Adaptive Design and Inference for a Step-stress Accelerated Life Test

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Abstract
Advancement in manufacturing has significantly extended the lifetime of consumer products, while at the same time made it harder to perform product life testing at the normal operating conditions due to the extensively long operational life spans. Accelerated life testing (ALT) mitigates this issue by testing units at higher stress levels so that the lifetime information can be acquired more quickly. The lifetime of a product at normal operation can then be estimated through extrapolation using a regression model. However, there are potential technical difficulties involved since the units are subjected to higher stress levels than normal. In this work, we develop an adaptive design of a step-stress ALT in which stress levels are determined sequentially based on the information obtained from the preceding steps. After each stress level, the estimates of the model parameters are updated and the decision is made on the direction of the next stress level by using design criteria such as D- and C-optimality. Assuming the popular log-linear assumption between the mean lifetime and stress levels, this adaptive design and inference are illustrated based on exponential lifetimes with progressive Type-I censoring.

Key Words: accelerated life tests, adaptive design, Fisher information, progressive Type-I censoring, step-stress loading

1. Introduction

Life testing is a critical stage of product commercialization. Waiting for extended times to examine the life time and robustness of a product is not realistic. Accelerated life testing (ALT) enables getting the life estimates of a product in quick and reliable manner (e.g., Tang et al. (2007), Tseng (1994)). Some examples of ALT include subjecting devices to extreme temperatures, pressure, vibrations, voltage, or other environmental conditions at a fast pace in order to accelerate failure mechanisms and thus enable failures identification faster. The main goal of ALT is to estimate the product life distribution under normal conditions in shorter time so products maintain a short path to the marketplace associated with full product reliability characterization. For a comprehensive review on the ALT models and methods, see, for example, Escobar and Meeker (2006) and references therein.

Accelerated stress-testing takes several forms. Some are time dependent such as Ramp and cyclic stress. Others are time independent such as constant step stress. Constant step stress is most commonly used due to the availability of existing theoretical methods (e.g. Miller and Nelson (1983), Meeker and Escobar (1993), Khamis and Higgins (1996), Zhu and Elsayed (2013), Han and Ng (2013)). Adaptive ALT combines ALT and decision making after each step. The main advantages of such an approach are to significantly shorten testing time and to estimate the parameters of interest with better precision and efficacy, in addition to Cost saving (e.g., Lee et al. (2018)).

Several studies have addressed the adaptive constant stress testing. For instance, (Tang and Liu (2010)) planned sequential tests on constant stress levels by proposing a Bayesian framework. The objective is to optimize both sample allocations and stress combinations at lower stress levels of subsequent accelerated tests. Cramer and Illiopoulos (2010) developed adaptive stress to extend the progressive Type-II censoring model with an objective to choose next censoring numbers based on previous censoring numbers and previous failure

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times. Shen et al.(2017) used order statistics technique to derive likelihood by assuming a general log-location scale distribution for product lifetime. These authors worked on increasing stress level adaptively in order to avoid insufficient failures. Furthermore, You and Pham (2017) developed self-adaptive stress accelerated life tests (SAS-ALT) that determines the next step stress based on the failure data of the previous step-stress.

In this paper, we develop an adaptive step-stress accelerated life test entitled ada-ALT for an exponential lifetime distribution with a single step-stress variable. The ada-ALT algorithm assumes that a log linear relationship exists between the mean time of failure and stress level, along with an exponential distribution from cumulative exposure model for the effect of changing stress in a step-stress ALT. At each stress step, we formulate an optimization function that minimizes the variance of the MLE estimates by using the information matrix. A decision on the next move of the stress level in direction (increase or decrease) is then made with a dependence on C-optimality and D-optimality design criteria. An illustration based on simulation is presented and discussed in Section 4. It is important to emphasize that one ought to be wise in choosing the stress test to be within the design limits of the product. The rest of the paper is organized as follows. Section 1 presents the model formulation and derivation of MLE of the model parameters and the associated Fisher information for the step-stress ALT. In Section 2, C-optimality and D-optimality are defined based on the determinant of the variance. In Section 3, we present the results of illustrative numerical study.

1.1 Problem Formulation

We start by a simple step stress. Randomly, we select a standardized stress level X_1 belongs to $]X_0, X_{(n)}[$, where X_0 is the normal operation stress level at which we want to estimate the mean lifetime of units using extrapolation. $X_{(n)}$ is the maximum standardized stress level we can test unit on. Initializing at X_1, we start with N_1 units. We follow the unit timeline until the first failure time occurrence. Assuming that the first failure time y_1 occurs during monitoring time lag τ_1, then, we move to the next standardized stress level X_2. We start with $N_2 = N_1 - 1$ total units. The units are monitored until first failure occurred at this level y_2, assuming the failure time at this level is τ_2. After this, a decision should be made to stay on the same level or to move up or down in stress level direction. Figure 1 illustrates the formulation of the problem.

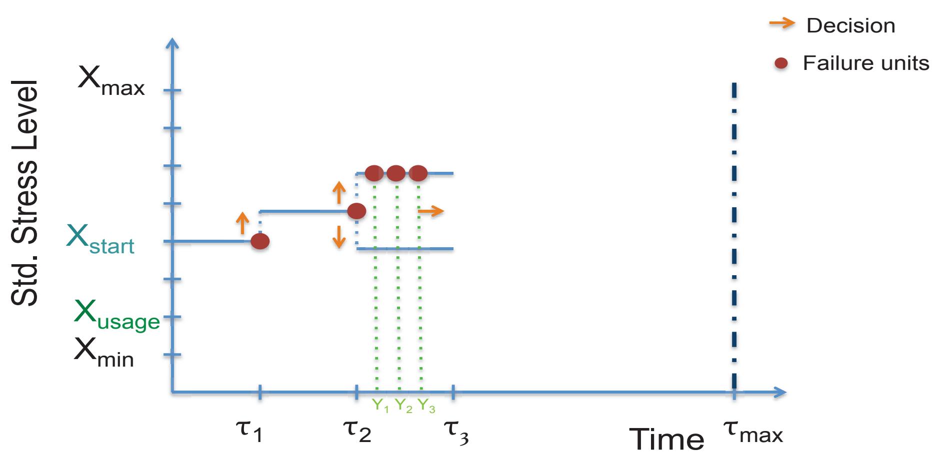


Figure 1: An illustration of the adaptive step-stress ALT.

The following assumptions are used while constructing the sequential adaptive stress level:

1. Lifetime of units follows an exponential distribution at each stress level, with the PDF given by
$$f_i(t) = \frac{1}{\theta_i} \exp\left(-\frac{t}{\theta_i}\right), \qquad 0 < t < \infty$$

2. The stress level at time t is parameterized as
$$X(t) = \frac{s(t) - s_H}{s_U - s_H}, \qquad t \geq 0$$
where s_H is the highest stress level and s_U is the used stress where we would like to predict the lifetime distribution of the units.

3. At any stress level X_i, the mean failure time to failure (MTTF) of a test unit θ_i is a log-linear function of stress represented as
$$\log \theta_i = \alpha + \beta X_i$$

The Accelerated Failure Time (AFT) of units is considered exponential such as

$$\begin{aligned} f(t) &= \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] f_i(t - \tau_{i-1}) \\ & \text{if} \begin{cases} \tau_{i-1} \leq t < \tau_i & \text{for} \quad i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \leq t < \infty & \text{for} \quad i = k \end{cases} \end{aligned}$$

The cumulative distribution function is derived as

$$\begin{aligned} F(t) &= 1 - \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] S_i(t - \tau_{i-1}) \\ & \text{if} \begin{cases} \tau_{i-1} \leq t < \tau_i & \text{for} \quad i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \leq t < \infty & \text{for} \quad i = k \end{cases} \end{aligned}$$

The joint distribution function of the failure counts $n = (n_1, n_2, n_3, ..., n_k) = (1, 1, n_3, ..., n_k)$ and $y_i = (y_{i,1}, y_{i,2}, y_{i,n_i})$ is derived as

$$\begin{aligned} f(y, n) &= \prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!} f(t_i)[S(t_i)]^{N_i - n_i} \\ f(y, n) &= \left[\prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!}\right] \left[\prod_{i=1}^{k} \theta_i{}^{-n_i}\right] \exp\left(\sum_{i=1}^{k} \frac{U_i}{\theta_i}\right) \end{aligned}$$

where

$$\begin{aligned} & n_1 = 1,\ n_2 = 1,\ n_3 = n_3, \\ & N_1 = n,\ N_2 = N_1 - 1 = n - 1,\ N_3 = N_2 - 1 = n - 2, \\ & \tau_i = y_{i,1},\ \Delta_i = \tau_i - \tau_{i-1} \end{aligned}$$

Therefore,

$$U_i = \sum_{l=1}^{n_i} (y_{i,l} - \tau_{i-1}) + (N_i - n_i)\Delta_i$$

is the i-th total time on Test Statistics at stress level X_i.

Using assumption 3, the log-likelihood of (α, β) is given by

$$l(\alpha, \beta) = -\alpha \sum_{i=1}^{k-1} n_i - \beta \sum_{i=1}^{k-1} n_i X_i - \sum_{i=1}^{k-1} U_i \exp(-\alpha - \beta X_i) \tag{1}$$

Since the logarithmic function is a monotonic, differentiating with respect to α, and β, and solving simultaneously provides the Maximum Likelihood Estimators (MLE) $(\hat{\alpha}, \hat{\beta})$. The MLE $(\hat{\alpha}, \hat{\beta})$ are obtained by solving the following score function.

$$\sum_{i=1}^{k-1} U_i e^{-\alpha-\beta X_i} = \sum_{i=1}^{k-1} n_i \tag{2}$$

$$\sum_{i=1}^{k-1} U_i X_i e^{-\alpha-\beta X_i} = \sum_{i=1}^{k-1} n_i X_i \tag{3}$$

Solving for (2), we get

$$\hat{\alpha} = \log\left(\frac{\sum_{i=1}^{k-1} U_i e^{-\beta X_i}}{\sum_{i=1}^{k-1} n_i}\right) \tag{4}$$

Replacing (4) in (3), the MLE of β or $\hat{\beta}$ the solution of

$$\frac{\sum_{i=1}^{k-1} U_i X_i e^{-\hat{\beta} X_i}}{\sum_{i=1}^{k-1} U_i e^{-\hat{\beta} X_i}} = \frac{\sum_{i=1}^{k-1} n_i X_i}{\sum_{i=1}^{k-1} n_i} \tag{5}$$

The explicit form of $\hat{\beta}$ is the solution of (5). Because it is not straight forward to show the exact form of $\hat{\beta}$, we refer to Han and Bai (2019) to show existence and uniqueness of $\hat{\beta}$, solution of (5). Once we get $\hat{\beta}$, we can now plug it in (4) to get the form of $\hat{\alpha}$.

1.2 The Hybrid Information

Due to non-linear nature of the MLE, statistical inference with MLE is based on the asymptotic result of MLE. Thus $(\hat{\alpha}, \hat{\beta})$ is approximately distributed as bivariate normal with mean $E\left[(\hat{\alpha}, \hat{\beta})^\top\right] = (\alpha, \beta)^\top$ and variance matrix $I^{-1}(\alpha, \beta)$, where $I(\alpha, \beta)$ is the expected Fisher information matrix of (α, β). The information matrix is

$$I(\alpha, \beta) = \begin{pmatrix} I_\alpha & I_{\alpha\beta} \\ I_{\alpha\beta} & I_\beta \end{pmatrix}$$

By utilizing the log likelihoods obtained in (2) and (3), Fisher information matrix can be expressed as a sum of observed information I_{obs} and expected information I_{exp}. We start by showing derivation of each term.

$$\begin{aligned} I_\alpha &= E\left[-\frac{\partial^2 l(\alpha, \beta)}{\partial \alpha^2}\right] \\ &= \sum_{i=1}^{k-1} \frac{U_i}{\theta_i} + \frac{1}{\theta_k} E[U_k] \\ &= \sum_{i=1}^{k-1} \frac{U_i}{\theta_i} + \frac{1}{\theta_k}\left(E\left[\sum_{l=1}^{n_k} (Y_{k,l} - \tau_{k-1})\right] + E[N_k - n_k](\tau_k - \tau_{k-1})\right) \end{aligned}$$

It should be noted that $Y_{k,1}, Y_{k,2}, Y_{k,3}, Y_{k,4}, \ldots, Y_{k,n_k}|n_k$ are distributed jointly as order statistics from a random sample of size n_k with the distribution left truncated at τ_{k-1} and right truncated at τ_k. Therefore, in deriving the expected value of $E\left[\sum_{i=1}^{n_k} Y_{k,1}\right]$, we use the property that

$$E\left[\sum_{l=1}^{n_k} Y_{k,l}\right] = E\left[\sum_{l=1}^{n_k} E[Y_{k,l}|n_k]\right] = \theta_k - \tau_k \frac{S_k(\Delta_k)}{F_k(\Delta_k)} + \frac{\tau_{k-1}}{F_k(\Delta_k)}$$

Therefore,

$$E[U_k] = E[N_k] F_k(\Delta_k)\theta_k$$

with $\Delta_k = \tau_k - \tau_{k-1}$. For $k = 1, 2, 3, \ldots$, the hybrid information matrix $I_n(\alpha, \beta)$ can be expressed as

$$I_n(\alpha, \beta) = \begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} \\ I_{\alpha\beta} & I_{\beta\beta} \end{pmatrix}$$

where

$$\begin{aligned}
I_{\alpha\alpha} &= \sum_{i=1}^{k-1} \frac{U_i}{\theta_i} + \frac{E[U_k]}{\theta_k} = \sum_{i=1}^{k-1} \frac{U_i}{\theta_i} + N_k F_k(\Delta_k) \\
I_{\alpha\beta} &= \sum_{i=1}^{k-1} x_i \frac{U_i}{\theta_i} + x_k \frac{E[U_k]}{\theta_k} = \sum_{i=1}^{k-1} x_i \frac{U_i}{\theta_i} + x_k N_k F_k(\Delta_k) \\
I_{\beta\beta} &= \sum_{i=1}^{k-1} x_i^2 \frac{U_i}{\theta_i} + x_k^2 \frac{E[U_k]}{\theta_k} = \sum_{i=1}^{k-1} x_i^2 \frac{U_i}{\theta_i} + x_k^2 N_k F_k(\Delta_k)
\end{aligned}$$

2. Ada-SSALT Proposed Algorithm

Figure 2 below summarizes the steps we used in the ada-ALT algorithm. First we identify the user defined variables as inputs. These are the total number of objects we start the experiment with denoted by n, the time of experiment τ_{max}, the first level randomly chosen to start with X_1, and the stress step size ΔX. We enter first stress level X_1 chosen randomly until first failure of objects is observed. We record time of failure τ_1 and move up to the next stress level denoted by X_2. We observe the remaining $n - 1$ objects until first failure occurs at this level. Therefore, $n_2 = 1$ and we record time of failure τ_2. At this point, MLE estimates of the coefficient represented in (1) is calculated by using (4) and (5). We then check whether the total number of failures recorded is greater than total number of objects, we start the experiment with n or whether time of failures τ_i is greater than or equal τ_{max}. If the conditions are achieved, we stop the experiment and report the MLE used. Otherwise, we proceed to calculate the optimality criteria.

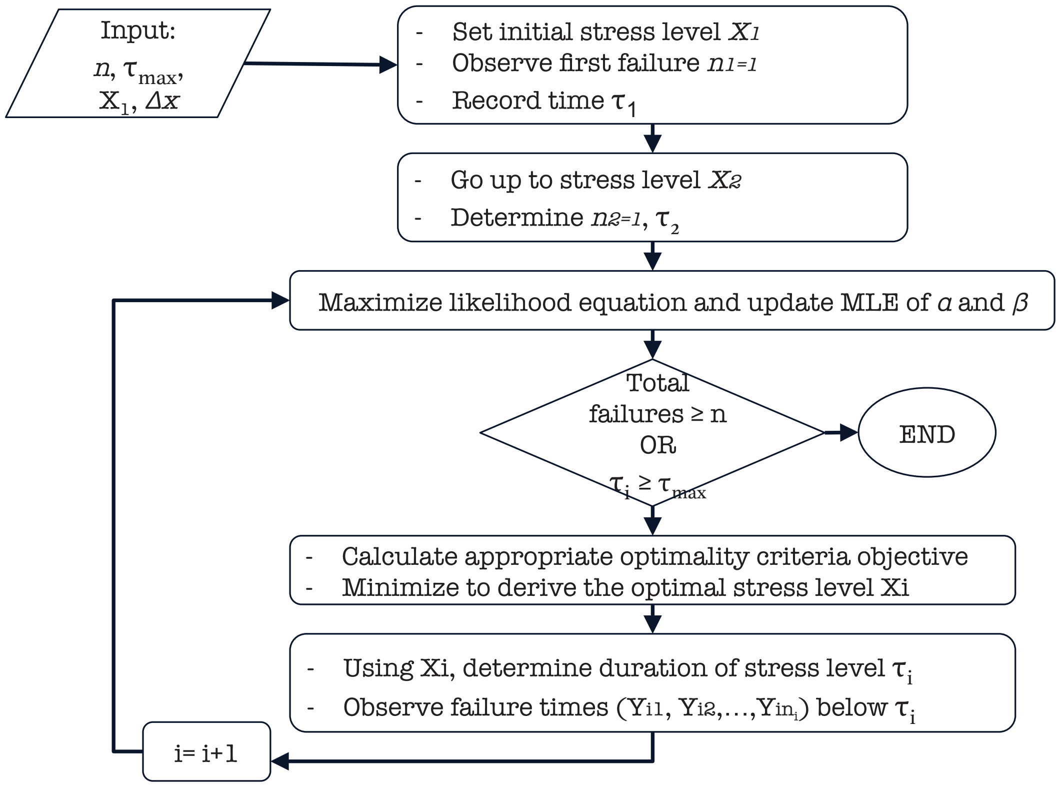


Figure 2: Ada-SSALT Algorithm Steps

At this step, a decision on which direction to move next in step stress should be defined. This is done by minimizing the joint precision of MLE by using different optimality criteria. In this section, we will be using the D-optimality and C-optimality criteria.

2.1 D-optimality Criteria

Using D-optimality, the main objective function is designed based on the determinant of the inverse of the Fisher information matrix or equivalently the reciprocal of the determinant of Fisher information matrix. The objective function to minimize can be represented as

$$\phi_D(\Delta) = |I(\alpha, \beta)|^{-1} = 1/|I(\alpha, \beta)|$$

where

$$|I(\alpha, \beta)| = a x_k^2 + b x_k + c$$

with

$$\begin{aligned} a &= N_k F_k(\Delta_k) \sum_{i=1}^{k-1} \frac{U_i}{\theta_i} \\ b &= -2 N_k F_k(\Delta_k) \sum_{i=1}^{k-1} x_i \frac{U_i}{\theta_i} \\ c &= \left(\sum_{i=1}^{k-1} x_i^2 \frac{U_i}{\theta_i} \right) \left(\sum_{i=1}^{k-1} \frac{U_i}{\theta_i} + N_k F_k(\Delta_k) \right) - \left(\sum_{i=1}^{k-1} x_i \frac{U_i}{\theta_i} \right)^2 \end{aligned}$$

Or equivalently, by using induction, D-optimality can be written as

$$\begin{aligned}\phi_D(x_k) &= |I_n^{-1}(\alpha,\beta)| \\ &= \left(\sum_{i=1}^{k-1}\sum_{j=i+1}^{k-1}(x_i-x_j)^2\frac{U_i}{\theta_i}\frac{U_j}{\theta_j} + N_kF_k(\Delta_k)\sum_{i=1}^{k-1}(x_i-x_k)^2\frac{U_i}{\theta_i}\right)^{-1} \quad (6)\end{aligned}$$

and

$$\begin{aligned}\phi'_D(x_k) &= N_k|I_n(\alpha,\beta)|^{-2} \\ &= \frac{\beta\Delta_k}{\theta_k}S_k(\Delta_k)\sum_{i=1}^{k-1}(x_i-x_k)^2\frac{U_i}{\theta_i} + 2F_k(\Delta_k)\sum_{i=1}^{k-1}(x_i-x_k)\frac{U_i}{\theta_i}\end{aligned}$$

2.2 C-optimality Criteria

The C-optimality minimizes the (asymptotic) variance of the estimator of the lifetime at normal stress level ($i.e.$, when $x_0 = 0$). Using the invariance property of MLE, the objective function of the C-optimality is $V(\log\hat{\theta}_0) = V(\hat{\alpha})$ to be minimized. An objective function to serve this purpose is expressed as

$$\begin{aligned}\phi_C(x_k) &= AVar(\log\hat{\alpha}) = (1\ \ 0)\ I_n^{-1}(\alpha,\beta)\begin{pmatrix}1\\0\end{pmatrix} \\ &= \phi_D(\Delta)I_{\beta\beta} = \frac{1}{|I(\alpha,\beta)|}\left(\sum_{i=1}^{k-1}x_i^2\frac{U_i}{\theta_i} + x_k^2N_kF_k(\Delta_k)\right) \quad (7)\end{aligned}$$

and

$$\begin{aligned}\phi'_C(x_k) &= \phi'_D(x_k)\left(\sum_{i=1}^{k-1}x_i^2\frac{U_i}{\theta_i} + x_k^2N_kF_k(\Delta_k)\right) \\ &\quad - x_kN_k\phi_D(x_k)\left(x_k\frac{\beta\Delta_k}{\theta_k}S_k(\Delta_k) - 2F_k(\Delta_k)\right)\end{aligned}$$

We calculate $\phi_D(x_k)$ and $\phi_C(x_k)$ as presented in (6) and (7) at three stress levels: current stress level (X_2), above one level (X_3) and below one level (X_1). Keep in mind the move in stress level should be slow in order to avoid achieving the shock event fast. The stress level that minimizes the objective of interest ($\phi_D(x_k)$ or $\phi_C(x_k)$) is the next stress level to choose. Using X_i, we determine the duration of stress level τ_i. We observe failure times $(Y_{i1}, Y_{i2}, Y_{i3}, \ldots, Y_{in_i})$ under the condition to stay below τ_i. Then we repeat the process.

3. Illustrative Results

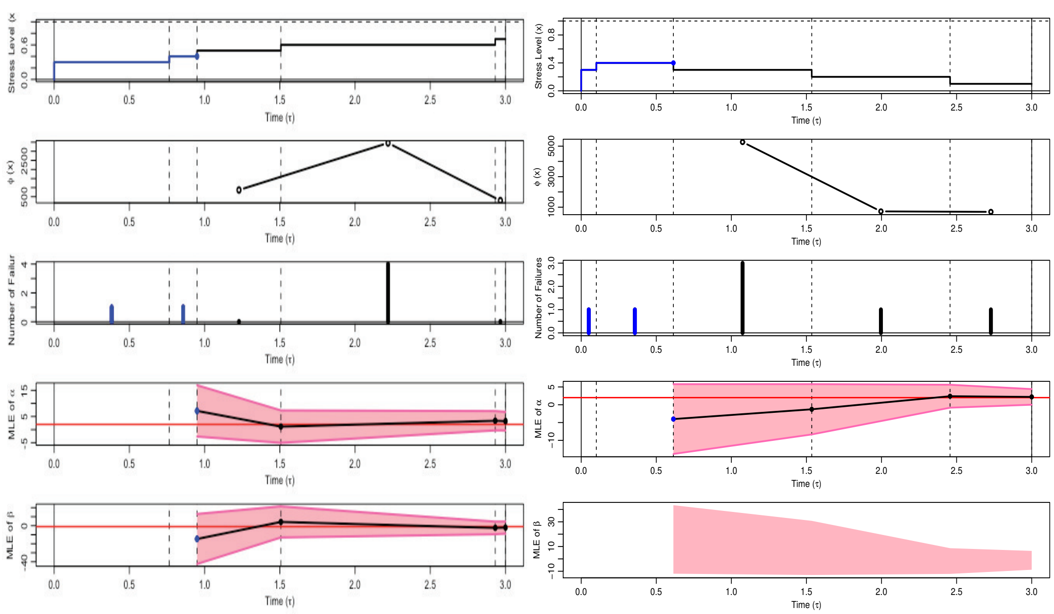
A simulation study is performed to test the ada-SSALT algorithm by using D-optimality and C-optimality designs optimization. Illustrations are shown in Figures 3 and 4. We start with $n = 20$ total units, the starting stress level is 0.3 with an increment of 0.1 such as $X_K = X_0 + (k-1)\Delta X$. The termination time is set to 3. The step duration at iteration k is set to

$$\Delta_k = \min\{w\ \tau_2,\ -\theta_k\log(1-p)\}$$

with adjustments for max step duration $w = 0.5$ and median lifetime for step duration $p = 0.5$.

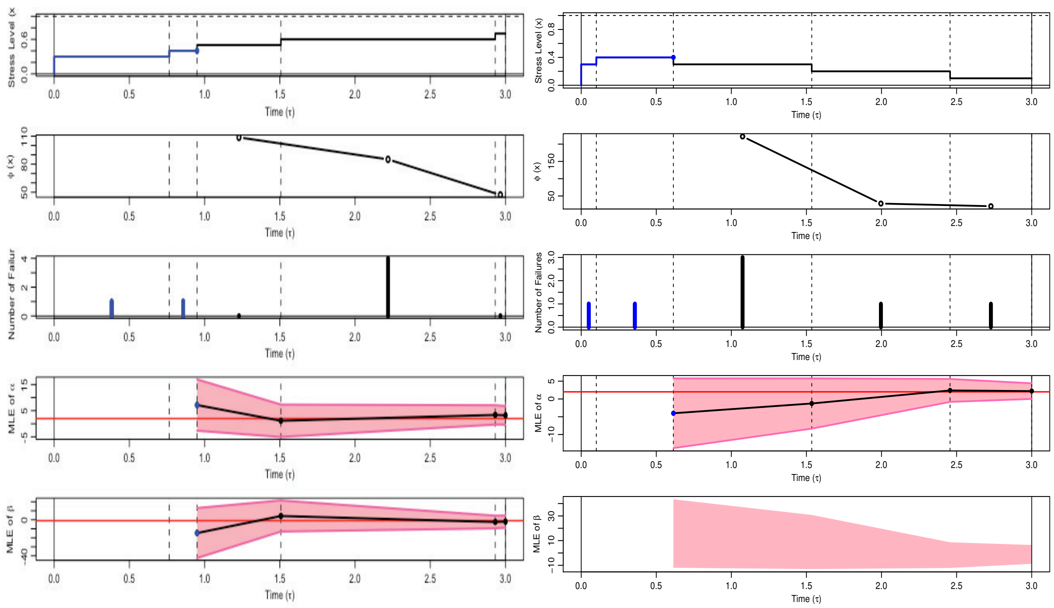
We set the initial parameters of interest to be $\alpha = 2$ and $\beta = -1$, and taking into consideration the negative relationship that exists between log of mean lifetime and the stress level. Different results were obtained out of the simulation. We display the results by setting the seed to '1' and '12345' as shown in Figure 3 (D-optimality) and Figure 4 (C-optimality), respectively. Noting that the seeds are preset to enable reproduction of the results. In each result you can see a panel with 5 different plots. The horizontal axis for all plots displays the experiment time τ. Looking at Figure 3 with D-optimality objective function, a walk through each plot along with interpretation is provided. Top panel shows the stress level trend chosen after the first two levels. The first two of these levels are color coded by blue and the rest with black. After the second level, we can see the decision made based on (6). The decision is to move upward in direction till the end of the experiment. The second panel shows the D-optimality objective function at each stress level chosen for each iteration. This is calculated based on (6). The third panel records the number of failures. In the first two iterations, we have observed one failure at each level. At third iteration, no failures were observed. As a consequence, the algorithm decides to go level up in direction where 4 failures were recorded. In the fourth and fifth panels, we display the MLE estimates of intercept ($\hat{\alpha}$) and slope ($\hat{\beta}$) at each iteration.

As we can see, using (4) and (5), at the third iteration we start by bad estimators with $\hat{\alpha} = 7.1481$ and $\hat{\beta} = -14.733$ with wide confidence intervals. Then as it progresses and goes to the last iteration, the algorithm converges to initial set for α and β and the confidence interval became narrower. Similarly, one can see the results of D-optimality criteria using a different seed '12345', illustrated in Figure 3 (b). In the first panel, we now a see downward trend as opposed to upward when seed was set to 1. Different seed combinations result in a set of decisions such as moving monotonically upward, monotonically downward, or mix of up and down directions. The results based on C-optimality design criteria are presented in Figure 4 (a) and (b). Similar description and interpretation of the results is applicable, similar to that of the D-optimality.



(a) Seed 1 (b) Seed 12345

Figure 3: D-optimality Simulation Results



(a) Seed 1 (b) Seed 12345

Figure 4: C-optimality Simulation Results

4. Conclusion

We developed an adaptive step-stress accelerated life test (ada-SSALT) algorithm in which we adaptively obtain stress levels based on the information from the preceding steps. The decision on direction and value of the next step is derived based on C-optimality and D-optimality designs. The formulation and derivation of the problem is provided in Section 1 and 2. Preliminary results of the simulation show that depending on the first stress level chosen and the predetermined time spent at each step, the direction of the next stress level can be derived. This direction depends on the initial failure time setting.

The next step is to optimize the adaptive lag in time stepping using different quantile ranges. We will also consider different lifetime distributions, censoring schemes with additional optimality designs such as A-, M-, and E-criteria. A test of efficiency of the algorithm will be examined by comparing this algorithm to the optimal simple step stress ALT in the presence of shock effect.

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