Design Optimization for the Step-Stress Accelerated Degradation Tests based on Exponential Dispersion Process

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Abstract
In order to assess the lifetime characteristics of highly reliable products, the step-stress accelerated degradation test (ADT) is a practical and effective solution, especially when there are very few items available for testing. During the past decades, the step-stress ADT has been studied by many researchers based on the assumption that the underlying degradation path follows one of the well-known but restricted stochastic processes such as Wiener, gamma, and inverse Gaussian. In practice, however, the degradation path of a product/device may not follow these specific processes, and the researchers are calling for a more flexible but unified approach toward generalized degradation models. To address this issue, the exponential dispersion process has been proposed, which is a generalized stochastic process including Wiener, gamma, and inverse Gaussian processes as special cases. In this work, we develop the step-stress ADT of products/devices when the underlying degradation path follows a class of the exponential dispersion processes. Based on this framework, the design optimization for the step-stress ADT is formulated under the C-optimality. Under the constraint that the total experimental cost does not exceed a pre-specified budget, the optimal design parameters such as measurement frequency and test termination time are determined via minimizing the approximate variance of the estimated mean time to failure of a product/device under the normal operating condition.

Key Words: accelerated degradation test, design optimization, exponential dispersion process, step-stress loading

1. Introduction

Several decision variables should be determined carefully in the planning stage of ADT in order to conduct ADT efficiently with constrained resources in practice. These design parameters include the sample size, the stress levels, the sample allocation proportions, the measurement intervals and frequencies, the stress change time points, the total test duration, etc. This is an important decision problem for reliability engineers and practitioners as these decision variables affect both the precision of the parameter estimates and the experimental costs; see Han (2015, 2019, 2021a, 2021b). During the past decades, the design optimization of ADT has been investigated extensively based on the assumption that the underlying degradation path follows one of the stochastic processes mentioned above. For designing a constant-stress ADT with non-monotonic degradation paths, Pan et al. (2009), Lim and Yum (2011), Tsai et al. (2014) adopted the Wiener process. Kim et al. (2018) extended the work by including two stress variables while Chen et al. (2016) proposed a nonlinear generalized Wiener process. The design optimization of a step-stress ADT based on the Wiener process was investigated by Liao and Tseng (2006), Hu et al. (2015), Sung and Yum (2016). Ge et al. (2010) examined the stress optimization while Zhao et al. (2019) explored the Bayesian designs under various optimality criteria. For planning a constant-stress ADT with non-decreasing degradation paths, Ling et al. (2015), Zhang et al. (2015), Duan and Wang (2019) considered the gamma process. Jiang et al. (2019) addressed the inferential issues in this regard while Tsai et al. (2012) included the random effects for the parameter heterogeneity and Tsai et al. (2016) studied a two-variable ADT design.

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Tseng et al. (2009) considered planning a step-stress ADT based on the gamma process while Pan and Sun (2014) studied the problem with multiple performance characteristics. Wang and Xu (2010) introduced the inverse Gaussian process as an alternative to model the monotonic degradation path, and Ye et al. (2014) investigated the design optimization of a constant-stress ADT with this stochastic process. Using this framework, Wang et al. (2017) proposed the M-optimal design while Wu et al. (2019) studied a multi-objective design. Wang et al. (2016) developed the optimal step-stress ADT based on the inverse Gaussian process, and Duan and Wang (2018) approached the problem based on a proportional degradation rate model. The Bayesian perspective was also considered by Peng et al. (2014), Li et al. (2017).

Based on its strong model flexibility as a generalization of the exponential family, here we study the design optimization for a general k-level step-stress ADT when the underlying degradation path follows the TED process. The saddle-point approximation is employed to produce a tractable distribution of the TED model. With a given degradation threshold for a soft failure, the cumulative distribution function (CDF) and the probability density function (PDF) of FPT/FHT at the normal usage stress level are also approximated using the Birnbaum-Saunders distribution. Subsequently, the mean time to failure (MTTF) is obtained, and the design optimization for planning a step-stress ADT is then investigated under the constraint that the total experimental cost does not exceed a pre-specified budget. The decision parameters such as the stress change time points and the inspection/measurement frequencies are to be tuned under various criteria. Assuming an evenly spaced inspection/measurement period, a real engineering case study is provided to illustrate the methods developed in this study. The numerical results show that the proposed model performs well, and in general, the power parameter ρ of the TED process plays a significant role in determining the optimum design points.

The rest of this article is organized as follows. The degradation model for a k-level step-stress ADT based on the TED process is formulated in Section 2. The newly proposed parameterization for the stress-response link is discussed, which maintains the additive and reproductive properties of the TED process even under non-constant stress regimes. The saddle-point approximation to the distribution of a degradation increment/decrement is provided as well as the lifetime distributions approximated by the Birnbaum-Saunders distribution. The cost functions for the constrained designs are defined in Section 3, and upon deriving the expected Fisher information, the C-optimal design criterion is defined under the time and cost constraints. Section 4 demonstrates the design efficiency of the proposed methodology using a case study.

2. Model Formulation

2.1 *Exponential Dispersion Process*

Let $s(t)$ denote the specified function of (transformed) stress loading for the ADT under consideration, which is a deterministic function in the natural time scale $t \geq 0$. We define s_U to be the normal use-stress level and s_H to be the upper bound of stress level below which the degradation mechanism is unchanging. Then, the stress loading is standardized as

$$x(t) = \frac{s(t) - s_U}{s_H - s_U}, \qquad t \geq 0$$

such that the range of $x(t)$ is between 0 and 1 inclusive. Now, let $0 \leq x_1 < x_2 < \cdots < x_k \leq 1$ to be the ordered k standardized stress levels to be used for the (step-up) step-stress ADT. Let $0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k$ be the corresponding stress change time points with τ_k

denoting the termination time of test. Also, let $\tau_{i-1} \equiv t_{i,0} < t_{i,1} < \cdots < t_{i,m_i} \equiv \tau_i$ specify the m_i inspection times points at stress level x_i for $i = 1, 2, \ldots, k$. Then, let $Y_i\big(\Upsilon(t)\big)$ for $t \geq 0$ describe the degradation path of a product or test item placed at stress level x_i over time. Here, $\Upsilon(t)$ is a known time-transformation function, which is non-negative and monotonically increasing with $\Upsilon(0) = 0$. It provides additional model flexibility by accelerating or decelerating the natural time scale, and it can be a piece-wise function with each piece corresponding to a unique stress level. As a default, the natural time scale is represented when $\Upsilon(t)$ is an identity function ($viz.$, $\Upsilon(t) = t$). Now, it is understood that $Y_i(0) = 0$ ($i.e.$, no initial degradation), and non-overlapping increments of $Y_i\big(\Upsilon(t)\big)$ are stationary and statistically independent according to the properties of the exponential dispersion process. Also, each increment/decrement $\Delta Y_{ij} = Y_i\big(\Upsilon(t_{i,j})\big) - Y_i\big(\Upsilon(t_{i,j-1})\big)$ for $j = 1, 2, \ldots, m_i$ follows the exponential dispersion distribution with parameters θ and $\lambda_i \Delta t^\star_{ij}$, where $\Delta t^\star_{ij} = \Upsilon(t_{i,j}) - \Upsilon(t_{i,j-1}) > 0$ is the corresponding increment of the transformed time. The density of ΔY_{ij} is then defined as

$$f(y;\ \theta, \lambda_i \Delta t^\star_{ij}) = c(y; \lambda_i \Delta t^\star_{ij}) \exp\big(\theta y - \lambda_i \Delta t^\star_{ij}\ \kappa(\theta)\big) \tag{1}$$

with suitable functions $c(y;\cdot)$ and $\kappa(\cdot)$. The moment-generating function (MGF) of ΔY_{ij} is then obtained as

$$M_{\Delta Y_{ij}}(z) = \exp\Big(\lambda_i \Delta t^\star_{ij}\big[\kappa(z+\theta) - \kappa(\theta)\big]\Big)$$

and its cumulant generating function is

$$K_{\Delta Y_{ij}}(z) = \lambda_i \Delta t^\star_{ij}\big[\kappa(z+\theta) - \kappa(\theta)\big]$$

Based on this, the mean and variance of ΔY_{ij} are obtained as

$$\begin{aligned} E[\Delta Y_{ij}] &= K'_{\Delta Y_{ij}}(0) = \lambda_i \Delta t^\star_{ij}\ \kappa'(\theta) \\ Var[\Delta Y_{ij}] &= K''_{\Delta Y_{ij}}(0) = \lambda_i \Delta t^\star_{ij}\ \kappa''(\theta) > 0 \end{aligned}$$

which implies that with $\lambda_i > 0$, $\kappa''(\theta) > 0$. That is, $\kappa'(\theta)$ is increasing in θ and so is $E[\Delta Y_{ij}]$.

Using the MGF, it can be easily shown that $\Delta Y_i = Y_i\big(\Upsilon(\tau_i)\big) - Y_i\big(\Upsilon(\tau_{i-1})\big) = \sum_{j=1}^{m_i} \Delta Y_{ij}$ also follows the exponential dispersion distribution with parameters θ and $\lambda_i \Delta\tau^\star_i$ where $\Delta\tau^\star_i = \Upsilon(\tau_i) - \Upsilon(\tau_{i-1}) = \sum_{j=1}^{m_i} \Delta t^\star_{ij}$ is the transformed duration of stress level x_i for $i = 1, 2, \ldots, k$. Since a step-stress ADT implements a non-constant loading of stress, it requires an additional model to incorporate the effect of changing stresses over the course of a test. Based on the additive accumulative damage model, the total degradation observed at time $t_{i,j}$ can be expressed as $Y\big(\Upsilon(t_{i,j})\big) = \sum\limits_{i'=1}^{i-1} \sum\limits_{j'=1}^{m_{i'}} \Delta Y_{i'j'} + \sum\limits_{j'=1}^{j} \Delta Y_{ij'} = \sum\limits_{i'=1}^{i-1} \Delta Y_{i'} + Y_i\big(\Upsilon(t_{i,j})\big) - Y_i\big(\Upsilon(\tau_{i-1})\big)$, and it follows the exponential dispersion distribution with parameters θ and $\sum_{i'=1}^{i-1} \lambda_{i'} \Delta\tau^\star_{i'} + \lambda_i\big(\Upsilon(t_{i,j}) - \Upsilon(\tau_{i-1})\big)$. Ultimately, $Y\big(\Upsilon(\tau_k)\big) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \Delta Y_{ij} = \sum_{i=1}^{k} \Delta Y_i$ follows the exponential dispersion distribution with parameters θ and $\sum_{i=1}^{k} \lambda_i \Delta\tau^\star_i$ as the exponential dispersion model is additive and reproductive with shared θ. Although the exponential dispersion model is closed under convolution, it not closed under scale transformation in general. To resolve this, an important class of the exponential dispersion model, known as Tweedie exponential dispersion (TED), was proposed by recognizing that the exponential dispersion model can be characterized by its variance function; see Jørgensen (1987, 1992). For the TED model, the unit variance function is defined as a power function of the mean. That is $\kappa''(\theta) = [\kappa'(\theta)]^\rho$ where $\rho \in (-\infty, 0] \cup [1, \infty)$ is a shape parameter. Then, with θ_L as the hypothetical lower bound of θ, the function $\kappa(\theta)$ can be derived depending on the value of ρ.

2.2 *Saddle-point Approximation*

Still the analytic form of (1) is difficult to obtain without the specification of ρ for the TED model. In turn, the likelihood function cannot be written in an explicit form, and the MLE of the model parameters cannot be computed. In order to tackle this problem, the saddle-point approximation method based on the MGF is suggested, which approximates the PDF of any distribution with a high degree of accuracy; see Lugannani and Rice (1980). According to Daniels (1954), the saddle-point approximation to $f(y;\ \rho, \theta, \lambda_i \Delta t^\star_{ij})$ can be obtained as

$$f(y;\ \rho, \theta, \lambda_i \Delta t^\star_{ij}) \approx \frac{1}{\sqrt{2\pi K''_{\Delta Y_{ij}}\left(z^*_{ij}\right)}} \exp\left(K_{\Delta Y_{ij}}\left(z^*_{ij}\right) - z^*_{ij} y\right)$$

where z^*_{ij} is the solution to $K'_{\Delta Y_{ij}}\left(z^*_{ij}\right) = y$. Since $K'_{\Delta Y_{ij}}(z) = \lambda_i \Delta t^\star_{ij}\ \kappa'(z+\theta)$ and $K''_{\Delta Y_{ij}}(z) = \lambda_i \Delta t^\star_{ij}\ \kappa''(z+\theta)$, one can derive z^*_{ij}, $K_{\Delta Y_{ij}}\left(z^*_{ij}\right)$ and $K''_{\Delta Y_{ij}}\left(z^*_{ij}\right)$ depending on the value of ρ. Therefore, a closed-form expression of (1) is deduced as

$$f(y;\ \rho, \mu_0, \lambda_i \Delta t^\star_{ij}) = \frac{1}{\sqrt{2\pi (\lambda_i \Delta t^\star_{ij})^{1-\rho}}} y^{-\rho/2} \exp\Big(- \lambda_i \Delta t^\star_{ij}\ d_{ij}(y)\Big) \tag{2}$$

where the unit deviance function is

$$d_{ij}(y) = \begin{cases} \dfrac{y}{\lambda_i \Delta t^\star_{ij}} \log\left(\dfrac{1}{\mu_0}\dfrac{y}{\lambda_i \Delta t^\star_{ij}}\right) - \dfrac{y}{\lambda_i \Delta t^\star_{ij}} + \mu_0 & \rho = 1\ ; \\ -\log\left(\dfrac{1}{\mu_0}\dfrac{y}{\lambda_i \Delta t^\star_{ij}}\right) + \dfrac{1}{\mu_0}\dfrac{y}{\lambda_i \Delta t^\star_{ij}} - 1 & \rho = 2\ ; \\ \dfrac{1}{(1-\rho)(2-\rho)}\left(\dfrac{y}{\lambda_i \Delta t^\star_{ij}}\right)^{2-\rho} - \dfrac{\mu_0^{1-\rho}}{1-\rho}\dfrac{y}{\lambda_i \Delta t^\star_{ij}} + \dfrac{\mu_0^{2-\rho}}{2-\rho} & \rho \in (-\infty, 0] \cup (1,2) \cup (2,\infty) \end{cases}$$

with $\mu_0 = \kappa'(\theta)$. It can be shown that $d_{ij}(y)$ is continuous on $\rho \in (-\infty, 0] \cup [1, \infty)$ by using the fact that $\lim\limits_{h\to 0} \dfrac{y^h - 1}{h} = \log y$ for $y > 0$.

It should be also noted that although (2) is an approximated distribution, it still yields the exact distributions for the well-known special cases of $\rho = 0, 2, 3$, resulting in the exact stochastic processes; see Jørgensen (1992). When $\rho = 1$, it also produces the Poisson process upon utilizing Stirling's approximation. For other values of ρ, Zhou and Xu (2019), Chen et al. (2020) illustrated numerically that the approximated distribution is pretty much identical to the one based on Dunn and Smyth (2005, 2008), implemented in the statistical programming language R. Thus, the TED model can describe more complex and diverse degradation processes with higher flexibility and wider applicability compared to the conventional stochastic models that have been studied in the literature.

2.3 *Stress-Response Link*

To speed up the degradation of test items, more severe operating environments or conditions can be imposed by non-normal levels of stress factors such as temperature, pressure, humidity, vibration, loading, chemical agents, voltage, cycles, etc. This change can affect the rate and/or volatility of the degradation evolution. For the reduction of the parameter dimension, it is considered that at any stress level x_i, λ_i has a log-linear stress-response relationship, specified as

$$\log \lambda_i = \beta_0 + \beta_1 x_i \tag{3}$$

where the regression parameters (β_0, β_1) need to be calibrated. This log-linear link is a very popular and well-supported model. Besides its simplicity, a number of physics-based life-stress relationships all validate this log-linear link, which includes (modified) Arrhenius, temperature-humidity, (inverse) power law, Eyring, and temperature-non-thermal relationships; see Li et al. (2020). Using this stress-response link, the degradation paths can be extrapolated from the accelerated conditions so that the lifetime information at the normal operating condition can be deduced analytically.

2.4 *Lifetime Distribution*

Let $Y_0\big(\Upsilon(t)\big)$ describe the degradation path of a product under the normal operating condition ($viz.$, stress level x_0) for $t \geq 0$. When $Y_0\big(\Upsilon(t)\big)$ crosses a pre-specified critical threshold y_D, the (soft) failure occurs, and the product's lifetime T_0 is defined to be the time of this event. It is known as the first passage time (FPT) or the first hitting time (FHT) of a stochastic process, defined as

$$T_0 = \inf\left\{t > 0 \,\middle|\, Y_0\big(\Upsilon(t)\big) \geq y_D\right\}$$

Since the TED process includes numerous sub-models whose degradation evolution may not be monotonic, it is difficult to derive the distribution of T_0 in a unified fashion. For instance, the TED model with $\rho = 0$ ($viz.$, Wiener process) features a non-monotonic degradation dynamics whereas the cases of $\rho = 2$ ($viz.$, gamma process) and $\rho = 3$ ($viz.$, inverse Gaussian process) exhibit a monotone degradation path, resulting in different distributions for the corresponding FPT/FHT. It is well known that the Υ-transformed FPT/FHT $\big(viz.$, $\Upsilon(T_0)\big)$ of the Wiener process ($viz.$, $\rho = 0$) has an inverse Gaussian distribution. As $Y_0\big(\Upsilon(t)\big)$ is practically non-decreasing for other values of ρ, Hong and Ye (2017) suggested to approximate the probability distribution of $\Upsilon(T_0)$ by the Birnbaum and Saunders fatigue life distribution extensively used in reliability analyses. Accordingly, the CDF and PDF of T_0 are given as

$$\begin{aligned} F_{T_0}(t;\, y_D) &= P\Big(Y_0\big(\Upsilon(t)\big) \geq y_D\Big) \approx 1 - \Phi\left(\frac{y_D - \mu_0\lambda_0\Upsilon(t)}{\sqrt{\mu_0^{\rho}\lambda_0\Upsilon(t)}}\right) \\ &= \Phi\left(\frac{1}{\sqrt{\mu_0^{\rho}\lambda_0}}\left(\mu_0\lambda_0\sqrt{\Upsilon(t)} - \frac{y_D}{\sqrt{\Upsilon(t)}}\right)\right), \end{aligned} \tag{4}$$

and

$$f_{T_0}(t;\, y_D) \approx \frac{\mu_0\lambda_0\Upsilon(t) + y_D}{2\Upsilon(t)\sqrt{\mu_0^{\rho}\lambda_0\Upsilon(t)}}\Upsilon'(t)\phi\left(\frac{1}{\sqrt{\mu_0^{\rho}\lambda_0}}\left(\mu_0\lambda_0\sqrt{\Upsilon(t)} - \frac{y_D}{\sqrt{\Upsilon(t)}}\right)\right), \tag{5}$$

respectively, where $\Phi(\cdot)$ and $\phi(\cdot)$ are the standard normal CDF and PDF while $\lambda_0 = \exp(\beta_0)$ owing to the log-linear relationship, and $\Upsilon'(t) = \frac{d}{dt}\Upsilon(t)$. For $\Upsilon(T_0)$, the shape parameter of its distribution is $1\Big/\sqrt{y_D\mu_0^{1-\rho}}$ whereas $y_D/(\lambda_0\mu_0)$ is the scale parameter.

The median lifetime of T_0 is then $\Upsilon^{-1}\left(\frac{y_D}{\lambda_0\mu_0}\right)$, where $\Upsilon^{-1}(\cdot)$ is the unique inverse of $\Upsilon(\cdot)$. Since the first two moments of $\Upsilon(T_0)$ are approximated to be

$$\mu^{\star}_{T_0} := E\big[\Upsilon(T_0)\big] = \frac{1 + 2y_D\mu_0^{1-\rho}}{2\lambda_0\mu_0^{2-\rho}}$$

and

$$\sigma^{\star 2}_{T_0} := Var\big[\Upsilon(T_0)\big] = \frac{5 + 4y_D\mu_0^{1-\rho}}{4\big(\lambda_0\mu_0^{2-\rho}\big)^2},$$

the mean and variance of T_0 can be approximated to be

$$E[T_0] \approx \Upsilon^{-1}\big(\mu^{\star}_{T_0}\big) + \frac{\sigma^{\star 2}_{T_0}}{2}\Upsilon^{-1}_{(2)}\big(\mu^{\star}_{T_0}\big) \tag{6}$$

and

$$Var[T_0] \approx \sigma^{\star 2}_{T_0}\Big[\Upsilon^{-1}_{(1)}\big(\mu^{\star}_{T_0}\big)\Big]^2$$

using the delta method, where $\Upsilon^{-1}_{(r)}(t) = \frac{d^r}{dt^r}\Upsilon^{-1}(t)$ for $r = 0, 1, 2, \ldots$.

3. Cost-constrained ADT Design

In order to conduct an ADT experiment efficiently with constrained resources in practice, several decision variables such as the sample size, allocation proportions, stress levels and durations, inspection/measurement periods and frequencies should be determined carefully at the design stage. It is because these decision variables affect the experimental cost as well as the precision of the parameter estimates of interest. There is a body of literature addressing the model optimization related to certain cost functions. Under the constraint that the total experimental cost does not exceed a pre-specified budget, a typical decision problem of interest can be formulated as to optimize (minimize or maximize) an objective function of choice subject to $C_T \le C_B$, where C_B is the pre-specified budget and C_T is the total cost for running an ADT.

3.1 *Cost Function*

Assuming no complete loss of functionality of a test item ($i.e.$, hard failure) due to shocks or any other causes during the test, the total cost of a step-stress ADT with the sample size n can be expressed in general as

$$C_T = C_{set} + nC_{unit} + \sum_{i=1}^{k} C_{op}(x_i)\Delta\tau_i + nm_{\bullet}C_{ins} \tag{7}$$

where $\Delta\tau_i = \tau_i - \tau_{i-1} = \sum_{j=1}^{m_i}\Delta t_{ij}$ is the duration of stress level x_i for $i = 1, 2, \ldots, k$ while $\Delta t_{ij} = t_{i,j} - t_{i,j-1}$ is the inspection/measurement period for $j = 1, 2, \ldots, m_i$. Among the non-negative cost parameters in (7), C_{set} denotes the fixed cost for setting up an ADT experiment, which includes the costs of facility and testing chambers. C_{unit} is the cost of each test unit, including the costs of manufacturing, purchasing and/or installation plus post-test scrapping, waste management, refurbishing or disintegration. $C_{op}(x)$ is the operation cost of conducting an ADT per unit time under the given setup which depends on the applied stress level x. For the sake of simplicity, it is assumed that the operation costs at the (instantaneous) stress change times are negligible. Moreover, although both C_{set} and $C_{op}(x)$ may increase with the scale of ADT ($e.g.$, A larger n requires a larger facility to accommodate), here we assume that the changes in C_{set} and $C_{op}(x)$ are negligible in a neighborhood of n under the optimal condition, keeping these costs constant and uniform. This is a reasonable assumption as the fixed costs accommodate a range of the sample sizes by absorbing the scaling/sizing effects until it is necessary to require additional resources ($i.e.$, step-wise cost increments). Lastly, C_{ins} is the cost of each inspection and measurement per test unit.

Based on (7) with the cost parameters, the design space appears to be composed of both continuous and discrete decision variables, which are $\Delta\tau_i$ and m_i for $i = 1, 2, \dots, k$. A closer analysis reveals that the complete design space also includes all the inspection/ measurement periods Δt_{ij}'s in addition to the stress durations $\Delta\tau_i$ and the inspection/ measurement frequencies m_i. In order to reduce the dimensions of the decision variables as well as to restrict the design space to a discrete manifold so that the optimal designs can be located using a grid search algorithm, an evenly spaced inspection/measurement period Δt can be assumed across the stress levels. That is, $\Delta t = \Delta t_{ij}$ for all $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, m_i$ such that $\Delta\tau_i = m_i \Delta t$, or $\Delta t = \tau_k / m_\bullet$ where $m_\bullet = \sum_{i=1}^{k} m_i > 0$ is the total number of inspection/measurement. Under this setup, (7) can be rewritten as

$$C_T = C_{set} + nC_{unit} + \tau_k \sum_{i=1}^{k} \pi_i C_{op}(x_i) + n m_\bullet C_{ins} \tag{8}$$

where $\pi_i = m_i / m_\bullet$ is precisely the proportion of the total test duration τ_k assigned to the stress level x_i such that $\Delta\tau_i = \pi_i \tau_k$. Given the termination time of test τ_k along with the sample size n and the budget C_B, the design of a step-stress ADT can be completely specified by the inspection/measurement frequencies $(m_1, m_2, \dots, m_k)$, which is the k-tuples of non-negative integers as $m_i = 0, 1, 2, \dots$. With an alternative parameterization, the design can be also specified by $(\pi_1, \pi_2, \dots, \pi_{k-1}, m_\bullet)$ under the conditions $0 \le \pi_i \le 1$, $\sum_{i=1}^{k} \pi_i = 1$ and $m_\bullet \in \mathbf{N}$.

3.2 *C-optimal Design*

When planning ADT, it is often the aim of the experiment to estimate the parameters of interest with maximum precision and minimum dispersion possible. For a step-stress ADT, such a parameter of interest is $E[T_0]$, the mean lifetime of a test unit at the normal usage stress level x_0 as provided in (6). For convenience, assuming the identity function $\Upsilon(t) = t$ for the time-transformation, $E[T_0] = \mu^\star_{T_0} = \dfrac{1 + 2y_D \mu_0^{1-\rho}}{2\lambda_0 \mu_0^{2-\rho}}$ with $\lambda_0 = e^{\beta_0}$ since $x_0 \equiv 0$. Using the delta method with the inverse of the expected Fisher information matrix $\mathbf{I}_E^{-1}(\boldsymbol{\vartheta})$, the objective function of interest in this case is defined as

$$\begin{aligned} \varphi_C(\cdot) &= n \,\mathrm{AVar}\big(\hat{\mu}^\star_{T_0}\big) = n \,\nabla\mu^{\star\top}_{T_0}\, \mathbf{I}_E^{-1}(\boldsymbol{\vartheta})\, \nabla\mu^\star_{T_0} \\ &= \frac{n^2 \mu_0^{2-\rho}}{|\mathbf{I}_E(\beta_0, \beta_1)|} \sum_{i=1}^{k} \left[x_i \mu^\star_{T_0} + (1-\rho) \frac{(x_i - \bar{x}_m)}{\log \mu_0} \left(\frac{\partial \mu^\star_{T_0}}{\partial \rho} \right) \right]^2 \lambda_i \Delta\tau_i \\ &\qquad - \frac{4(1-\rho)}{m_\bullet \log \mu_0} \left(\frac{\partial \mu^\star_{T_0}}{\partial \rho} \right)^2 \end{aligned} \tag{9}$$

where AVar stands for the asymptotic or approximate variance, and $\hat{\mu}^\star_{T_0} = \dfrac{1 + 2y_D \hat{\mu}_0^{1-\hat{\rho}}}{2\hat{\lambda}_0 \hat{\mu}_0^{2-\hat{\rho}}}$ is the MLE of $\mu^\star_{T_0}$ according to the invariance property of the MLE. The gradient of $\mu^\star_{T_0}$ as a function of the model parameters is given as

$$\begin{aligned}\nabla\mu^\star_{T_0} &= \left(\frac{\partial\mu^\star_{T_0}}{\partial\rho}, \frac{\partial\mu^\star_{T_0}}{\partial\mu_0}, \frac{\partial\mu^\star_{T_0}}{\partial\beta_0}, \frac{\partial\mu^\star_{T_0}}{\partial\beta_1}\right)^\top \\ &= \left(\mu^\star_{T_0}\log\mu_0 - \frac{y_D}{\mu_0\lambda_0}\log\mu_0, -\frac{2-\rho}{\mu_0}\mu^\star_{T_0} + \frac{(1-\rho)y_D}{\mu_0^2\lambda_0},\ -\mu^\star_{T_0},\ 0\right)^\top\end{aligned}$$

The C-optimal design minimizes (9) for the maximal precision of $\hat{\mu}^\star_{T_0}$.

4. Illustrative Case Study

Here, the proposed ADT planning method is illustrated with the carbon-film-resistor data from Meeker and Escobar (1998). The resistance value of the carbon-film resistors increases over time, and changes in resistance will cause the reduction of the performance of the product or even system failures. The product lifetime is defined as the time when the percent increase in resistance (QC) hits a critical threshold $y_D = 5$ under the normal operating temperature $50°C$. With a specified budget C_B = \$1500 (ignoring the setup cost), it is desired to construct the optimal simple step-stress ADT ($viz.$, $k = 2$) with the sample size of $n = 9$. The Arrhenius model is assumed for the stress-response relationship with the test temperatures at $83°C$ and $133°C$. After standardization, these are $x_1 = 0.45$ and $x_2 = 1.00$. After fitting the carbon-film resistors ADT data from a pilot study, the parameters are set to be $(\rho, \mu_0, \beta_0, \beta_1) = (2.00, 0.06, -8.29, 2.54)$ along with the cost parameters C_{unit} = \$53/unit, $C_{op}(x_1) = C_{op}(x_2)$ = \$0.48/hour, and C_{ins} = \$1.30/measurement. Under the C-optimality, the optimal step-stress ADT design was determined with the linear time transformation $\Upsilon(t) = t$. The optimal design required the inspection/measurement period at $\Delta t^\star = 32$ with the measurement frequencies of $m_1^\star = 16$ and $m_2^\star = 22$ for each stress level, resulting in the total test duration of $\tau_2^\star = 1216$ hours. As the parameter estimates could depart from the true parameter values in practice, the sensitivity of the optimal designs on the estimated parameter values was also investigated via Monte Carlo simulations. The optimal designs were found quite stable and robust for a moderate departure from the assumed parameter values.

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