Projection of Argentina's Macroeconomic Aggregates through a Constrained Adaptive Expectations Model

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Abstract
The paper proposes an adaptive expectations model to jointly project the main macroeconomic aggregates of Argentina. The model is essentially an autoregressive vector with exogenous variables (VARX) with the novelty that its parameters are not estimated by OLS but by linear programming in order to incorporate equality and inequality constraints and prior information as well. These constraints are intended to bound the estimated parameters within the parametric space suggested by the economic theory, while previous estimates are introduced to minimize the discrepancies between projections made with homologous series from different base years. The study concludes that projections obtained from the adaptive expectations model fit well to macro aggregates of Argentina's National Accounts System, although they do not always outperform those made with specific ARIMA models for each aggregate.

Key Words: adaptive expectations, autoregressive vector, linear programming, GDP, macroeconomic aggregates, National Accounts.

1. Introduction

The projection of macroeconomic aggregates of the System of National Accounts (SNA) such as gross domestic product (GDP), consumption, investment, exports and imports, is complex and is carried out in general through specific econometric models for each aggregate.[1] However, when projecting any of these aggregates separately, that is when projecting each component independently of the others, it is not possible to guarantee that the resulting projections are consistent one with each other, even though each of the models used is "correct" in the sense that it satisfies selection criteria widespread in the econometric literature. In fact, by modeling each component of SNA separately, the analyst decides (consciously or unconsciously) to exclude from the modeling process any information other than that provided by the series itself, plus a reduced number of exogenous variables that presumably influence the component, which implies a significant loss of efficiency in the use of information.

The objective of this paper is to develop a general model to project the main macroeconomic aggregates of the SNA of Argentina as a whole and to compare these projections with those that arise from specific ARIMA models in order to detect possible estimation biases. It is not the objective of the paper to develop an alternative econometric model to those already known (see

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[1] The author, for example, has proposed alternative SARIMA models [5, 6] to project Argentine exports.

e.g. as [13], [14] or [8] just to name three) for Argentina, nor validate models developed for other countries but with local data. Our objective is to develop a model that is simple but consistent with economic theory, whose parameters also have a concrete economic meaning even outside of the general model. The model to be developed is based on adaptive expectations.

2. The Theoretical Model

We depart from the fundamental macroeconomic identity interpreted in the context of a simplified four-sector Keynesian model (see e.g. [3, cap. 14] or [16]) in which the aggregate demand is the sum of consumption (C), public spending (G), investment (I) and net exports ($X - M$), that is,

$$D = C + G + I + (X - M). \tag{1}$$

Since this is an open economy $X, M \geq 0$. In the model (i) consumption is a function of the *available* income $Y_d = Y - T$ (income minus taxes); (ii) public spending (alike taxes) is determined exogenously; and (iii), exports and imports are also exogenous variables but are determined through specific price functions.[2] We also assume (iv) that investment depends on available income (as is usually assumed in macro models) and on the real interest rate, which is exogenous to the model; and that (v) aggregate demand equals income ($Y = D$).[3] Formally,

$$\begin{cases} C = f(Y_d), \\ G \quad \text{is exogenous,} \\ I = f(R, Y_d), \\ X = f(P_X), \text{ and } M = f(P_M, Y) \\ Y = D, \end{cases} \tag{2}$$

Under this specification, exports are a function of real export prices, as in a typical supply function, while imports are a function of real import prices and income, as in a typical demand function. The function $f(.)$ that relates endogenous and exogenous variables is a linear function. For instance,

$$C = f(Y_d) = \mu + \alpha Y_d, \tag{3}$$

where μ is a constant and α is the so called *marginal propensity to consume*. An equivalent form of this function is

$$\frac{C}{C_0} = \left(\frac{\mu}{C_0}\right) + \left(\alpha \frac{Y_0^d}{C_0}\right) \frac{Y^d}{Y_0^d} \tag{4}$$

where consumption and available income are expressed as indexes scaled to unity in a certain period, say at $t = 0$. This representation has two advantages. First, that the new marginal propensity

[2]Logically, if there is fiscal equilibrium, $T = G$, that is, taxes equal public spending.

[3]We exclude employment as an explanatory variable assuming that in the recent economic context of Argentina the availability of workers is not limiting.

to consume is the elasticity of C with respect to Y_d at the base period. Second, the new expression enables us to equate Y_d with Y as long as the tax rates remain constant (that is, as long as $T = \theta Y$) since the tax rate θ cancels out. For these two reasons, from now on we will express all equations in terms of indexes scaled to unity in the base year. This representation has, however, one disadvantage: the new parameters (elasticities) are not bounded (except for the sign) even if the original parameter is.[4] For example, in the consumption function it would be expected that the original marginal propensity to consume lies within the interval $(0; 1]$. Nevertheless, the bounds of the elasticity of consumption with respect to income are not obvious even though the bounds of α appear quite clear. In order not to confuse the original variables with their respective indexes, we will replace uppercase with lowercase in all equations. Then, the complete system is

$$\begin{cases} c_t = \lambda_0^C + \lambda_Y^C \, y_t \\ i_t = \lambda_0^I + \lambda_R^I \, r_t + \lambda_Y^I \, y_t \\ x_t = \lambda_0^X + \lambda_{P_X}^X \, p_t^X \\ m_t = \lambda_0^M + \lambda_{P_M}^M \, p_t^M + \lambda_Y^M \, y_t \end{cases} \tag{5}$$

where c, i, x and m on the left hand side are consumption, investment, export and import indexes, and y, r and p on the right hand side are indexes of production, interest rate and prices. Note that the interest rates are introduced in the model as an index instead of simple rates. All indexes are based on the National Accounts base year (currently 2004) so the parameters λ are elasticities at the same year. This set of equations may be written in matrix form as

$$\begin{bmatrix} c_t \\ i_t \\ x_t \\ m_t \\ y_t \end{bmatrix} = \begin{bmatrix} \lambda_0^C & 0 & 0 & \dots & \dots & \lambda_Y^C \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ \lambda_0^M & 0 & 0 & \dots & \dots & \lambda_Y^M \\ \lambda_0^{Y*} & \lambda_G^Y & \lambda_{P_X}^Y & \dots & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ g_t \\ p_t^X \\ p_t^M \\ r_t \\ y_t \end{bmatrix}, \tag{6}$$

or $\mathbf{z}_t = \mathbf{\Lambda} \mathbf{x}_t$. For $t = 1, \dots, T$ periods, $\mathbf{z}_t$ and $\mathbf{x}_t$ may be written in single system as $\mathbf{Z} = \mathbf{\Lambda} \mathbf{X}$. This system, however, would be incomplete without a set of constraints that sets to zero the null parameters. Moreover, another set of constraints ought to be introduced to guarantee that the non-null parameters have the sign expected according to the economic theory. Then, the complete economic model can be written

$$\text{vec}(\mathbf{Z}) = \left(\mathbf{X}' \otimes \mathbf{I}_K\right) \text{vec}(\mathbf{\Lambda}) \quad \text{subject to} \quad \mathbf{R}^{\text{I}} \text{vec}(\mathbf{\Lambda}) = \mathbf{r}^{\text{I}}, \quad \mathbf{R}^{\text{II}} \text{vec}(\mathbf{\Lambda}) \geq \mathbf{0} \tag{7}$$

where $\otimes$ means Kronecker product and vec(.) is an operator that vectorizes the matrix in the argument. Note that in model (6) the transformation of exogenous into endogenous variables is instantaneous, following the principle of market emptyness. This simultaneity is useful for reasoning in a context of macroeconomic equilibrium, although this equilibrium is not always reached in

[4] We speak of parameters in an econometric rather than an economic sense.

practice. To support the simultaneity hypothesis, we will assume that the demand components are a function of the *expected* or *desired* level of the exogenous variables and income. That is, we will replace the variables on the right hand side of the model (6) by their expectations, also assuming that these expectations are adaptive.

3. The Econometric Model

The econometric version of (6) arises immediately by transcribing the mathematical relationship into a statistical relationship in which the K macro aggregates become random variables governed by a set of exogenous variables and a first-order autoregressive process. Formally, the econometric model is

$$\mathbf{z}_t = \boldsymbol{\mu} + \mathbf{A}_1\, \mathbf{z}_{t-1} + \mathbf{B}_0\, \mathbf{x}_t + \mathbf{C}\, \mathbf{d}_t + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim N\left(\mathbf{0}_{K\times 1}, \sigma^2 \mathbf{I}_K\right), \tag{8}$$

where $\mathbf{z}_t$ is a vector of $K \times 1$ macro aggregate indixes, $\boldsymbol{\mu}$ is a vector of $K \times 1$ constants, $\mathbf{A}_1$ is an array of $K \times K$ parameters associated with the vector of lagged variables $\mathbf{z}_{t-1}$, $\mathbf{B}_0$ is a matrix of $K \times M$ parameters associated to M exogenous variables contained in $\mathbf{x}_t$, and $\mathbf{C}$ is a matrix of $K \times N$ parameters associated with atypical events unrelated to the theoretical model. Then, model (7) becomes (8) by replacing $\mathbf{x}_t$ on the right side of (6) by its expectation $\mathbf{x}_t^*$ and this latter vector by a function of its past values. If model (8) is true, matrices $\mathbf{A}_1$ and $[\boldsymbol{\mu}, \mathbf{B}_0]$ are equal to $(1-\pi)\,\mathbf{I}_K$ and $\pi\,\boldsymbol{\Lambda}_1$, respectively. This particular specification of $\mathbf{A}_1$ determines that the model parameters cannot be estimated directly by ordinary least squares (OLS) as in any traditional VARX model since in (8) the parameters are subject to inequality constraints that are infeasible with any closed analytic solution as the OLS estimator. Instead, we propose a less conventional although feasible two-stage estimator. In the first stage, we estimate π and a preliminary estimate of $[\boldsymbol{\mu}, \mathbf{B}_0]$ minimizing the sum of absolute deviations of errors (LAD estimate). In the second stage, we add a set of linear constraints to fix the diagonal elements of $\mathbf{A}_1$ to $(1 - \hat{\pi})$ and addsome more restrictions to ensure that the elasticities of each equation add to unity. For the first estimation we propose the model

$$\left[\begin{array}{ccc} \mathbf{z}_2 & \ldots & \mathbf{z}_T \end{array}\right] = \left[\begin{array}{cccc} \boldsymbol{\mu} & \mathbf{A}_1 & \mathbf{B}_0 & \mathbf{C} \end{array}\right] \left[\begin{array}{ccc} 1 & \ldots & 1 \\ \mathbf{z}_1 & \ldots & \mathbf{z}_{T-1} \\ \mathbf{x}_2 & \ldots & \mathbf{x}_T \\ \mathbf{d}_2 & \ldots & \mathbf{d}_T \end{array}\right] + \left[\begin{array}{ccc} \boldsymbol{\epsilon}_2 & \ldots & \boldsymbol{\epsilon}_T \end{array}\right]. \tag{9}$$

This model can be written more compactly as $\mathbf{Y} = \mathbf{B}^*\mathbf{X}^* + \mathbf{E}$, where $\mathbf{Y}$ is an matrix of macro aggregate series of dimension $K \times (T-1)$, $\mathbf{B}^*$ is a matrix of parameters of dimension $K \times (K + M + N + 1)$, and $\mathbf{X}^*$ is a matrix of regressors of dimension $(K + M + N + 1) \times (T-1)$. Note that vector $\mathbf{x}_1$ of exogenous variables in (9) was lost when we introduced the first order autoregressive term. Then, the vectorized form of (9) is

$$\text{vec}(\mathbf{Y}) = \left(\mathbf{X}^{*\prime} \otimes \mathbf{I}_K\right) \text{vec}(\mathbf{B}^*) + \text{vec}(\mathbf{E}) \quad \text{subject to} \quad \mathbf{R}\, \text{vec}(\mathbf{B}^*) \geq \mathbf{0}. \tag{10}$$

To estimate $\text{vec}(\mathbf{B}^*)$ we use a two-stage procedure. In the first stage we solve the following linear program to estimate π, while in the second stage we estimate the elasticities keeping the diagonal elements of $\mathbf{A}_1$ equal to $1-\hat{\pi}$.

$$\min_{\text{vec}(\mathbf{E})} \left\{ \mathbf{0}'_{K(K+M+N+1)}\text{vec}(\mathbf{B}^*) + \mathbf{1}'_{K(T-1)}\text{vec}(\mathbf{E})^- + \mathbf{1}'_{K(T-1)}\text{vec}(\mathbf{E})^+ \right\}$$

subject to

$$\begin{bmatrix} (\mathbf{X}^{*\prime} \otimes \mathbf{I}_K) & -\mathbf{I}_{K(T-1)} & \mathbf{I}_{K(T-1)} \\ \mathbf{R}^{\text{I}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \text{vec}(\mathbf{B}^*) \\ \text{vec}(\mathbf{E})^- \\ \text{vec}(\mathbf{E})^+ \end{bmatrix} \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} \text{vec}(\mathbf{Y}) \\ \mathbf{0} \end{bmatrix},$$

and

$$\begin{bmatrix} \text{vec}(\mathbf{E})^- \\ \text{vec}(\mathbf{E})^+ \end{bmatrix} \geq \mathbf{0}, \qquad (11)$$

where $\mathbf{R}^{\text{I}}$ is a matrix of $K(K+M+N+1)$ rows by $K(T-1)$ columns. The objetive function of this linear program is the sum of errors in absolute values. The supra-index of $\text{vec}(\mathbf{E})$ refers to the sign of the slack variables associated to the observations of $\text{vec}(\mathbf{Y})$. For further details about this optimization criterium we refer the reader to the text of Williams [18, pp. 32-34]. The expansion of the system $\mathbf{R}^{\text{I}}\text{vec}(\mathbf{B}^*) \geq \mathbf{0}$ is the following

$$\begin{bmatrix} \mathbf{R}^{\text{I}}_{\mu} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{\text{I}}_{\text{A}_1} & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{R}^{\text{I}}_{\text{B}_0} & \mathbf{0} \end{bmatrix} \text{vec}(\mathbf{B}^*) \begin{matrix} \geq \\ = \\ = \end{matrix} \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \qquad (12)$$

where $\mathbf{R}^{\text{I}}_{\mu} = \mathbf{I}_K$, to ensure the positivity of the values of $\boldsymbol{\mu}$, and the block $\mathbf{R}^{\text{I}}_{\text{A}_1}$ is a matrix of $K^2 \times K^2$ whose diagonal is $\text{vec}(\mathbf{1}_K\mathbf{1}'_K - \mathbf{I}_K)$. The system $\mathbf{R}^{\text{I}}_{\text{A}_1} = \mathbf{0}$ restricts to 0 the off diagonal elements of $\mathbf{A}_1$, but does not impose any restrictions on the elements of the diagonal. The block $\mathbf{R}^{\text{I}}_{\text{B}_0}$ is a diagonal matrix of $KM \times KM$ whose $K(j-1)+i$ element (in the diagonal) is equal to 1 if $\lambda_{ij} = 0$, or 0 otherwise. Logically, by λ_{ij} we refer to the elasticities of matrix $\boldsymbol{\Lambda}$ excluding the first column of constants, which we called before $\boldsymbol{\Lambda}_1$.

In the second stage, we extend the constraint system to equalize the elements of the diagonal of $\mathbf{A}_1$ to $1-\hat{\pi}$, and the sum of elasticities of each aggregate, plus the constant, to unity. The extended system is as follows

$$\begin{bmatrix} \mathbf{R}^{\text{I}}_{\mu} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{\text{I}}_{\text{A}_1} & \ddots & \vdots \\ \vdots & \ddots & \mathbf{R}^{\text{I}}_{\text{B}_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{\text{II}}_{\text{A}_1} & \mathbf{0} & \vdots \\ \mathbf{R}^{\text{II}}_{\mu} & \mathbf{0} & \mathbf{R}^{\text{II}}_{\text{B}_0} & \mathbf{0} \end{bmatrix} \text{vec}(\mathbf{B}^*) \begin{matrix} \geq \\ = \\ = \\ = \\ = \end{matrix} \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{r}_{\text{A}_1} \\ \mathbf{1} \end{bmatrix}, \qquad (13)$$

where $\mathbf{R}^{\text{II}}_{\text{A}_1}$ is a diagonal matrix of $K^2 \times K^2$ whose diagonal elements correspond to those of vector $\text{vec}(\mathbf{I}_K)$ and $\mathbf{r}_{\text{A}_1} = (1 - \hat{\pi})\,\text{vec}(\mathbf{I}_K)$. Matrices $\mathbf{R}^{\text{II}}_{\mu}$ and $\mathbf{R}^{\text{II}}_{\text{B}_0}$ are, respectively, $\mathbf{I}_K$ and $\mathbf{1}'_M \otimes \mathbf{I}_K$. Note that except for the addition of these restrictions, the linear program of the second stage is exactly the same as that of the first one.

4. Information Sources and Parameter Estimation

4.1 The Data

In the previous section we described the theoretical model that underlies the macroeconomic aggregates and the econometric model that enables the estimation of its parameters, which are essentially the elasticities multiplied by π. In this section we estimate these elasticities with real data. They are:

- Endogenous variables: annual indexes of C, I, X, M and Y at constant prices (2004-2020 period) calculated from the SNA and scaled to unity in 2012. We chose this base year because it is an intermediate period in the series and because the gap between the financial and commercial exchange rates was narrow.
- Exogenous variables: (i) annual index of public spending G calculated from from the SNA, scaled to unity in 2012; (ii) import and export price indexes published by INDEC, multiplied by the Central Bank's multilateral real exchange rate (TCRM), and scaled to unity in 2012; and (iii) sovereign risk index (EMBI+) on the same scale as the previous indices. The latter was downloaded from the database of the financial newspaper Ámbito Financiero.[5]
- We also included a dummy variable for pandemic years (H1N1 pandemic in 2009 and COVID-19 in 2020) to check for outliers.

4.2 Parameter estimation

To build the matrix system and solve the linear program (11) we wrote a code in Euler Math Toolbox matrix language and used the `simplex` algorithm of the software.[6] Table 1 shows the the estimated parameters after fitting the current series of the SNA whose base year is 2004. For comparative purposes, we also fitted the series of C, I, X, M and Y from the 1993 SNA and series of public spending, foreign trade price and sovereign risk of the period 1993-2012.[7] Then we combined both samples into a single estimator that would presumably yield better estimate.

[5] The EMBI+ database is daily, so the annual series was calculated by averaging within months and among months. See `https://www.ambito.com/contenidos/riesgo-pais-historico.html`

[6] Euler Math Toolbox is a free software downloadable from `http://euler-math-toolbox.de/`

[7] The sources of these series were the same of those of the 2004-2020 period except for the sovereign risk that had to be completed between 1993 and 1996 with proxy figures from [2, p. 126].

Table 1: Parameters estimated by LAD from the 2004 SNA series. Results of the second stage of the estimation process, without prior information.

Aggregate	$\hat{\hat{\boldsymbol{\mu}}}$	$\hat{\hat{a}}_{ii}$	$\hat{\hat{\mathbf{B}}}_0$					$\hat{\hat{\mathbf{c}}}_1$	$\hat{\hat{\mathbf{c}}}_2$
			g_t	p_t^X	p_t^M	r_t	y_t		
Consumption	0.1634	-0.3180	–	–	–	–	1.1027	-0.0241	0.0521
Investment	0.2172	-0.3180	–	–	–	-0.0996	1.1314	-0.0470	0.0443
Exports	0.1740	-0.3180	–	1.0887	–	–	–	-0.0922	0.0644
Imports	–	-0.3180	–	–	-0.6451	–	1.9631	-0.0307	-0.0049
GDP	0.1511	-0.3180	0.8210	0.1939	0.1212	-0.0173	–	-0.0341	0.0258

To do so we formulated a version of the linear program (11) with prior information. The prior information was introduced as a a stochastic constraint in the following fashion

$$\text{vec}(\mathbf{B}^*) + \text{vec}(\mathbf{E})^- + \text{vec}(\mathbf{E})^+ = \text{vec}(\hat{\hat{\mathbf{B}}}_0^*) \tag{14}$$

where $\hat{\hat{\mathbf{B}}}_0^*$ is the matrix of estimated parameters with the SNA93 series and $\text{vec}(\mathbf{E})$ are *error* vectors analogous to those of the program (11).[8] This set of equations is introduced into the program as follows

$$\min_{\text{vec}(\mathbf{E})} \left\{ \mathbf{0}'_{K(K+M+N+1)}\text{vec}(\mathbf{B}^*) + \mathbf{1}'_{K(T-1)}\text{vec}(\mathbf{E}_1)^- + \mathbf{1}'_{K(T-1)}\text{vec}(\mathbf{E}_1)^+ + \mathbf{1}'_{KM}\text{vec}(\mathbf{E}_2)^- + \mathbf{1}'_{KM}\text{vec}(\mathbf{E}_2)^+ \right\}$$

subject to

$$\begin{bmatrix} (\mathbf{X}^{*\prime} \otimes \mathbf{I}_K) & -\mathbf{I}_{K(T-1)} & \mathbf{I}_{K(T-1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}^* & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{KM} & \mathbf{I}_{KM} \end{bmatrix} \begin{bmatrix} \text{vec}(\mathbf{B}^*) \\ \text{vec}(\mathbf{E}_1)^- \\ \text{vec}(\mathbf{E}_1)^+ \\ \text{vec}(\mathbf{E}_2)^- \\ \text{vec}(\mathbf{E}_2)^+ \end{bmatrix} \begin{matrix} = \\ \geq \\ = \end{matrix} \begin{bmatrix} \text{vec}(\mathbf{Y}) \\ \mathbf{0} \\ \text{vec}(\hat{\hat{\mathbf{B}}}_0^*) \end{bmatrix}, \tag{15}$$

At this point the estimation procedure with prior information might appear somehow confusing. To clarify it let us state the whole procedure as an algorithm, step by step:

(1) We solve the program (11) with the series of the SNA93 and compute the matrix of elasticities λ_{ij} dividing each element of b_{ij}^* by $\hat{\pi}$. Recall that estimating b_{ij}^* involves a two stage procedure, the first one to estimate π and the second one to estimate $\text{vec}(\mathbf{B}^*)$ with a proper diagonal matrix $\mathbf{A}_1$.

[8] ¡Caution! $\hat{\hat{\mathbf{B}}}_0^*$ is not the parameter estimation that arises directly from solving the linear program with the SNA93 series, but an estimation in a different base year as will be explained later.

Table 2: Parameters estimated by LAD from the SNA04 series. Results of the second stage of the estimation process using the estimates of the SNA93 series as prior information.

Aggregate	$\hat{\hat{\boldsymbol{\mu}}}$	$\hat{\hat{a}}_{ii}$	$\hat{\hat{\mathbf{B}}}_0$					$\hat{\hat{\mathbf{c}}}_1$	$\hat{\hat{\mathbf{c}}}_2$
			g_t	p_t^X	p_t^M	r_t	y_t		
Consumption	0.2845	-0.3180	–	–	–	–	0.9431	-0.0028	0.0901
Investment	0.4721	-0.3180	–	–	–	-0.0914	0.7873	-0.0048	0.0871
Exports	0.4219	-0.3180	–	0.7619	–	–	–	0.0038	0.1267
Imports	0.1121	-0.3180	–	–	-0.4053	–	1.5755	-0.0969	0.0381
GDP	0.1319	-0.3180	0.8929	0.1580	0.1052	-0.0120	–	-0.0288	-0.0037

(2) We change the basis of the elasticities found in the previous step by multiplying each one by the ratio of the indexes of its associated variables. for instance, to change the base of λ_Y^C we multiply by the ratio $(y_{2012}/y_{2001})/(c_{2012}/c_{2001})$. This relationship becomes evident when we verify that the expression (4) is completely equivalent to

$$\frac{C}{C_1} = \frac{\mu}{C_1} + \left[\left(\alpha\,\frac{Y_0}{C_0}\right)\frac{Y_1/Y_0}{C_1/C_0}\right]\frac{Y}{Y_1}. \tag{16}$$

where the coefficient in brackets is the elasticity expressed in the new base.

(3) Finally, we multiply the elasticities in the new base by the estimate of π with the series from SNA04 and construct the matrix $\hat{\hat{\mathbf{B}}}_0^*$. Note that we only use previous estimates of elasticities to enrich the current estimates. We do not include in the prior information the constants, or the parameter π.

In table 2 we present the estimated parameters using the elasticities of SNA93 (translated to base 2012) as prior information. To these parameters we imposed negativity restrictions on sovereign risk and import price elasticities. Let us recall that we had omitted these restrictions for the computation of table 1 to check the signs of these elasticities if they were not restricted.

4.3 Goodness of fit

The restricted LAD estimator is not a conventional estimator and therefore there is not a theoretical development that enables testing the “significance” of the model coefficients. Instead, we focused on evaluating the goodness of fit of the estimated to the observed series through two widely disseminated metrics in econometric studies: the mean absolute percentage error or MAPE and Theil’s U statistic [12, pp. 360-367]. The first is computed bu the formula

$$\text{MAPE} = \frac{1}{T-1}\sum_{j=1}^{T-1} 100\left|\frac{z_{ij}-\hat{z}_{ij}}{z_{ij}}\right| \tag{17}$$

Table 3: Statistics of goodness of fit test, MAPE (mean absolute percetage error) and U of Theil from the series of SCN04. Results of the second stage with and without prior information.

Macro aggregate	MAPE s/prior	u_i s/prior	MAPE c/prior	u_i c/prior
Consumption	5,3933	0,0738	5,8424	0,0810
Investment	5,7387	0,0828	6,5162	0,0998
Exports	8,2012	0,1306	9,6060	0,1309
Imports	7,0259	0,1188	8,9750	0,1262
GDP	2,8054	0,0481	3,4816	0,0518

while the second is computed

$$U_i = \frac{\sqrt{\frac{1}{T-1}\sum_{j=1}^{T-1}\left(\hat{z}_{ij} - z_{ij}\right)^2}}{\sqrt{\frac{1}{T-1}\sum_{j=1}^{T-1}\hat{z}_{ij}^2} + \sqrt{\frac{1}{T-1}\sum_{j=1}^{T-1} z_{ij}^2}}. \tag{18}$$

The Theil statistic is scaled such that $0 \leq U \leq 1$. Values close to 0 indicate perfect fit, while values close to 1 indicate a practically null predictive capacity of the model.[9] Although Theil did not establish tolerance limits for U - but he showed that for small values of U, e.g. $U < 0.3$ - the variance of the statistic is $var(U) \approx U^2/(T-1)$ (again we replace T by $T-1$ for our particular case) which would enable $E(U) = 1$ to be rejected if $u < 1 - 2/\sqrt{T-1}$ approximately, as long as T is large enough. In our case, the critical value of the Theil statistic would be $u^* = 0,4836$ although this value should be considered with caution since $T = 16$ cannot be considered a "large" sample. On the other hand, let us remember that the X-13 ARIMA program [17, p. 138], through the `pickmdl` function, considers acceptable specifications with MAPE less than 15 % in the last three years of the series. Table 3 shows the statistics calculated for each series.

Figure 1 shows the series of private consumption, investment, exports and imports, estimated with the model (9) and observed series. Figure 2 shows the series of GDP estimated directly through (9) nd indirectly through a Laspyres index of its components, including public spending, and the observed GDP as well.

5. Conclusion

The adaptive expectations model explains reasonably well the behavior of consumption, investment, exports and imports, as well as the GDP, all of them evaluated through the MAPE and Theil's U statistic, and also by simple graphic inspection. Of these aggregates, consumption is

[9]The above formulas assume the loss of an observation due to the lagged term of the endogenous variables.

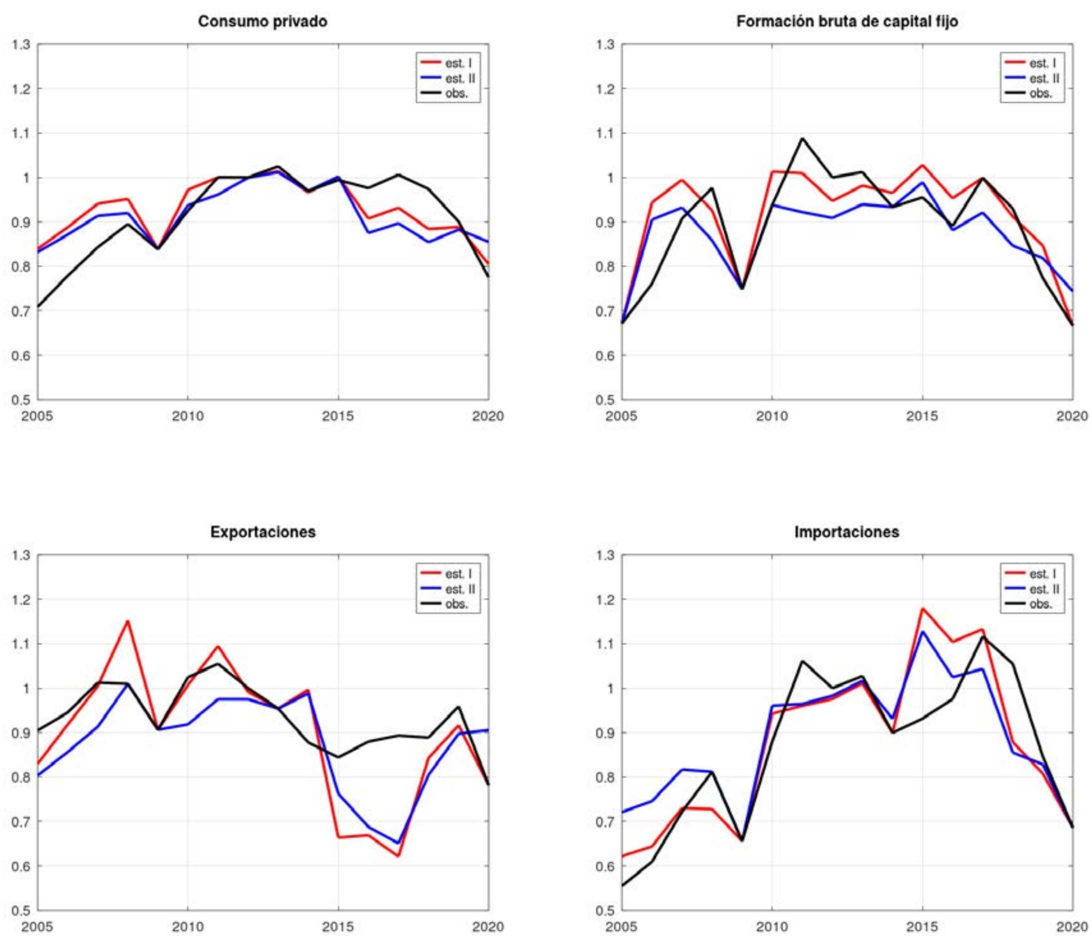

Figure 1: Estimated versus observed series of 2004 SNA, period 2005-2020. Indixes based on 2012 = 1. Estimation I, result of the second stage without prior information. Estimation II, result of the second stage with prior information.

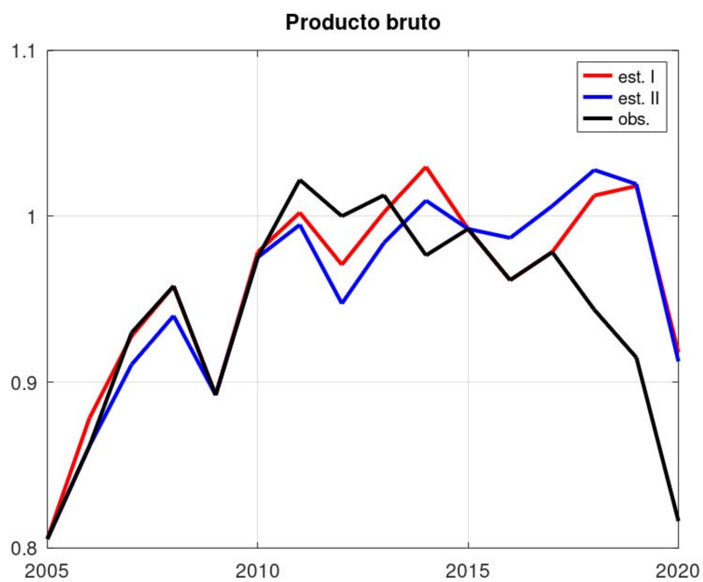

Figure 2: Estimated versus observed income series (GDP) of SNA04, period 2005-2020. Indices based on 2012 = 1. Estimation I, result of the second stage without a priori information. Estimation II, result of the second stage with a priori information.

the one that best fits the data while exports is the worst, although the fit of the five aggregates is statistically "significant". A detailed inspection of figure 1 suggests that the mismatch between the estimated and the observed exports is mainly due to deviations in years 2015 and 2016. In those years a number of overlaping events that justify the lack of fit: (i) an abrupt devaluation of the local currency from the end of 2015 to mid-2016 as a result of the abolition of the exchange rates control of the Central Bank; (ii) the elimination of taxes on exports of wheat, corn, meat, and regional products, and the reduction of 5 points in soybean exportation taxes; and (iii) the recession in Brazil (Argentina's main trading partner) which recorded in 2015 the worst drop in decades in its GDP.[10] Possibly, we should consider these events as atypical and incorporate them into the model as dummy variables. However, due to the small size of the sample we omitted these variables to avoid the risk of overfitting.

By construction, the matrix $\mathbf{X}^{*\prime} \otimes \mathbf{I}_K$ has multiple collinearity relationships. In fact, the condition number of this matrix is in the order of $\kappa(\mathbf{X}) \approx 2 \times 10^3$, exceeding by far the tolerance levels given by [9] for reliable matrix inversion in the context of least squares estimation. In linear programming, such colinear relationships produce an effect known as "quasi-optimal alternative solutions" [15]. In the extreme case of perfect collinearity relationships among the columns of the left hand side matrix of the constraint systtem, it can be proved [4] that the linear program will have infinite solutions. Then, by introducing a priori information, we reduce this possibility and therefore the solution of table 2 should be more reliable than that of table 1. Note e.g. that when comparing tables 1 and 2 we see that the estimates of the parameters associated with export and import prices vary markedly, contrary to the estimated parameters of public consumption and country risk. The result should not be surprising since the prices of imports and exports are related to each other through the so-called terms of trade. As long as the terms of trade remain stable, the export and import price series will be highly collinear, leading to the numerical instabilities already mentioned. The advantages of incorporating prior information to estimate the model parameters without the complications derived from high levels of collinearity in $\mathbf{X}$, however, are not reflected in a better predictive capacity of the macro aggregates. In table 3 it canbe seen that the model with prior information shows slightly worse adjustments than that without previous information, suggesting that in the adaptive expectations model there would be a kind of trade-off between precision in the parameter estimation and predictive capacity.

Finally, when comparing the adjustments achieved with the adaptive expectations model (table 3) to those of traditional seasonal ARIMA models (table 4), we see that the latter outperform the adaptive expectations model only in private consumption but not in the rest of the macro aggregates. These conclusions, however, should be taken with caution because in the table we compare MAPEs of an annual model with those of quarterly models.

[10]Let us remember that the price indixes prepared by the DNESE consider FOB prices, so that our estimated elasticity does not capture the exporter's response to foreign trade tax changes.

Table 4: MAPE of seasonal ARIMA models used to project macroeconomic aggregates for Argentina (see models proposed in [7]), period I-2004 to I-2021

Macro aggregate	MAPE
Private Consumption	3.81
Public Consumption	1.14
Investment	8.38
Exports	9.53
Imports	10.90
GDP	7.11

A. The Adaptive Expectations Model

We deduce Koyck's adaptive expectations model for a generic function $z_t = f(\mathbf{x}_t)$. In general, econometric texts present this deduction for the simple case of a function with a single exogenous variable. Here we extend the deduction to vector and matrix format following the notation of the macroeconomic model described above in order to facilitate the reader the reinterpretation of the Koyck model in the context of the macro model. We depart from the function

$$z_t = \boldsymbol{\lambda}^{Z\prime}\mathbf{x}_t^* + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2)$$

where $\mathbf{x}_t^*$ represents the vector of expected levels for the exogenous variables, which are unobservable. Koyck's model is based on the following hypothesis

$$\mathbf{x}_t^* - \mathbf{x}_{t-1}^* = \mathbf{\Pi}\left(\mathbf{x}_t - \mathbf{x}_{t-1}^*\right) \iff \mathbf{x}_t^* = \mathbf{\Pi}\,\mathbf{x}_t + \left(\mathbf{I} - \mathbf{\Pi}\right)\mathbf{x}_{t-1}^*$$

where $\mathbf{\Pi}$ is a diagonal matrix such that $\pi_{ij} = \pi$, for all $i = j$, y $\pi \in (0;1]$. This relationship suggests that expectations regarding exogenous variables change or are updated based on the discrepancy between the value that these variables actually take and their expected value in the previous period. The adjustment rate π is constant and is the same for all exogenous variables. Continuing with the development, if we replace $\mathbf{x}_t^*$ for this last expression in c_t

$$z_t = \boldsymbol{\lambda}^{Z\prime}\left[\mathbf{\Pi}\,\mathbf{x}_t + \left(\mathbf{I} - \mathbf{\Pi}\right)\mathbf{x}_{t-1}^*\right] + \epsilon_t.$$

If we propose an analogous expression for c_{t-1} pre-multiplied on both sides of the equality by $(1-\pi)$

$$(1-\pi)\,z_{t-1} = (1-\pi)\,\boldsymbol{\lambda}^{Z\prime}\mathbf{x}_{t-1}^* + (1-\pi)\,\epsilon_{t-1}.$$

and we subtract it term by term from c_t, we get

$$\begin{aligned} z_t - (1-\pi)\,z_{t-1} &= \boldsymbol{\lambda}^{Z\prime}\left[\mathbf{\Pi}\,\mathbf{x}_t + \left(\mathbf{I} - \mathbf{\Pi}\right)\mathbf{x}_{t-1}^*\right] - (1-\pi)\,\boldsymbol{\lambda}^{Z\prime}\mathbf{x}_{t-1}^* + \epsilon_t - (1-\pi)\,\epsilon_{t-1} \\ &= \pi\,\boldsymbol{\lambda}^{Z\prime}\,\mathbf{x}_t + \epsilon_t - (1-\pi)\,\epsilon_{t-1} \end{aligned}$$

which implies that

$$z_t = \pi\,\boldsymbol{\lambda}^{Z\prime}\,\mathbf{x}_t + (1-\pi)\,z_{t-1} + \nu_t, \quad \nu_t \sim N\left(0, \left[1 + (1-\pi)^2\right]\sigma^2\right)$$

which is Koyck's adaptive expectations model. The extension of the expectations model to the set of endogenous variables is immediate as long as the matrix $\mathbf{\Pi}$ is *scalar*.

$$\begin{aligned} \mathbf{z}_t - \left(\mathbf{I} - \mathbf{\Pi}_0\right)\mathbf{z}_{t-1} &= \mathbf{\Lambda}\left[\mathbf{\Pi}_1\,\mathbf{x}_t + \left(\mathbf{I} - \mathbf{\Pi}_1\right)\mathbf{x}_{t-1}^*\right] - \left(\mathbf{I} - \mathbf{\Pi}_0\right)\mathbf{\Lambda}\mathbf{x}_{t-1}^* + \boldsymbol{\epsilon}_t - \left(\mathbf{I} - \mathbf{\Pi}_0\right)\boldsymbol{\epsilon}_{t-1} \\ &= \pi\,\mathbf{\Lambda}\,\mathbf{x}_t + \boldsymbol{\epsilon}_t - \left(\mathbf{I} - \pi\mathbf{I}\right)\boldsymbol{\epsilon}_{t-1} \end{aligned}$$

The final expression is

$$\mathbf{z}_t = \left(\pi\,\mathbf{\Lambda}\right)\mathbf{x}_t + (1-\pi)\,\mathbf{z}_{t-1} + \boldsymbol{\nu}_t, \quad \boldsymbol{\nu}_t \sim N\left(\mathbf{0}, \left[1 + (1-\pi)^2\right]\sigma^2\mathbf{I}\right).$$

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