Modified Control Chart for Monitoring the Process Dispersion

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Abstract
Control charts are powerful tools used by many industries to monitor the quality of their processes and detect special causes of variation. They are often used to monitor the mean and the variance of some process quality characteristic with the well-known $\bar{X}$ and S^2 Control Charts, respectively. In their original formulation, if the actual process mean (μ) or variance (σ^2) are different or larger from their specified in-control values (μ_0 and σ_0^2, respectively), the process is declared out of control. However, in many practical situations, even though the process may be declared out of control, it might be still capable from a practical point of view in terms of the proportion of nonconforming items produced. Thus, it may not be necessary to stop the process and start looking for assignable causes, which can save time and resources. The Modified Control Charts were designed to monitor the process mean in such a capable situation. With this background as motivation, in this paper, we propose the use of the Modified Control Charts for monitoring the process dispersion.

Key Words: Type I and Type II Errors, False Alarm Rate, Average Run Length, Acceptance and Modified Control Chart, S^2 Control Chart.

1. Introduction

Control charts are powerful tools used by many industries to monitor the quality of processes and detect special cause of variations on them. The S^2 Control Chart is one of the most used tools to monitor if the variance of some quality characteristic (X), which is assumed to be normally distributed, may change from an in-control (IC) to an out-of-control (OCC) situation. The main objective of this chart is to detect increases of any magnitude in the process variance, as soon as possible. In this context, if the actual process variance is larger than an in-control single level point, the process is considered to be in an out-of-control state.

The basic procedure of the S^2 Control Chart is the following: samples of size n (of some quality characteristic, X, of the product being produced) are collected at regular intervals so the sample variance (S^2) can be computed. This sample variance is compared with a control limit. If S^2 is above the control limit, chances should be high that the process is out of control, or in other words, chances should be high that the actual process variance is larger than the nominal in-control value.

However, in some situations, even if a process is declared out of control, it might still be capable from a practical point of view in the sense that it still produces an acceptably low proportion of non-conforming items and hence the process does not need to be stopped in order to look for assignable causes. This can save valuable time and resources. In other words, if the process variance is permitted to be a bit larger than the in-control variance value and yet the rate of non-conforming items being produced is small enough, this may be a tolerable situation from a practical point of view.

In summary, it is of interest to monitor the process mean and variance with control charts with a broader definition of "in-control" together with the capability of the process. Unfortunately, the original Shewhart $\bar{X}$ and S^2 control charts are not designed for this type of monitoring. Instead, in this situation, the Modified and the Acceptance charts, (which are Shewhart-type charts) introduced respectively by Hill (1956) and Freund (1957), are more appropriate tools, since they allow the process mean to vary between two specified/tolerated limits (Montgomery, 2009) and yet ensure that only a small proportion of non-conforming items are produced so there is no need to declare the process out-of-control and start a search for assignable causes.

Modified and Acceptance charts are also powerful tools to avoid many false alarms and this is especially important nowadays when there are several systems with many control charts being used simultaneously, as emphasized recently by Woodall and Faltin (2019). Modified (and Acceptance) control charts generate less false alarms (compared with the Shewhart X chart) because, as explained above, they are designed to detected only genuinely important changes in the process mean (changes that generate a rate of non-conforming items larger than what is specified). So, even though these charts were created a long time ago (in the 50's), they may be still of great value in practice today. More applications of these types of charts can be found in Mohammadian and Amiri (2012), Oliveira et al. (2018) and Wu (1998).

Unfortunately, the Modified and Acceptance Control charts were designed only focusing on monitoring the process mean. As emphasized by several authors, see for example Montgomery (2009), monitoring the process variance is also important to avoid the production of an undesirable amount of non-conforming units. Given this background as motivation, this work extent the idea of the Modified and Acceptance Control Charts by focusing on monitoring the process dispersion.

This paper provides further development of the S^2 Modified Control Chart presented by Landim, Jardim and Oprime (2021) and study this chart for the case where it is designed with estimated variance parameter. This paper and Landim, Jardim and Oprime (2021) paper share parts of the same text.

The rest of the paper will be presented in four parts. First, we provide an overview of the well-know S^2 Control Chart emphasizing the problem of not considering the process capability in its formulation. In sequence, we briefly present the model of the Modified Control Chart for monitoring the process variance, already presented in details by Landim, Jardim and Oprime (2021), introduces the S^2 Modified Controle Chart for unknown variance and brings an illustrative example (using simulation) of the application of the new proposed Modified Control Chart for monitoring the variance and finally the conclusions.

2. A Review Of The S^2 Control Chart

The S^2 Control Chart is one of the most well-known tools to monitor the variance of a quality characteristic (X) of processes in many industries. As presented in the Introduction, the main objective of the S^2 Control Chart is to detect increases (of any magnitude) in the process variance (σ^2), as soon as possible. In this context, if the actual process variance (σ^2) is larger (by any magnitude) than an in-control single level point (σ_0^2), the process is considered to be in an out-of-control state, otherwise the process is declared in control. Figure (1) illustrates this situation.

This chart shall be used together with the mean control chart, in order properly detect changes for the mean. To monitor the process variance (σ^2) with the S^2 Control Chart samples of size n of the quality characteristic (X) are collected at regular intervals so the sample variance (S^2) can be computed. S^2 is also known as the plotting statistic of the chart and it is given by

$$S^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_j - \bar{X}\right)^2, \tag{1}$$

where X_j is the j-th observation of the quality characteristic of each sample being collected at regular intervals ($j = 1,2,\dots,n$). X_j is considered normally distributed with mean μ_0 and variance σ^2, n is the size of each sample being collect at regular intervals and $\bar{X}$ is the sample mean of each sample given by

$$\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j \,. \tag{2}$$

Note that σ^2 is the actual variance of the process (the one which is being monitored). Here we assume that the process mean remains at the in-control value (μ_0) and in the exact middle point between the specification limits, consistently with the purpose of detecting relevant increases in the process variance only.

The plotting statistic (S^2) given by Equation (1), should be compared with the Upper Control limit (UCL_{S^2}) of the S^2 Control Chart which is given by

$$UCL_{S^2} = \sigma_0^2 \,\frac{\chi^2_{n-1,\,1-\alpha}}{n-1}, \tag{3}$$

where σ_0^2 is the nominal in-control process variance, $\chi^2_{n-1,\,1-\alpha}$ is the (1-α)-quantile of a chi-square distribution with $\mathrm{n}-1$ degrees of freedom and α is the nominal false alarm rate (or in other words, the false alarm probability) chosen by the practitioner (usually, $\alpha = 0.0027$).

A false alarm is defined as a signal (alarm) when the process is in control. The maximum false alarm rate happens when $\sigma^2 = \sigma_0^2$. So, note that the Control Limits given by Equation (3) is derived in order to provide a maximum false alarm rate equal to α, as shown in Equations (4) and (5) below.

$$Maximum\ \ False\ Alarm\ Rate = 1 - P(S^2 < UCL_{S^2} \mid \sigma^2 = \sigma_0^2)$$
$$= 1 - P\left(S^2 < \sigma_0^2 \frac{\chi^2_{n-1,1-\alpha}}{n-1} \middle| \ \sigma^2 = \sigma_0^2\right)$$
$$= 1 - P\left(\frac{(n-1)S^2}{\sigma_0^2} < \sigma_0^2 \frac{(n-1)\,\chi^2_{n-1,1-\alpha}}{\sigma_0^2(n-1)}\right), \quad (4)$$

where $\frac{(n-1)S^2}{\sigma_0^2} = \chi^2_{n-1}$ is a random variable that follows a chi-squared distribution with n-1 degrees of freedom, so

$$Maximum\ False\ Alarm\ Rate\ = 1 - P\left(\chi^2_{n-1} < \chi^2_{n-1,1-\alpha}\right) = \alpha. \quad (5)$$

When the actual variance of the process (σ^2) is exactly at the in-control process variance (σ_0^2) value, the proportion of non-conforming units being produced should be small. In other words, the probability of the quality characteristic (X) be smaller than the lower specification limits (LSL) or larger than the upper specification limits (USL), should be small. Figure (2) illustrates this situation. Note that these specification limits are provided by the project/manager.

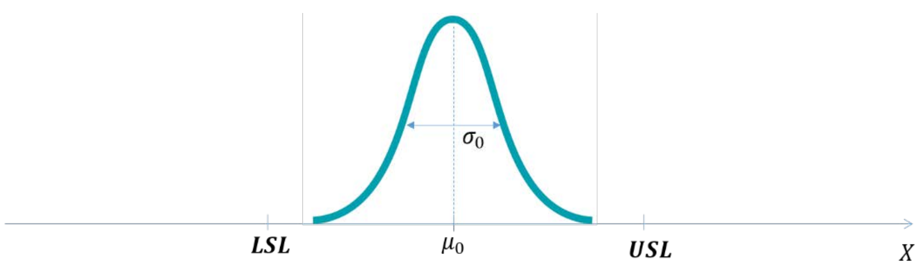
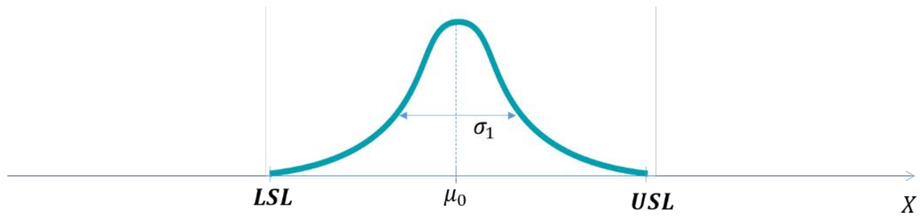


Figure 2: Process running with the nominal in-control variance ($\sigma^2 = \sigma_0^2$) with all the item being produced within the specification limits. Source: The authors themselves.

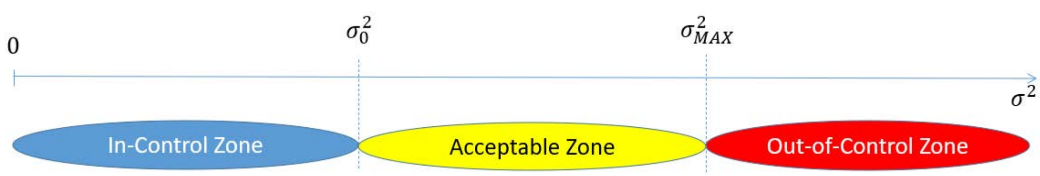
The S^2 Control Chart is designed to detect increases (larger than σ_1^2) of any magnitude in the actual process variance (σ^2), even increases that does not affect the rate of non-conforming items being produced. These increases will tend to produce a signal (alarm) on the control chart. Consider the illustration provided by Figure (3) where the actual process variance is larger than σ_0^2 ($\sigma^2 = \sigma_1^2 > \sigma_0^2$), but yet the rate of non-conforming items is still small.



Note that since σ_1^2 is larger than σ_0^2, from the perspective of the S^2 Control Chart, the process should be declared out of control. In this case, the chart will tend to signal an alarm. However, this may be a problem because, as can be seen in Figure (3), the process is still not producing a large number of non-confirming items (almost all the items being produced are still within the specification limits even though $\sigma^2 = \sigma_1^2 > \sigma_0^2$. So, trying to fix this increase on the variance may be a waste of time and money, since in most of the cases, the process would have to be paused. So, as seen in the Introduction, it is of interest to monitor the process variance with a control chart with a broader definition of "in-control" which considers the specification limits. In the next section, we develop such kind of Control Chart for variance, we named this chart as the S^2 Modified Control Chart [in consonance with the Modified Control chart for monitoring the process mean introduced by Hill (1956)].

3. The S^2 Modified Control Chart Model

As discussed in the Introduction, the main idea of the chart we are proposing in this paper is that the process variance (σ^2) is allowed to be larger than the in-control variance value (σ_0^2) until a maximum value (σ_{MAX}^2), as long as the process remains capable, in the sense that it produces a specified (tolerated) small fraction of non-conforming items (γ). In the situation we are concerned with, instead of the in-control situation be represented by $\sigma^2 \leq \sigma_0^2$ (where σ_0^2 represents the specified in-control target value for the process variance) as in the traditional process control setting, we allow the process to be "roughly in-control" or acceptable when $\sigma^2 \leq \sigma_{MAX}^2$ (where $\sigma_0^2 \leq \sigma_{MAX}^2$). If σ^2 assumes a value larger than σ_{MAX}^2, the process is deemed out-of-control (OOC). Figure 4 illustrates this situation.



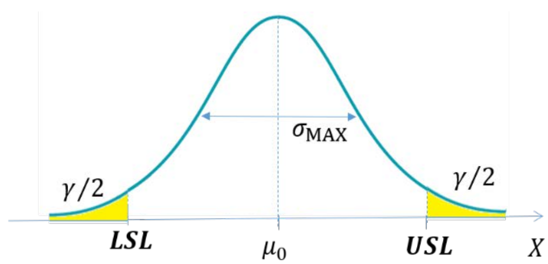
The major concern of the chart user is to detect increases in process variance, for this reason, this research aims at upper one-sided charts (without a lower control limit) only.

The σ_{MAX}^2 value must be chosen with care, depending on the lower and upper specification limits, LSL and USL, respectively, and the maximum rate (probability) of non-conforming units produced (denoted here by γ) that may be tolerated (or allowed). LSL, USL and γ are specified by the management/project and have the following relationship:

$$\gamma = 1 - P(LSL < X < USL | \sigma^2 = \sigma_{MAX}^2), \tag{6}$$

where X is the quality characteristic of the process and follows a normal distribution with mean μ_0 and variance σ^2. As considered in the traditional S^2 Control Chart, it is assumed that the process mean remains at the in-control value (μ_0).

So, γ is the maximum tolerated probability of X being smaller than the LSL or greater than the USL that can be tolerated in a specific application. Figure (5) illustrates when the process is running at the maximum allowed tolerated rate of nonconforming units (γ), which happens when $\sigma^2 = \sigma_{MAX}^2$. Note that if $\sigma^2 > \sigma_{MAX}^2$ the rate of nonconforming units produced will be larger than the specified γ, and consequently, the process will be declared OOC.



According to Landim, Jardim and Oprime (2021), the maximum tolerated variance (σ_{MAX}^2), when the in-control mean μ_0 is centered between the specification limits, can be calculated by known or given parameters, that are the specification limits and the maximum tolerated rate of non-conforming units (γ) as follow:

$$\sigma_{MAX} = \frac{USL - LSL}{2\, z_{1-\gamma/2}} \tag{7}$$

For the case where the process variation is known, to calculate the upper control limits (UCL_{Mod}) of the S^2 Modified Control Chart, one just need to replace σ_0^2 in the original control limit equation of the S^2 Control Chart [see Equation (3)] by σ_{MAX}^2, as shown below:

$$UCL_{Mod} = \sigma_{MAX}^2 \frac{\chi^2_{n-1,\,1-\alpha}}{n-1} = \frac{(USL-LSL)^2\, \chi^2_{n-1,1-\alpha}}{4(n-1)\,(z_{1-\gamma/2})^2}. \tag{8}$$

An useful illustrative example of S^2 Modified Control Chart for known variance and the in-control mean μ_0 is provided and further discussed by Landim, Jardim and Oprime (2021).

For the case where the process mean is in-control value (μ_0) and centered between the specification limits, and process variance unknown, the estimation of the process variance is traditionally done by collecting m samples with size n elements from an in-control (IC) process during Phase I. Considering that the estimation is done from sampling data, there is a data variability when compared to the entire process which is known as "practitioner-to-practitioner variability" (SALEH, *et al.*, 2015) since each chart practitioner can get a different sample. This influences the parameter estimation, where the control limits become conditioned to these estimates and according to Chakraborti (2006), some operational properties of the control chart, such as FAR and ARL are compromised. Some of the parameter estimation effects on FAR calculation is the increase of false alarms.

According to Chakraborti and Graham (2019), after the control chart design phase (Phase I), when the reference samples are collected and the control limits have been calculated from the estimated parameters (recalling that in this study, the process mean is known), the Phase II starts, which is when the process monitoring actually takes place, assessing process samples and if there are special causes that move the process from an in-control state (IC) to an out-of-control state (OOC). Here, the chart provides information to process user to act on the process, so it moves back to a control state.

Because the process monitoring is done based on sampling, a particular sample collected may show that the process is in control when actually it is not, and vice versa, it is important that the process manager assess the control chart performance in order to minimize or prevent unnecessary process stop, which contributes for lower process efficiency and increased costs.

Once the estimator $\hat{\sigma}_0^2$ is calculated, the UCL for the one sided S^2 chart to be used during Phase II monitoring can be defined, for a specified or nominal False Alarm Rate (FAR) and α_{NOM}, Equation (9) can be rewritten as:

$$\widehat{UCL}_{S^2} = \hat{\sigma}_0^2 \; \frac{\chi^2_{n-1,\,1-\alpha}}{n-1} \tag{9}$$

In this paper, the estimator chosen and recommended is the pooled estimator $\left(S_p^2\right)$ based on the variances of the Phase I samples, that according to Mahmoud (2010), this is the best estimators for σ_0^2. The pooled variance $\left(S_p^2\right)$ is calculated by the sample variance means of the samples collected in Phase I, as shown:

$$S_p^2 = \frac{1}{m}\sum_{i=1}^{m} S_i^2, \text{ where } S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{i,j} - \bar{X}_i\right)^2 \tag{10}$$

The Modified control limit however, is not a random variable because it is not estimated, being calculated by given parameters such upper and lower specification limits that are defined by process manager or consumers and acceptance rate of nonconforming units, that are also defined by process manager in order to fulfil customer requirements. This being said, the Modified Chart for variance performance has the same behavior found for Shewhart'd S^2 Chart when parameters are known.

In order to evaluate how the UCL_{MOD} and $\widehat{UCL}_{S^2}$ correlates, we propose a additional Phase 0, before well known traditional Phase I. In addition, to provide to process practitioners guidance to define if the chart developed fits to real use, we also propose the assessment of the ratio between σ^2_{MAX} and σ_0^2 as detailed in the following paragraphs.

When we need to estimate σ_0^2, it means that the $\widehat{UCL}_{S^2}$ may vary because the variance now is a random variable. However, the UCL_{MOD}, considering that the mean is between the USL and LSL, does not shift. Thus, if the $\widehat{UCL}_{S^2}$ is located below UCL_{MOD} most of the time, it is possible to conclude that the issues caused by the estimations and its inherent errors could be less severe. Hence, we calculate the probability of $\widehat{UCL}_{S^2}$ is less than UCL_{MOD}, that is $P(\widehat{UCL}_{S^2} < UCL_{MOD})$, recalling that the UCL_{MOD} is calculated by using given parameters [see Equation (8)].

The estimation of Upper Control Limit is given by Equation (9) and as stated previously, according to Mahmoud (2010), the pooled variance (S_p^2) is the best estimators for σ_0^2 and is calculated by Equation (10). Replacing $\hat{\sigma}_0^2$ by S_p^2 we have the equation (11).

$$\widehat{UCL}_{S^2} = S_p^2 \, \frac{\chi^2_{n\text{-}1,\, 1\text{-}\alpha}}{n\text{-}1} \tag{11}$$

Therefore, $P(\widehat{UCL}_{S^2} < UCL_{MOD}) = P\left(S_p^2 \, \frac{\chi^2_{n\text{-}1,\, 1\text{-}\alpha}}{n\text{-}1} < \sigma^2_{MAX} \, \frac{\chi^2_{n\text{-}1,\, 1\text{-}\alpha}}{n\text{-}1}\right)$

$$= P\left(S_p^2 < \sigma^2_{MAX}\right) \tag{12}$$

Developing the Equation (12) we have the following:

$$P(\widehat{UCL}_{S^2} < UCL_{MOD}) = P\left(\frac{m(n-1)S_p^2}{\sigma_0^2} < \frac{m(n-1)\,\sigma^2_{MAX}}{\sigma_0^2} \right)$$

where $\frac{m(n-1)S_p^2}{\sigma_0^2} = \chi^2_{m(n-1)}$, here denoted as Y, is a random variable that follows a chi-squared distribution with $m(n-1)$ degrees of freedom, so we have the following Equation (13). Is important to notice that here we are still in Phase I, thus we have the in-control variance σ_0^2.

$$P(\widehat{UCL}_{S^2} < UCL_{MOD}) = P\left(Y < m(n-1)\,\frac{\sigma^2_{MAX}}{\sigma_0^2}\right) \tag{13}$$

Recalling that UCL_{MOD} has all parameters known and is fixed, after proper derivations, we found that the Equation (13) depends on the in-control variance σ_0^2, which is unkwown. Assuming that the process practitioner does not known the actual value of σ_0^2, is useful to note that $\sigma^2_{MAX}/\sigma_0^2$, is the ratio between σ^2_{MAX} (the maximum value that the process variance is allowed to be, compared with the in-control process value, in the sense that the process produces a tolerated small fraction of nonconforming items) and the in-control variance σ_0^2. By dividing σ^2_{MAX} by σ_0^2, we may define how much larger the maximum allowed variance is required to be when compared with the the target variance.

Hence, what should this ratio be so that the probability of the traditional S^2 control limit of the estimated parameter is less than the modified control limit, at a given probability of 95%, 90% or 99%? That is, 95% of all possible $\widehat{UCL}_{S^2}$, due to S_p^2 is a random variable, will be smaller than UCL_{MOD}.

The Figure 6 below shows graphically this behavior, for Phase I, where the green lines representing $\widehat{UCL}_{S^2}$ were estimated for a $m = 25, n = 5$ and the parameters listed in Table 1, and the red line representing UCL_{MOD}, which is fixed and calculated in terms of σ_{MAX}, specification limits, degrees of freedom and α_{nom}, all given parameters.

Table 1: Parameters used for calculation of control limits of Figure 6

μ_0	S_p	σ_{MAX}	USL	LSL	α
74,000	0,0117	0,0128	74,050	73,950	0,27%

Source: The author themselves.

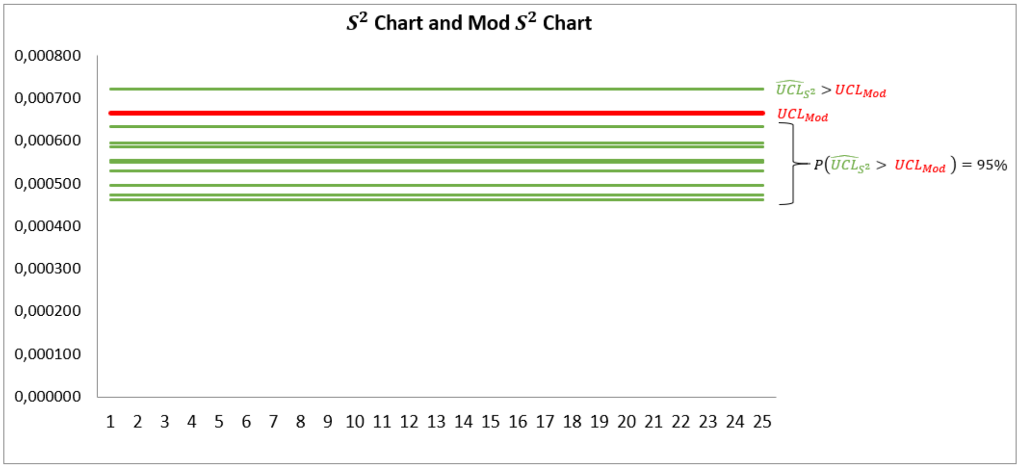


Figure 6: Illustration of how the $\widehat{UCL}_{S^2}$ locates around UCL_{MOD} Source: The author themselves.

Is also important to notice that we can find different required ratios for $\sigma^2_{MAX}/\sigma^2_0$ by varying m and n. Based on these informations, we can raise the following question:

What should be the value of $\sigma^2_{MAX}/\sigma^2_0$ so that $P\left(Y < m(n-1)\frac{\sigma^2_{MAX}}{\sigma^2_0} \right) = 95\%$ (or any other desired probability)?

The chart user defines this probability, recalling that the ideal scenario would be that σ^2_0 is known, however it is an unknown process parameter, that can be either given for calculation matters or replaced by the pooled variance (S^2_p).

For $m = 25$, $n = 5$ and $P = 95\%$, we have that $\sigma^2_{MAX}/\sigma^2_0 = 1{,}2434$, that means if the practitioner collects 25 samples of size 5 during Phase I, the σ^2_{MAX} shall be at least 24% larger than the in-control variance, or in this case, the pooled variance (S^2_p) from Phase I, given by Equation (10). If the ratio is 1,2434, we have that 95% possible estimated S^2 control limits $(\widehat{UCL}_{S^2})$, will locate below the modified control limit (UCL_{MOD}). In a new scenario we raise $m = 100$ and keep $n = 5$ and $P = 95\%$, and now we have that now $\sigma^2_{MAX}/\sigma^2_0 = 1{,}1191$, that means, if the process practitioner wants to ensure that all possible 95% possible estimated S^2 control limits $(\widehat{UCL}_{S^2})$, will locate below the modified control limit (UCL_{MOD}).

In Table 2 below, there are several ratios to be considered varying m, n and the given probability of $\widehat{UCL}_{S^2}$ being less than UCL_{MOD}. Is possible to notice that as m and n get larger, the required ratio gets smaller, that means that σ^2_{MAX} is required to be closer in magnitude to σ^2_0, which can contribute to achieve the required ratio easier.

Table 2: Ratio $\sigma^2_{MAX}/\sigma^2_0$ calculation varying m, n and the given probability of $\widehat{UCL}_{S^2}$ being less than UCL_{MOD}

m	n	Degrees of Freedom $m(n-1)$	Ratio (90%)	Ratio (95%)	Ratio (99%)
25	3	50	1,2633	1,3501	1,5231
	5	100	1,1850	1,2434	1,3581
	9	200	1,1301	1,1700	1,2472
50	3	100	1,1850	1,2434	1,3581
	5	200	1,1301	1,1700	1,2472
	9	400	1,0916	1,1191	1,1718
100	3	200	1,1301	1,1700	1,2472
	5	400	1,0916	1,1191	1,1718
	9	800	1,0646	1,0836	1,1200
200	3	400	1,0916	1,1191	1,1718
	5	800	1,0646	1,0836	1,1200
	9	1600	1,0456	1,0589	1,0841
500	3	1000	1,0577	1,0747	1,1070
	5	2000	1,0407	1,0526	1,0750
	9	4000	1,0288	1,0371	1,0528
1000	3	2000	1,0407	1,0526	1,0750
	5	4000	1,0288	1,0371	1,0528
	9	8000	1,0203	1,0261	1,0372
2000	3	4000	1,0288	1,0371	1,0528
	5	8000	1,0203	1,0261	1,0372
	9	16000	1,0144	1,0185	1,0262

Source: The author themselves.

Here, as stated previously, we propose one additional phase for chart design, when compared to what is traditionally accepted. In Phase 0, we determine UCL_{MOD} and $\widehat{UCL}_{S^2}$, following by the control limits and chart build. Then we have Phase I, that comprises to take m samples with n elements and plot each s^2_i. If all points are within the two control limits, we approve the process to move on to monitoring phase (Phase II).

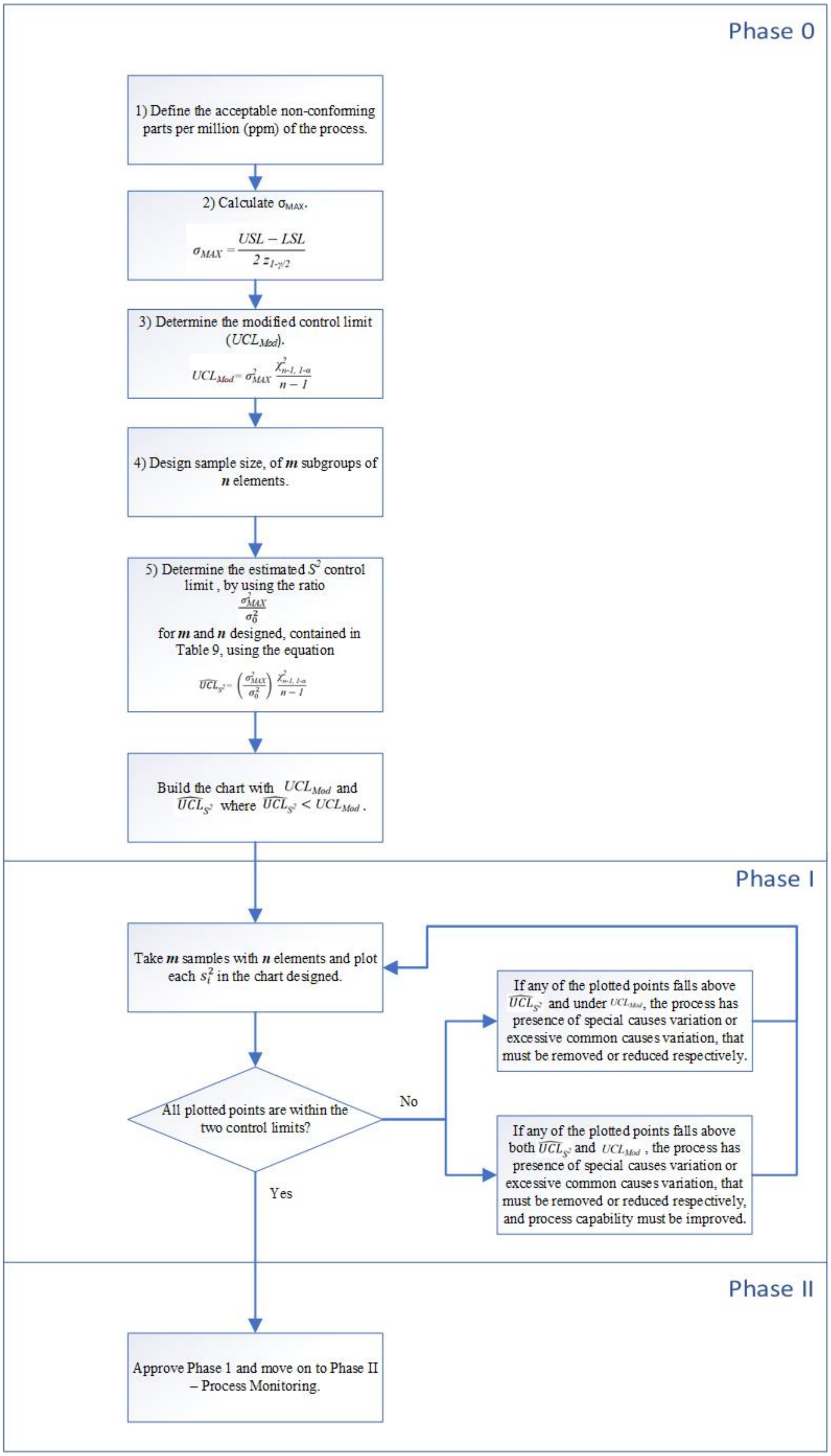


Figure 7: Proposed steps to build the Modified Chart for unknown variance

However, if any of the plotted points falls above $\widehat{UCL}_{S^2}$ and under UCL_{Mod}, the process has presence of special causes of variation or excessive common causes of variation, that must be removed or reduced, respectively. In addition, if any of the plotted points falls above both $\widehat{UCL}_{S^2}$ and UCL_{Mod}, the special causes of variation, if present, must be removed as low as possible, the excessive common causes of variation must be reduced and process capability must be improved. After all corrections required are completed, take another m samples with n elements and plot each s_i^2, and verify if all them fall under both control limits. The Figure 7 shows the steps to be followed to build the Modified Chart for unknown variance.

In the next section an illustrative example is provided to show the use of the proposed Modified Chart and its building steps when the in-control mean is centered between LSL and USL and the variance is unknown.

3.1 An Illustrative Example for unknown variance (σ_0^2) and in-control mean μ_0 centered between LSL and USL

In this section, we illustrate the use S^2 Modified Control Chart for unknown variance using the following example. In an automobile engine manufacturing process, that uses forged piston rings. A more detailed description of this example is given in Montgomery (2009, p.251). The quality characteristic variable (X) is the internal diameter of the piston rings, which has a two-sided specification limits of 74.000 $\pm$ 0.050 mm. It is assumed that the piston rings diameter (X) follows a normal distribution and the in-control mean (μ_0) is known, centered in the middle of specification limits. However, the in-control standard deviation (σ_0) of the piston rings diameter is unknown and therefore it needs to be estimated, being the pooled variance $\left(S_p^2\right)$ used. In this case, the Phase I step needs to be executed, where according to Montgomery (2009), typically $m = 20$ or 25 subgroups are used in this phase and in here, the Phase I will change to the proposed method explained in the previous section, with addition of one previous step.

The process leadership defined as acceptable up to 96 nonconforming parts per million (ppm) of units produced. The chart design starts with the Phase 0, when the chart control limits (UCL_{Mod} and $\widehat{UCL}_{S^2}$) will be defined. Starting with UCL_{Mod}, being USL, LSL and the acceptable nonconforming parts rate known, σ_{MAX} and UCL_{Mod} are calculated as follows:

$$\sigma_{MAX} = \frac{USL - LSL}{2\, z_{1-\gamma/2}} = \frac{74{,}05 - 73{,}95}{2\times 3{,}9} = \frac{0{,}1}{7{,}8} = 0{,}0128$$

$$UCL_{Mod} = \sigma_{MAX}^2 \frac{\chi^2_{n-1,\,1-\alpha}}{n-1} = 0{,}0128^2\ \frac{16{,}25}{5-1} = 0{,}00067$$

During Phase I, the process manager decided to take $m = 25$ subgroups with $n = 5$ elements each. In addition, it is defined the 95% probability of S^2 estimated upper control limit ($\widehat{UCL}_{S^2}$) is less of UCL_{Mod}. Thus, based on Table 2 the ratio $\sigma_{MAX}^2/\sigma_0^2$ shall be at least 1,2434 and hence, considering that σ_{MAX}^2 is 0,000164 ($\sigma_{MAX} = 0{,}0128$), the maximum in-control process tolerable variance σ_0^2 is 0,000132. Therefore, the $\widehat{UCL}_{S^2}$ is calculated as follows:

$$\widehat{UCL}_{S^2} = \left(\frac{\sigma^2_{MAX}}{1{,}2434}\right)\frac{\chi^2_{n-1,\,1-\alpha}}{n-1} = 0{,}000132\frac{16{,}25}{5-1} = 0{,}00054$$

Hence, after Phase 0, we have the following designed control chart:

Figure 4.1 - Graphical demonstration of how the $\widehat{\boldsymbol{UCL}}_{S^2}$ locates around $\boldsymbol{UCL_{MOD}}$

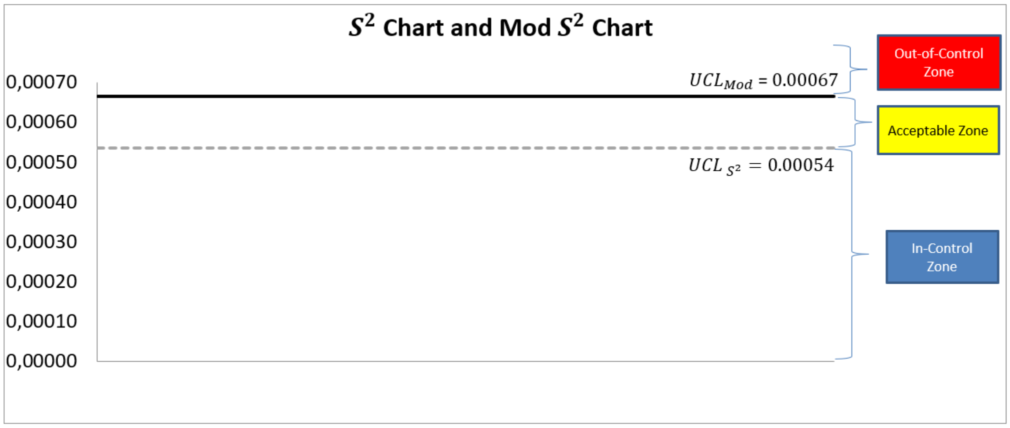


Figure 8: Graphical demonstration of how the $\widehat{\boldsymbol{UCL}}_{S^2}$ locates around $\boldsymbol{UCL_{MOD}}$. Source: The author themselves.

In the next phase, we will take m samples with n elements and plot each Phase I s_i^2, in the chart designed in Figure 8.

To examine the behavior of Phase I, we simulate 25 independent samples of the piston ring diameter (each sample with size 5), from a normal distribution with mean $\mu_0 = 74{,}000$ and a standard deviation equals to $\sigma^2_{MAX}/1{,}2434$, based on Table 2 for $m = 25$, $n = 5$ and $P = 95\%$ was considered as the unknown process variance. The simulated sample variances (S^2) and the control limits (UCL_{S^2} and UCL_{Mod}) are shown in Figure 9. UCL_{S^2} is shown in a dashed grey line and UCL_{Mod} in a solid black line.

As we can see in Figure 9, signals above the $\widehat{UCL}_{S^2}$ dashed line were not found and therefore, the practitioner can move to Phase II, process monitoring. Now in Figure 10, we simulate 1,000 Phase II samples variances with size 5.

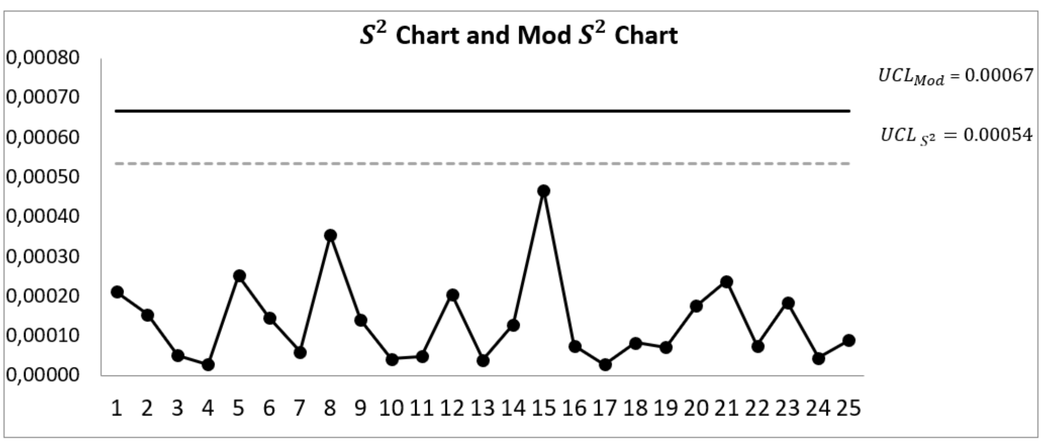


Figure 9: The S^2 Control Chart and the S^2 Modified Control Chart for Phase I execution of a process $X \sim N(\mu_0 = 74,\ \sigma^2 = 0{,}000132)$ given the in-control parameters in the example. Source: The author themselves

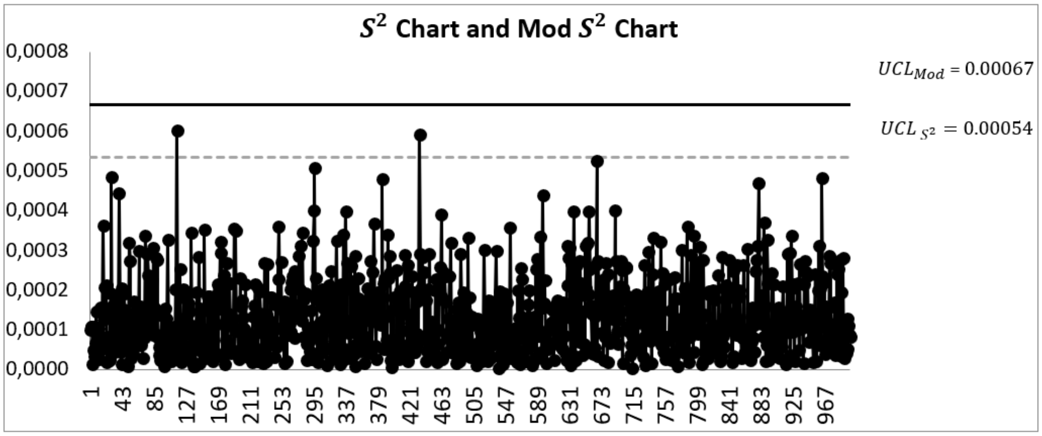


Figure 10: The S^2 Control Chart and the S^2 Modified Control Chart for monitoring the variance of a process $X \sim N(\mu_0 = 74, \sigma^2 = 0{,}000132)$. Source: The author themselves.

We can also see in Figure 10 signals above the $\widehat{UCL}_{S^2}$ dashed line, and again, instead of process practitioner suspect that an assignable cause has occurred, by using UCL_{Mod}, the process variance is still smaller than the maximum variance allowed ($\sigma^2_{\text{MAX}} = 0{,}0128^2$), so the proportion of nonconforming items being produced is still acceptable according to the specification of the project.

The S_p^2 for the data shown in Figure 10is $0,0001322$, so the probability of a false alarm is smaller than $\alpha_{nom} = 0{,}0027$, as shown below.

$$False\ Alarm\ Rate = 1 - P\left(S^2 < UCL_{Mod} \middle| S_p^2 = 0{,}0001322\right)$$
$$= 1 - P\left(S^2 < \sigma^2_{MAX} \frac{\chi^2_{n-1,\ 1-\alpha}}{n-1} \middle| S_p^2 = 0{,}0001322\right)$$
$$= 1 - P\left(\frac{(n-1)S^2}{S_p^2} < \frac{\sigma^2_{MAX}}{S_p^2}\ \chi^2_{n-1,\ 1-\alpha}\right),$$

where $\frac{(n-1)S^2}{\sigma_1^2} = \chi^2_{n-1}$ is a random variable that follows a chi-squared distribution with n-1 degrees of freedom, so

$$False\ Alarm\ Rate = 1 - P\left(\chi^2_{n-1} < \frac{0{,}000164}{0{,}0001322}\ 16{,}25\right) = 1 - P(\chi^2_{n-1} < 20{,}1365)$$
$$False\ Alarm\ Rate = 0{,}00047 < \alpha_{nom} = 0{,}0027$$

Conclusions

This article shows the reasons why the modified chart is useful when running high capable processes this is because, differently from the well-known S^2 Control Chart. This new tool considers in its formulation, the process specifications limits provided by the project/manager and not only the nominal in-control process variance.

This allowed range for variation shift is defined based on the acceptable nonconforming index, which protects the process from produce a rate of defective items larger than what is accepted. The practical implication is that the S^2 Modified Control Chart, designed to detect only genuinely increases in the process variance, may preventing unnecessary process stop and assessment for assignable causes and contribute to higher process efficiency. This is desirable in the sense that small increases in the variance may not affect much the rate of not-conforming items being produced and pausing the process generate extra costs.

Despite being a relevant tool for process monitoring, none other work was found related to provide a range where the process variance may shift without compromising its ability to fulfil quality requirements. Hence, the present work aimed to define the variables and parameters necessary for the construction of the S^2 Modified Control Chart for unknown variance. This work also presented an additional step before Phase I, when the process variance is unknown, providing a practical approach on how to define the monitoring control limits, supporting the process manager decision.

Thus, the major contributions of this work is provide evidence that for highly capable processes, a certain room for variance shift may be allowed without harm quality performance and show how the modified chart can contribute to avoid process over-control and improve management by reducing false alarms. In addition, the use of the modified control chart may minimize the impact of the False Alarm Rate when the control limits are estimated due to estimation due to the modified control limit be defined based on known or target parameters.

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