500 data spanned from January 01, 2002, to December 31, 2019. Daily values for the opening price, closing price, high price, low price, and adjusted price are reported over the span of the study period. The data set was obtained from the Yahoo Finance (https://finance.yahoo.com/) by using the '*quantmod*' package in R software. The data set is divided in to two sub samples: the first sub sample which is also known as in-sample period and this sample is used to estimate the model parameter and in-sample predictions. In-sample periods for S&P 500 spanning from January 01, 2002, to December 29, 2017. The second sub sample, which was also called as out-of-sample period, and this sample is used for out-of-sample forecasting. Out-of-sample periods for S&P 500, elapsed from January 1, 2018, to December 31, 2019. Table 2 presents the summary statistics of the S&P 500 range data which is calculated as given in section 2.

Table 2: Summary statistics for S&P 500 data

Statistics	01/01/2002-12/31/2019	01/01/2002-12/29/2017	01/01/2018-12/31/2019
Number of Days	4531	4028	503
Minimum	0.1456	0.1456	0.1699
Mean	1.2484	1.2770	1.0193
Median	0.9804	1.0046	0.7928
Maximum	10.9041	10.9041	5.0352
Standard Deviation	0.9931	1.0167	0.7408
Skewness	3.3472	3.3689	2.1125
Q (1)	2151***	1919.3***	204.43***
Q (5)	9912.4***	9012.9***	669.47***
Q (22)	32270***	30222***	1092.2***
Q (252)	90970***	87955***	3037.3***

Note: *** indicates 1% significance level.

Summary statistics for the S&P 500 stock index for full sample period, in-sample period, and out-of-sample period are presented in the Table 2. The Ljung-Box test for serial correlation in the range data is carried out for lags 1, 5, 22, and 252. The lag selection is done based on the insight from the financial literature where these lag orders represent a single trading day, week, month and year respectively. According to the Ljung-Box test results for all lags, show that all period of sample data exhibited highly significant correlations. This concludes the high persistence on the S&P 500 stock data. Positive skewness and non-negativity in range data suggests positively skewed nonnegative support pdf, such as an exponential distribution, must be used to model the S&P 500 price range data.

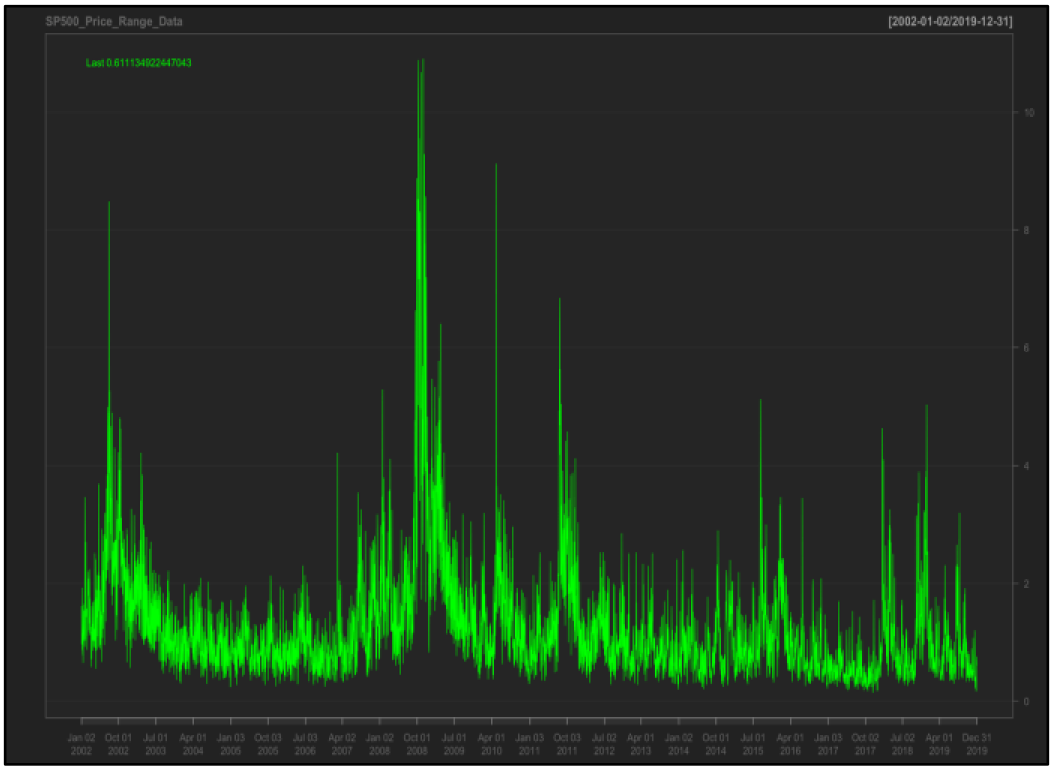

Figure 1: Daily price range for S&P 500 data from 01/01/2002 to 12/31/2019.

According to the figure1, height of the spike, represents the daily price range volatility. For a given trading day having a taller spike implies higher volatility when it compares to a day with a shorter spike. One noticeable feature that presents in the figure 1 is the volatility clusters, which means that periods with taller (shorter) spikes are clustered together. For an example the price volatility picked up quickly, reached its maximum, and kept it high during the economic recession that was taken place on 2007-2009.After the economic bubble ended volatility dropped down.

7.2 In-sample Estimating

In this section we compare the performance of the proposed HYCARR process and CARR process and discuss pros and cons about each model. To illustrative purposes we use the exponential version of both models that means Exponential HYCARR (EHYCARR), and Exponential CARR (ECARR) models are used. The autocorrelation function of the in-sample data (Figure 2) indicates the serial correlation in the price range data. The slow decaying autocorrelation pattern suggests that the long-memory (long range dependence) in the range data.

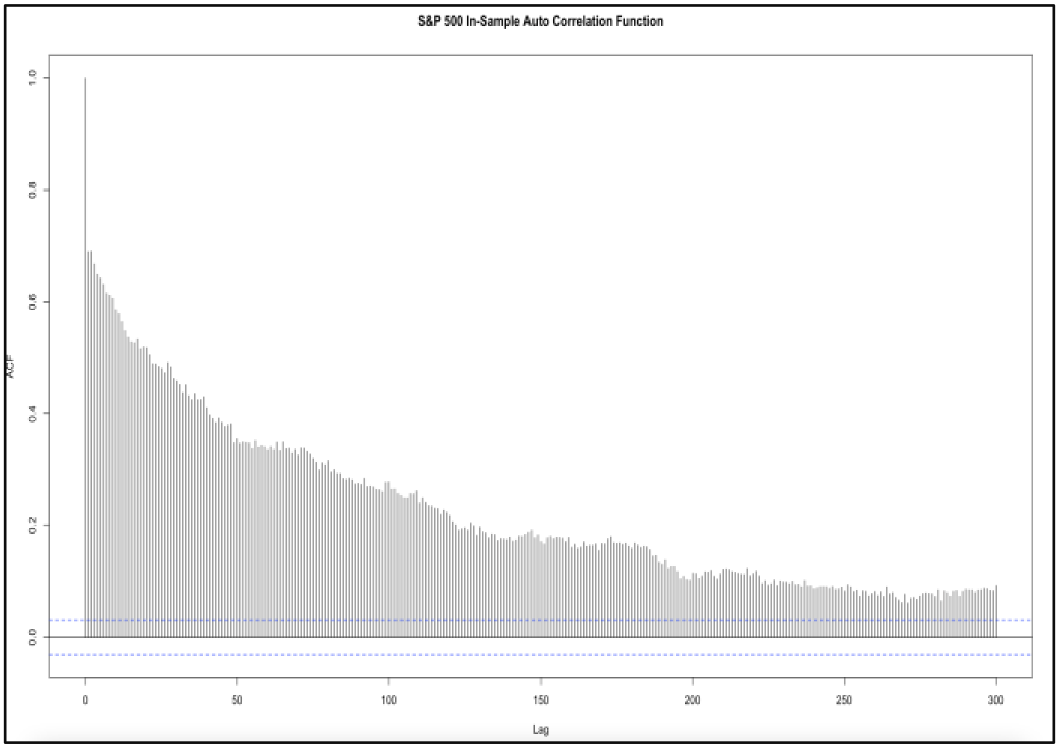

Figure 2: Autocorrelation Function (ACF) up to lags 300 for the S&P 500 daily price range data from 01/01/2002 - 12/31/2017.

Next, the EHYCARR (1, d, 1) and model ECARR (1,1) model are fitted to the in-sample data and model parameters are estimated. In this study $K = 1000$ lagsare accounted to calculate the weights (ψ_i^{HY}) which will be then used to calculate the conditional mean range term of the EHYCARR (1, d,1) process. Furthermore, residual series is going through series of tests such as Kolmogorov Simonov (KS) test to check whether the residuals are followed an exponential distribution and diagnostic test such as the Ljung-Box test, for residuals to check whether residuals are independent and identically distributed. Finally, we calculate the LLF (Log Likelihood Function), and AIC (Akaike Information Criteria), for each model and compare the results. The model with smaller AIC, value and larger LLF value are considered to be significantly better model than the other. Parameter estimation results for the S&P 500 price range data for the ECARR and EHYCARR are summarized in the Table 3.

According to the parameter estimation results, lim $\eta \to 1$ implies that the HYCARR process is behaves like EFICARR process. Moreover, LLF value for the EHYCARR process is slightly higher than that of the ECARR and AIC value of the EHYCARR model is slightly lower than that of the ECARR, suggest that EHYCARR fits better to the data. According to the Ljung-Box test for ECARR and EHYCARR residual series suggest that, both series are independently and identically distributed. However, the Kolmogorov Simonov test results indicates that the residuals are not exponentially distributed (test statistics for EHYCARR is closer to 0 than the ECARR) and this will be addressed in the full paper.

Table 3: Estimation and diagnostic test results of the ECARR (1,1) and EHYCARR (1, *d*, 1) for S&P 500 data.

Table 3A: Parameter Estimation Results		
	ECARR (1,1)	EHYCARR (1, *d*,1)
constant	0.0220	0.0303
α	0.1980	
β	0.7840	0.3200
θ		-0.0425
η		0.9919
d		0.5494
LLF	-4502.8585	**-4499.8657**
AIC	9011.7171	**9009.7314**
Table 3B: Diagnostic Test Results (P Value)		
	ECARR (1,1)	EHYCARR (1, *d*,1)
KS	0.3250 (<0.0001)	0.3232 (<0.0001)
Q (1)	0.9507 (0.3295)	0.0023 (0.9619)
Q (5)	10.9950 (0.0515)	1.0967 (0.9544)
Q (22)	27.2430 (0.2021)	16.4150 (0.7947)

According to the figure 3, except for few random spikes ACF values for HYCARR residual series lie inside the 95% confidence limits indicating that there is no significant serial correlation present in the residual series.

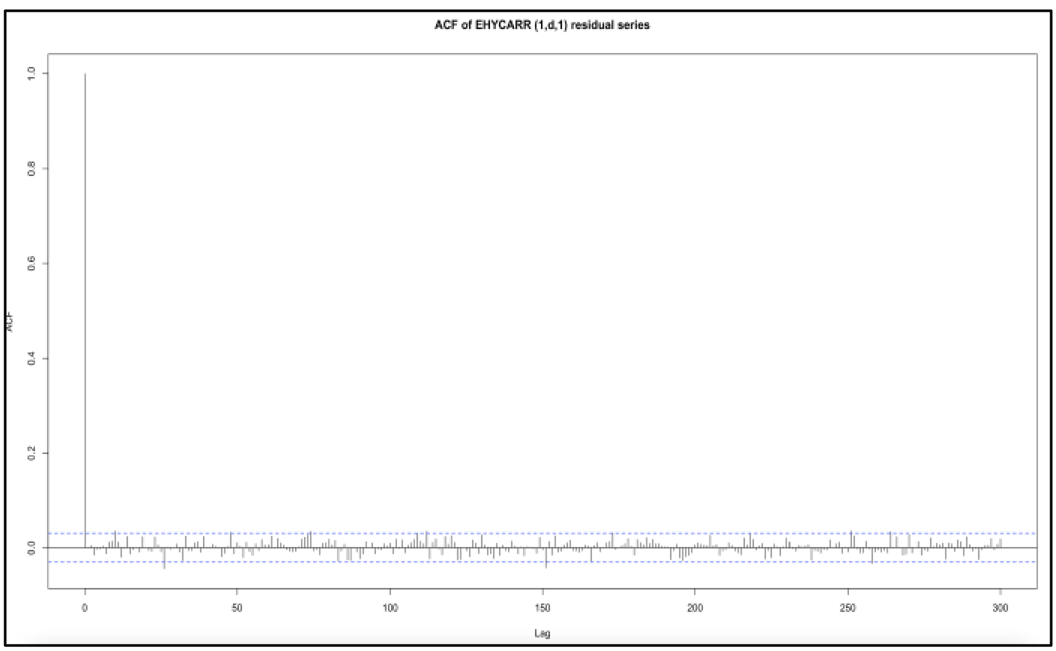

Figure 3: Autocorrelation Function (ACF) up to lags 300 for the residual series of the HYCARR (1, *d*, 1) process.

Table 4: In-Sample Comparison between ECARR (1, 1), and EHYCARR (1, *d*,1) for S&P 500 data.

Table 4A: Model Performance Comparison During Full Sample Period		
Statistic	ECARR (1,1)	EHYCARR (1, *d*, 1)
RMSE	0.6340	**0.6340**
MAE	0.4177	**0.4168**
Table 4B: Model Performance Comparison During Economic Recession Period		
Statistic	ECARR (1,1)	EHYCARR (1, *d*, 1)
RMSE	**1.2367**	1.2432
MAE	0.8509	**0.8496**

Since the RMSE and MAE values for EHYCARR are lower than that of the ECARR, therefore we can say that EHYCARR process fits better that the ECARR process to the S&P 500 in-sample data. We also check the performance of the proposed model during the recession period and based on the MAE value proposed EHYCARR model has more accuracy than the ECARR process. However, ECARR model has the lower RMSE value during the recission period (2007-2009).

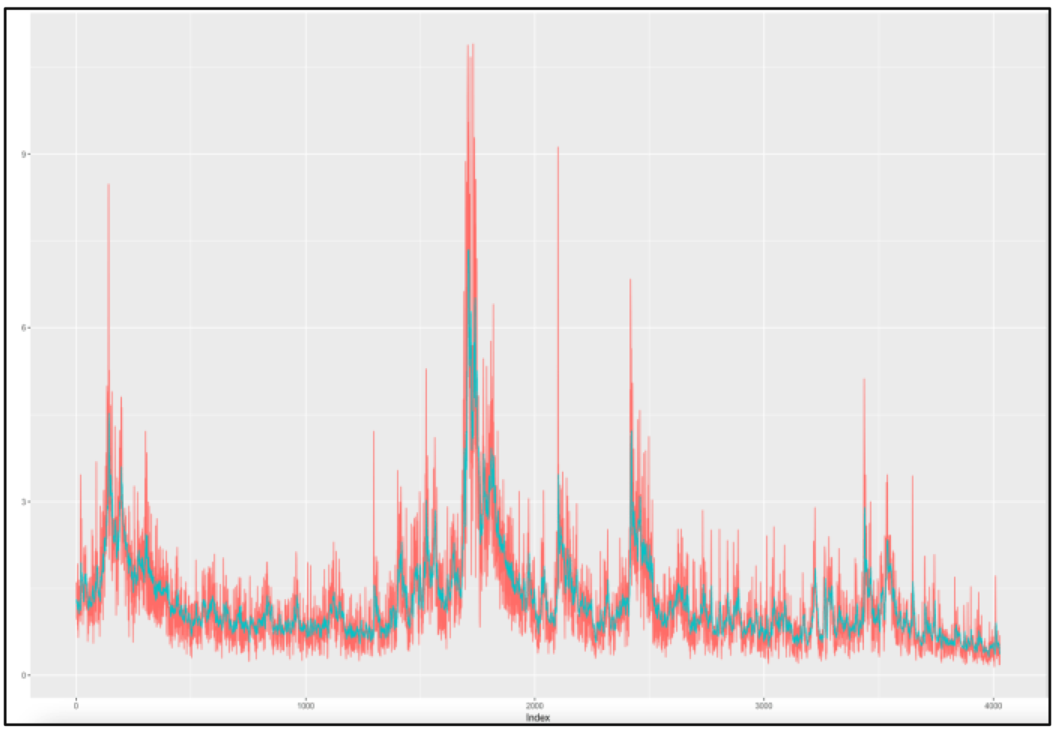

Figure 4: In-sample prediction (green) of the EHYCARR (1, *d*, 1) model for the S&P 500 price range data (red).

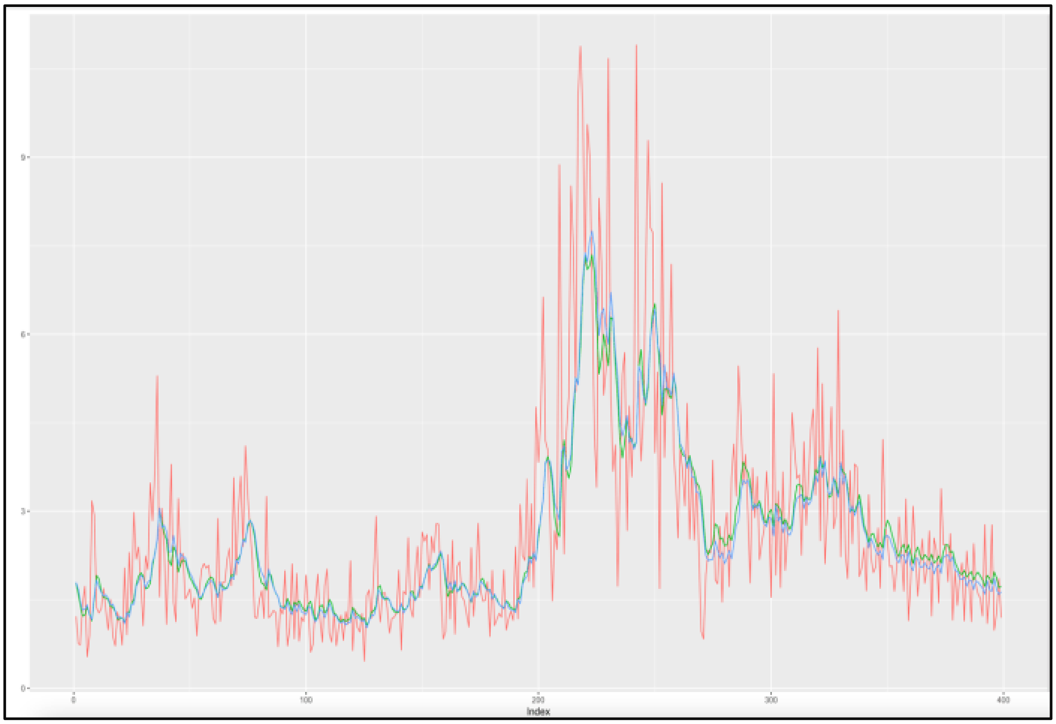

Figure 5: Prediction of the EHYCARR (1,*d*,1) model (**green**), and ECARR (1,1) model (**blue**) during the 2007-2009 recession period.

7.3 Out-of-Sample Forecasting

In this subsection, we focus on out-of-sample performance of proposed EHYCARR (1, *d*, 1) model and it compares with ECARR (1,1) model. Out of sample period span from January 1, 2018, to December 31, 2019. The out-of-sample performance is evaluated via using RMSE and MSE values and Table 5, summarizes the comparison results between two models. Finally, the Diebold- Marino (DM) test results for the forecast errors are presented.

According to the out-of-sample performance summarized in table 5, the long-memory EHYCARR (1, *d*, 1) model has lowest MAE and RMSE values when compared to the short-memory ECARR (1,1) model. This result suggests that the proposed EHYCARR (1, *d*, 1) does a better job than the ECARR (1,1) in forecasting S&P 500 price range data.

According to the DM test result (p-value = 0.0634), it can be concluded with 90% confidence that the one-step-ahead forecasting accuracy of the EHYCARR (1, *d*,1) model is higher than that of the ECARR (1,1) model.

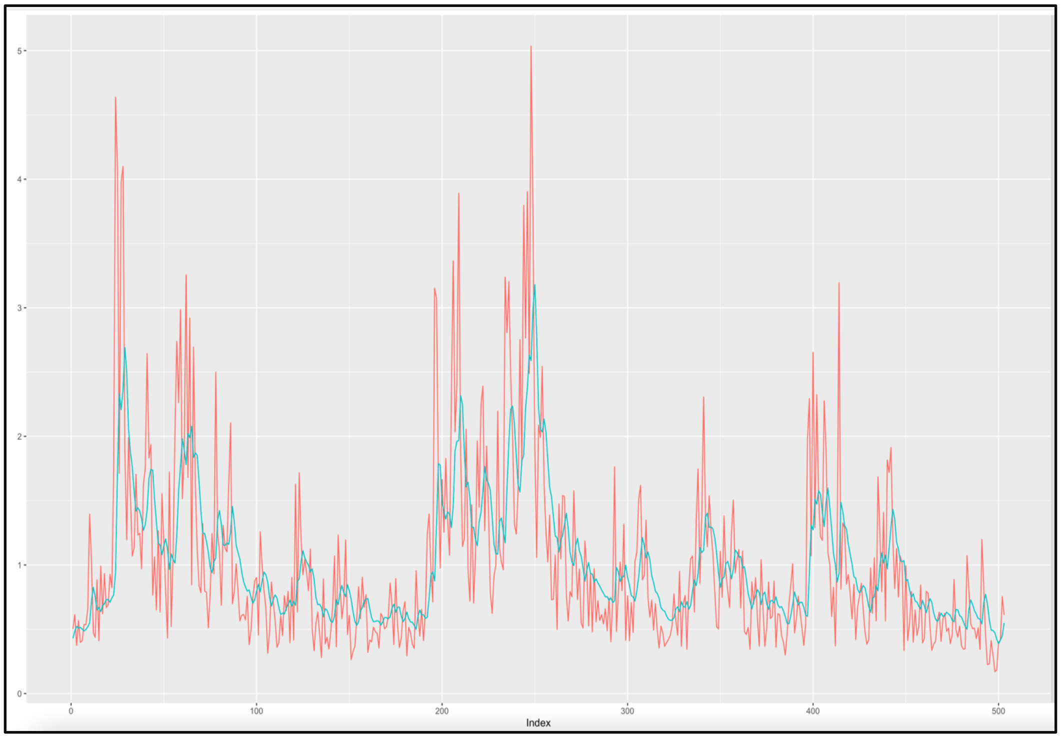

Figure 6: The one step ahead out of sample forecast of EHYCARR (1, d, 1) (green) for S&P 500 price range data (red).

Table 5: Out-of-Sample Comparison between ECARR (1, 1), and EHYCARR (1, *d*,1) for S&P 500 data.

Statistic	ECARR (1,1)	EHYCARR (1, *d*, 1)
RMSE	0.5684	**0.5637**
MAE	0.3873	**0.3805**

8. Conclusions

In this study we propose a long-memory conditional heteroscedastic range-based time series model to analyze the long run serial correlation that exhibits in the range series. The proposed model is named as Hyperbolic Conditional Autoregressive Range (HYCARR) and derived its moment properties. Further, the paper discussed sufficient conditions that ensure the non-negativity of the conditional mean range term. The MLE technique is used for parameter. The simulation study results indicated that the proposed method estimated parameters quite accurately. Finally. the empirical study is carried out by using the exponential versions of the HYCARR (1, *d*,1) and CARR (1,1) model by applying to the S&P 500 data. Kolmogorov Simonov test results suggest that residual series are not exponentially distributed instead are positively skewed positive support distributions. Hence, a lognormal distribution must be considered to model the error term. Based on the in-sample results proposed EHYCARR process performs slightly better than the ECARR. According to the MAE and RMSE results for the one-step ahead out-of-sample period, the proposed EHYCARR forecasts appear better than its competitor. Further, Diebold Marino (DM) test for forecast accuracy also agreed with the above conclusion with 90% confidence.

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