Estimating Inequality Process Parameters From Corporate Market Capitalizations

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Abstract
The Inequality Process (IP), a particle system, accounts for five frequently seen stock market statistical patterns, evidence motivating an IP model of corporate market capitalizations. Solutions of the IP are determined by its particle parameters. The IP's application to workers' labor income conditioned on their educations takes their educations as a self-evident indicator of IP particle parameters. The verbal theory from which the IP was abstracted posits that more productive workers are more sheltered from competition for wealth. There is no single self-evident indicator of corporate wealth productivity as education is for workers. Instead, there are many. Deciding which or which combination requires relating indicators of corporate wealth productivity to an initial estimate of IP particle parameters. The present paper describes how to do that initial estimate. This paper estimates an IP particle parameter for each of 921 large U.S. corporations by fitting that particle's wealth to that corporation's market capitalization. The data are from the 2013 Fortune Magazine 1000 list of large U.S. corporations.

Key Words: Inequality Process, market capitalization, quantitative finance, stochastic particle system

1.0 Modelling Corporate Market Capitalizations With the Inequality Process (IP)

This paper describes a method for estimating Inequality Process (IP) particle equivalence classes when the Inequality Process (IP) (Angle, 1983-2019) is fitted to market capitalizations of corporations in the 2013 Fortune Magazine 1000 (a list of 1,000 large U.S. corporations, chosen by size of revenue by Fortune magazine, a Time, Inc company). 921 of the these 1000 in 2013 have information on their market capitalization in the 2013 Fortune 1000 file. The data on the 2013 Fortune 1000 are produced by Fortune Magazine, a Time inc. company, a subset of which is published by the Someka Corporation, a software firm based in Izmir, Turkey. These data can be downloaded in Excel format from Someka at (https://www.someka.net/excel-template/fortune-1000-excel-list) .

See Appendix A for an introductory review of the basic Inequality Process (IP), i.e., unelaborated to explain particular social phenomena such as racial discrimination. The IP has been adopted in the econophysics literature as an early example, perhaps the earliest, of a particle system model of income and wealth distribution with demonstrated empirical relevance (Byrro Ribeiro, 2020).

1.1 Required information for an IP model of corporate market capitalization

To model the market capitalization of a corporation as an IP particle in a population of particles competing for wealth under IP rules (See Appendix A), the following information is required:

a) the parameters of all IP particles in the population of particles;
b) the simplification of knowing the number of particles in each distinct IP particle parameter equivalence class, or, nearly equivalent class, particularly useful if the equivalence classes correspond to a recognizable grouping of corporations;
c) the wealth of all particles, and,
d) a statistic of the fit of particle wealth to empirical wealth observations.

1.2 The basic IP (of Appendix A) has dilation symmetry (gauge invariance)

In the basic IP discussed in the present paper and stated in Appendix A, the grand mean of particle wealth appears only as a multiplicative scale factor in IP statistics, so it can, without loss of generality, be set to 1.0 to make it disappear from the statement of those IP statistics (and to facilitate computation in floating point arithmetic). In the basic IP, particle wealth relative to the grand mean of particle wealth is determined by each particle's parameter and the harmonic mean of particle parameters in the whole population. So, given the basic IP's dilation symmetry (gauge invariance in physics jargon), the natural way to model IP wealth is with a grand mean of 1.0 of particle wealth, i.e., with all particle wealth normalized. Fitting the IP to market capitalizations is facilitated by the normalizing of market capitalizations. In the present paper the grand mean of market capitalizations is taken as the mean market capitalization of the 921 corporations with information on market capitalizations in the data.

1.3 The usefulness of IP particle parameter equivalence classes

An example of an empirical grouping that can be assumed to be an IP parameter equivalence class is the education level of workers in the IP modelling of worker labor income. See Appendix B. The verbal theory from which the IP is abstracted asserts that workers more productive of wealth are more sheltered in the competition for wealth. 'Sheltered' in the IP is operationalized as having to give up a smaller fraction of wealth. That fraction is an IP particle's parameter. In the basic IP since winning or losing an encounter with another particle is independent and 50/50, particles that lose less have a higher expectation of wealth. Fits of IP particle statistics to those of labor incomes of the more educated yields smaller IP parameter estimates than fits of IP statistics to those of the labor incomes of the less educated, confirming the IP's hypothesis to the extent that it is reasonable to infer that the more educated, are, in general, more productive of wealth. What was a strategy to confirm the IP in its application of workers' labor income conditioned on education (Angle, 1983-2012) becomes an estimation strategy in the present paper. IP particle parameters will be estimated here from the clustering of similar particle wealth amounts around the mean particle wealth of particles in a particle parameter equivalence class. See how the wealth of similarly colored particles cluster in Figures 1 and 2. Figures 1 and 2 also show overlap of the wealth of particles in different equivalence classes even though those equivalence classes are chosen to have distinctly different parameters..

There is no obvious analogue of the role education plays in fitting the IP to workers' labor income statistics in fitting the IP to corporate market capitalization statistics. Instead of one self-evident indicator, there are many self-evident indicators of corporate wealth productivity. The situation prompts the question of how to weight one indicator against others in that combination. The answer consistent with the verbal theory from which the IP was abstracted is the combination with the largest correlation with corporate market capitalization and, (in absolute value) the largest negative correlation to IP particle parameters fitted to those corporate market capitalizations. This selection of indicators of corporate wealth productivity is similar to the practice of "factor analysis" (Ruppert and Matteson, 2015: 527ff) in contemporary quantitative finance. Some possible indicators of corporate wealth productivity:

a) return on investment,
b) market capitalization per employee,
c) revenue per employee,
d) net profit per employee,
e) basic economic viability, earnings before tax and financial penalties, per employee,
f the distribution of employee educations,
g) industry, or,
h) something else, or
i) some combination of the above

.

2.0 "Stylized Fact" Evidence that the Inequality Process (IP) Operates on Market Capitalizations

Compelling evidence that the Inequality Process (IP) operates on corporations in competition for wealth ('capital' is applicable financial term), as indicated by their market capitalization, is the ease with which the IP implies five "stylized facts" of stock prices (Angle, 2018). See Table 1. There is also some encouragement of the hypothesis that the IP operates in all markets in the list of seven verbal maxims of mainstream economics implied by the IP (Appendix C) as well as the usefulness of the IP in modelling workers' labor incomes.

Table 1: Five "Stylized Facts" (often seen empirical patterns) of Stock Market Statistics (quoted from Angle, 2018)

1.	Association between greater corporate market capitalization and a lower mean absolute value of the logarithm of its daily stock returns. Volatility is defined here as the mean absolute logarithm of daily returns. Source: Malkiel (2015:124).
2.	Big stock price movements down are associated with greater volatility, while big stock price movements up are associated with lower volatility. In finance this phenomenon is terms "leverage effect". Source: Tsay (2013:177).
3.	(t+k) autocorrelations of daily log returns to stocks of a particular corporation converge to near zero for k small beyond k = 1. Sources: Georgakopoulos (2015:115), Resnick (2007:6), Tsay (2013:178).
4.	t+k autocorrelations of squared daily log returns to stocks of a particular corporation show long term memory (i.e., do not converge to zero) as k increases. Sources: Georgakopoulos (2015:115), Resnick (2007:6).
5.	Bollinger Band-like bounded volatility of particle wealth. Source: Kaufman (2005: 294).

Given the statistical interchangeability of corporate market capitalization and corporate stock price (explained in Section 3.1 below), the basic Inequality Process (IP) of Appendix A implies the five "stylized facts", (frequently seen statistical patterns) of Table

1 in Figure 1, the phase diagram of the logarithm of particle wealth in five distinct particle parameter equivalence classes. Figure 1 is the phase diagram of log particle wealth, the change in the logarithm of particle wealth (y-axis) vs. the logarithm of particle wealth before the change (x-axis). Table 2 explains each of the five "stylized facts" of Table 1 by reference to Figure 1.

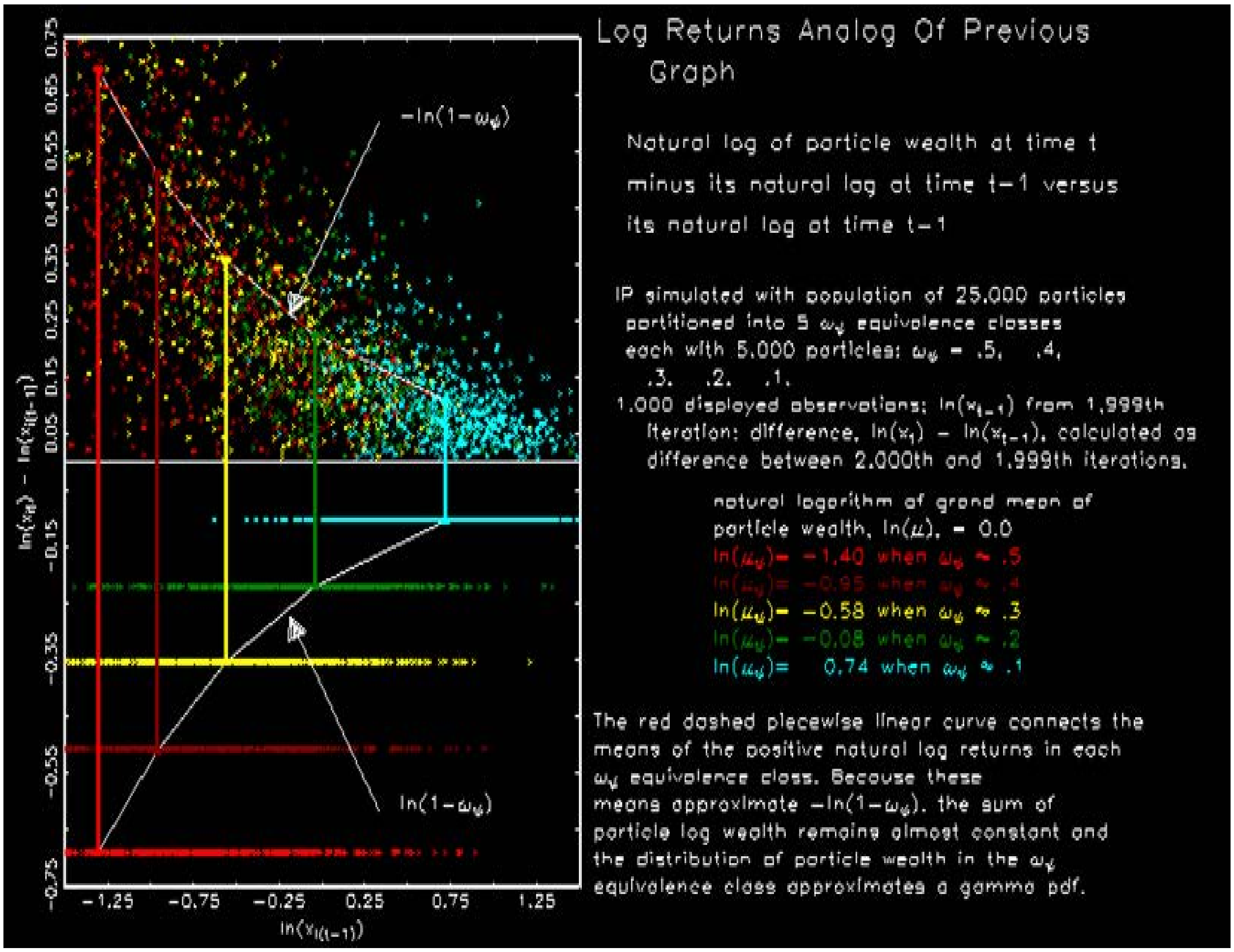

Figure 1

Table 2. How Figure 1 Explains the five "stylized facts" of Table 1

1.	In Figure 1 the particles with the smallest parameters are blue, the biggest red. Particles with smaller parameters have greater wealth and smaller changes in the log of particle wealth (in unlogged terms: smaller proportional changes, less volatility).
2.	A big, long-lasting, change downward in an IP particle's wealth is caused by a big change upward in its parameter. Such a change in parameter creates greater volatility, and vice versa. For example, see what happens in Figure 1 if a blue particle (smallest IP parameter) turns red (largest IP parameter), and vice versa.
3.	The wealth of particles in all particle parameter equivalence classes change at every tick of the IP's clock. Whether the change is an increase or a decrease is determined by the toss of a fair coin. Figure 2, the unlogged phase diagram of the basic IP of Appendix A, shows that if the change is a loss of wealth, a particle whose wealth

	equals that of the mean wealth of its parameter equivalence class equals its mean gain. Since most particles in a parameter equivalence class are in the vicinity of that class' mean wealth, their autoregressive correlation is the sum of products of roughly equal size with signs alternating according to the toss of a fair coin: the autoregressive correlation quickly converges to zero. Approximately ditto for log changes The log change is the log of the ratio of more recent wealth to previous wealth. It is positive for an increase, negative for decrease.
4.	When changes in IP particle wealth are squared, the signs of the changes are positive. Figure 1 shows that the log changes of particle wealth with smaller IP particles are smaller than those with larger IP particle parameters. In Figure 1, the magnitude of changes of log particle wealth depends on their particle parameter. They are smaller for particles with smaller parameters, and vice versa. Squaring makes this pattern positive and permanent in autoregressive summations. When all changes are positive, the dependence of magnitudes on particle parameters does not disappear, resulting in autoregressive correlations that do not converge to zero.
5.	In the absence of changes to particle parameters the wealth of IP particles fluctuate with a characteristic limit on volatility, less for particles with smaller parameters, more for particles with larger parameters.

One might add other stock price "stylized facts" replicated by the IP to Angle (2018)'s list in Table 1. For example, the ARCH phenomenon. 'ARCH' is the acronym for 'autoregressive conditional heteroskedasticity'. This phenomenon occurs when there is a market panic, usually caused by bad news, resulting in broadly lower prices. Not only does the price of some or all stocks plunge to a lower level, prices do not stay motionless at that lower level. Instead, they experience greater volatility around that lower level, including greater proportional increases off that lower level than they demonstrated before the fall around their higher level. The IP reproduces the ARCH phenomenon as a broadly shared increase in IP particle parameters. See in Figure 1 what a shift toward the red end of the color spectrum. 'Redder' means larger particle parameter and greater volatility around a smaller mean of particle wealth. See Ruppert and Matteson (2015, Chapter 14) for an overview of how complex econometric ARCH models have gotten. The IP's explanation is much simpler.

2.1 The stock trading strategy implied by the Inequality Process (IP) is already well known

The IP's symmetries (basic IP of Appendix A) have the consequence in Figure 2 of implying the following equation, asymptotically exact as the number of particles in the IP's population of particles increases:

$$\mu_{\psi} = \left(\frac{\tilde{\omega}}{\omega_{\psi}}\right) \mu \qquad (1)$$

where:

μ is the constant grand mean of particle wealth in the whole population of particles, possibly set to 1.0 because of the IP's dilation symmetry (gauge invariance);

μ_{ψ} is the mean of particle wealth in the ω_{ψ} the parameter equivalence class;

ω_{ψ} is a particle parameter value; the ω with the tilde over it is the harmonic mean of all parameters of all particles in the population.

So, if picking a stock is like picking a particle whose expected wealth is going to increase in an IP with a constant grand mean of particle wealth, the optimal particle picking strategy is to pick a particle whose particle parameter is decreasing.

This strategy is incorporated in a statistic known as the Sharpe Ratio, a statistic of the Capital Asset Pricing Model (CAPM), a model of decisions made by a rational investor in which risk of loss is perceived as an intrinsic aspect of profitable investment, i.e., an entirely different mathematical model from the Inequality Process. The Sharpe Ratio is the ratio of the volatility of the stock price of a corporation to the average volatility of stock prices in the market. As with IP particle parameters a smaller Sharpe Ratio is more desirable (Ruppert and Matteson (2015: 470). The IP explains stock prices because they are statistically interchangeable with corporate market capitalizations. The IP has no concept similar to stock price. Particle wealth is an IP concept with an empirical referent in corporate market capitalization: a particle' wealth. Less volatility is indicative in the IP of a smaller particle parameter and a higher expectation of particle wealth. A variant of the Sharpe Ratio is the Sortino Ratio, which uses downward price volatility alone in its numerator. The rationale for this variant of the Sharpe Ration is that upward volatility is good, downward bad, so less of the latter is good. In the IP both kinds of volatility is indicative of IP particle parameters: less is indicative of smaller IP parameters and greater particle wealth. As Figures 1 and 2 show there is much more information about IP parameters in downward volatility. The CAPM treats volatility as intrinsic to rapidly growing investments; the IP treats volatility as a bane.

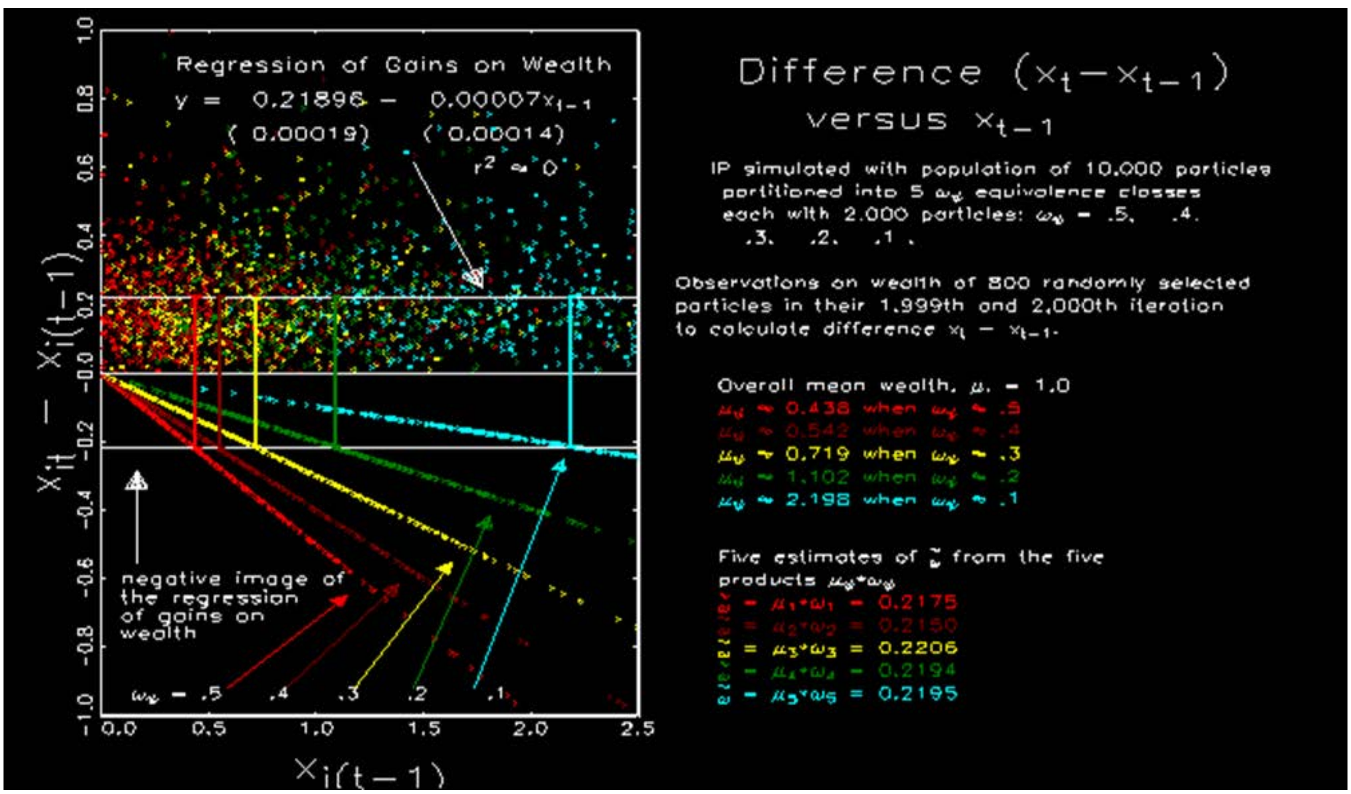

Figure 2

2.2 A criterion for how innovative a working Inequality Process (IP) model of corporate market capitalizations, if feasible, would be

If a working IP model of corporate market capitalizations is found, it would contradict a dictum of authors of major quantitative finance (qfin) textbooks. Their authors have written that there is no stochastic process, like a law of physics" operating in the stock market. See Table 3. Appendix D shows that the basic IP of Appendix A is similar to the particle system model of the Kinetic Theory of Gases, the oldest and best known statistical law of physics. Two substitutions into the transition equations of one model converts it into the other. Essentially, a working IP model of corporate market capitalization might, if found and be applicable to stock and securities trading, obsolesce sections of qfin textbooks with a simple particle system, whose implications appear in computer simulation.

Table 3: Four authors of textbooks on quantitative finance (quoted from Angle, 2018) on the impossibility of a statistical law, like a law of physics, in the stock market

1.	Emanuel Derman and Paul Wilmott. 2009. ***The Financial Modelers' Manifesto*** (2009:1) "The truth is that there are no fundamental laws in finance."[1]
2.	Harry Georgakopoulos. ***Quantitative Trading With R: Understanding Mathematical and Computational Tools from a Quant's Perspective*** (2015:147) "Market dynamics are also not bound by the laws of physics. Most patterns exist for a limited time, and are most likely not due to immutable underlying physical laws".
3.	Paul Wilmott and David Orrell. ***The Money Formula*** (2017:152) "Quantitative finance has no fundamental laws."

These experts should have asked themselves what produces those well known stock market "stylized facts", frequently seen statistical patterns in stock prices.

3.0 Modelling Corporate Market Capitalization with the Inequality Process (IP)

The evidence in Figures 1 and 2 is sufficient to suggest that the Inequality Process (IP) may be a useful model of market capitalizations of corporations and thus stock prices, given their statistical interchangeability. Interpretations of stock market "stylized facts" is only suggestive of the possibility the IP operates on corporate market capitalizations; it is not hard evidence. Hard evidence is a working IP model of corporate market capitalizations able to imply a variety of observed statistics and make testable predictions. The present paper describes the initial step toward creating the requisite IP model. It is mainly a task of finding the right IP particle parameter for each corporation. Given the absence of a priori information on natural IP particle parameter equivalence classes, the parameters have to be estimated. Then in a later step it can be determined if indicators of corporate wealth productivity cluster under a small number of understood rubrics. Firstly, IP concepts must be matched to empirical referents. Then a first attempt to find those parameter estimates must be made for each of the 921 corporations with data on market capitalization in the 2013 Fortune Magazine 1000 U.S corporations. See Table 4.

[1] Derman and Wilmott's qfin prominence is indicated by their admiring mention in ***The Physics of Wall Street*** (Weatherall, 2014), a history of contributions from applied mathematics to the quantitative finance of stock trading, something of an awe-struck a "fan" book.

Table 4: Empirical Referents of IP Concepts When the IP is Used to Model Labor Income Statistics and Stock Market Statistics

IP Concept	Labor Market Referent	Stock Market Referent
Particle	individual worker	Corporation
IP particle parameter referent	worker's education as measure of wealth productivity	indicators of corporate wealth productivity
IP particle parameter	semi-permanent	varies with whole market?
IP particle wealth	worker's human capital as indicated by worker's earnings	corporation's market capitalization

3.1 Why a corporation's market capitalization and its stock price are statistically interchangeable

A corporation's market capitalization is the multiplicative product of two factors. One factor is the number of shares of ownership in a corporation, its stock, that are, potentially, tradeable on a market, the "float". The other factor is the price of each share of stock. Their multiplicative product, the market capitalization, is the monetary valuation of the corporation as a whole by the stock market. A corporation's stock price fluctuates constantly, sometimes wildly, while the number of its tradeable shares may stay constant for months and often when it does change, for example, in a corporation's use of profit to buy back its own shares, reduces only a small proportion of its tradeable shares. Stock splits (e.g. two for one splits) or reverse splits (e.g. one for five shares) usually do not change a corporation's market capitalization immediately, but the former are indicative of a rising share price, and vice versa for the latter. So, in statistical terms, modelling a corporation's market capitalization is so closely related to the price of the corporation's stock that the two are statistically interchangeable, i.e., very closely correlated.

4.0 The Search Algorithm for the Estimating Inequality Process (IP) Parameter for each of Market Capitalizations

IP particle parameters are estimated from market capitalizations, based on the hypothesis that competition for capital in the stock market is governed by the IP. Consequently, the resulting estimates are hypothesis dependent and require a later test of validity. This search algorithm is the first step in a two step process. The algorithm produces estimates of IP particle parameters to be compared to various indicators of corporate wealth productivity. The second step is to find the combination of those indicators that best fits the IP parameter estimates of the first step. The second step is not described in this paper.

4.1 The Algorithm

The algorithm to estimate IP particle parameters from corporate market capitalizations is a stochastic search, a modification of the simulated annealing algorithm (Kirkpatric, Gelatt, and Vecchi, 1983). The algorithm is programmed in GAUSS21 (Aptech, 2021). The algorithm proceeds in cycles with a varying stochastic perturbation of a current optimum. The algorithm's objective function is the minimization of the sum of absolute differences between 921 ranked, normalized corporate market capitalizations and 921 ranked, normalized IP particle wealth amounts. The stochastic driver of the search is perturbations to the 921 IP particle parameters that generate the 921 ranked, normalized IP particle wealth amounts fitted to the 921 ranked, normalized corporate market caps. The algorithm proceeds in cycles. There are 6,000 major cycles.

4.1.1 The First Major Search Cycle:

1) The 921 market capitalizations of the corporations of the 2013 Fortune Magazine 1000 U.S. corporations with information on market capitalization are normalized, i.e., their mean is now 1.0;
2) These normalized market capitalizations are ranked from small to large; the identities of the 921 corporations remain known; the normalized market caps are never again altered in any way;
3) An IP simulation with 921 particles whose particle wealth is normalized (mean 1.0) is run with particle parameter values generated by a 0,1 continuous uniform random number generator;
4) The IP simulation is run through 300 iterations with the same particle parameters; 300 iterations is sufficient for particle wealth to converge and then fluctuate around its stationary distribution;
5) IP particle wealth after the 301st iteration of the IP is then ranked by wealth size from small to large; that ordering keeps the IP particle's parameter matched to the IP particle's particle wealth;
6) The absolute value of the difference between the vector of normalized and ranked market capitalizations and the vector of normalized and ranked particle wealth amounts, each matched rank to rank, is taken and the 921 absolute differences summed;
7) The absolute value of the difference between a vector of 20 statistics of the normalized and ranked market capitalizations and a vector of the same 20 statistics of the normalized and ranked particle wealth vector, each matched statistic to statistic, is taken and summed.
8) The sum of absolute value of the differences between the two vectors of statistics is multiplied by a factor of 57.39 so its maximum possible error equals 921, the maximum possible error of the differences between normalized market caps and IP particle wealth, i.e., so errors between the statistics equal in importance errors between the matched 921 market capitalization and IP particle wealth amounts.
9) Both sums of differences, errors in fitting the 921 market caps and their statistics, are added. Minimizing this grand sum of differences, whose possible maximum is 1,842, is the objective function of the search over the vector of 921 IP particle parameters.

4.1.2 The 5,999 Major Search Cycles Following

Each search continues in a sequence of 5,999 cycles after the first. The initial value of the objective function is 2,000, i.e., greater than the maximum error of 1,842. Consequently, whatever the sum of errors of the first search cycle is, its approximation to the corporate market caps, and its sum of errors, its fits, and its particle parameters become the current optimum. The next search cycle to produce a smaller sum of errors than the first cycle produces a new current optimum. The new optima are a) the average of the smaller sum of errors with the previous minimum, b) the average of the new, better fitting particle wealth amounts and the previous optimal wealth amounts, and c) the average of the new vector of 921 IP particle parameters that generate better fitting IP particle wealth with the 921 IP particle parameters that generated the previous optimal fits.

And so on for 5,999 major cycles of the search algorithm of this first search. The statistics of the average of 26 such searches of 6,000 major cycles each are given in Tables 5 and 6.

4.1.3 Why 20 Statistics of Market Cap Size are also Fitted

The reason that 20 statistics of the normalized market caps are fitted in addition to the 921 market caps is that the search to fit the 921 market caps by using 0,1 continuous uniform random numbers as IP particle parameters is slow to converge. Worse, it contains no information about how the composition of IP particles with different parameters change with market cap size, given the hypothesis that the IP determines market caps. Hence the need to fit 4 statistics from 5 quintile partitions of the sequence of market caps ordered from small to large.

The 921 ranked and normalized market caps are partitioned by the quintiles of market caps into five bins. The quintile boundaries are;

1) the smallest market capitalization to the 184^{th} by size,
2) the 185^{th} by size to the 368^{th},
3) the 369^{th} by size to the 552^{nd},
4) the 553^{rd} by size to the 736^{th},
5) the 737^{th} to the largest 921^{st}.

Four statistics in each of these five partitions of the 921 ranked and normalized market capitalizations are estimated: the first quartile market cap within the partition, the median market cap of the partition, and the fourth quartile of the partition.

The search algorithm is stochastic search with stochastic backups – stochastic convergence interrupted by large random perturbations to the current test vector of IP parameters, 'curdist' – perturbations large enough to slip the objective function off a current minimum it might be stuck on. The current optimum vector, 'bestdist' changes only if 'curdist' produces a better fitting vector of IP particle wealth to the vector of market caps. The GAUSS 21 code (Aptech, 2021) for the perturbation of the current optimum IP parameter vector, 'bestdist', is:

```
curdist = bestdist   +   toler[grid,.] * (rndu(rows(bestdist),1) .*  bestdist)

                -   toler[grid,.] * (rndu(rows(bestdist),1) .*  bestdist);

curdist = (curdist  .* (curdist  .ge .01)) + ((curdist  .lt .01) * rndu(1,1));

curdist = (curdist  .* (curdist  .le .99)) + ((curdist  .gt .99) * rndu(1,1));
```

where:

a) 'rndu(rows(bestdist),1)' is GAUSS code generating here a vector of 921 0,1 continuous uniform random variables
b) 'bestdist' is the current optimum vector of IP particle parameters;
c) 'bestdist' is perturbed to yield the current attempt, 'curdist', to generate a vector of IP particle parameters; 'curdist' is then tested to see if it yields a vector of IP particle wealth that fits the vector of market caps better than the vector of IP particle wealth generated by 'bestdist';
d) 'toler[grid,.]' varies the magnitude of the perturbation.

e) if occasionally a perturbed IP particle parameter is outside of the open interval (0,1) on which IP parameters are defined, it is replaced by a 0,1 continuous uniform random variable.

5.0 Findings

26 cycles of the stochastic search algorithm are averaged together in Tables 5 and 6. Table 5 averages the 26 inter-correlation matrices of three variables of the search: a) the vector of 921 normalized and ranked market caps of 921 large U.S. corporations, b) the vector of 921 normalized and ranked IP particle wealth amounts, the ones that best fit the vector of 921 normalized and ranked market caps, and c) the vector of 921 IP particle parameters that generated the vector of the 921 IP particle wealth amounts that best fit the vector of 921 corporate market caps. The search is over the vector of IP particle parameters.

Table 5 shows that the average correlation of the vector of 921 IP particle wealth amounts with the vector of 921 corporate market caps is 0.9952, a correlation so large that the two vectors are statistically interchangeable. The correlation between the estimated vector of 921 IP particle parameters and the vector of the 921 IP particle wealth amounts implied by that vector of IP parameters is only -.6466 in absolute value. The negative sign is expected since IP particle wealth varies inversely with IP particle parameters. The relatively small absolute value of this correlation, -.6466, indicates that estimating particle parameters from IP particle wealth produces only crude estimates of the parameters. -.6466 is the correlation between the IP particle parameters and the IP particle wealth generated by those same parameters, i.e., at the limit of perfectability of inference of particle parameters from particle wealth, given the distribution of market caps, without the use of equation (1) which would yield only expected particle wealth rather than actual particle wealth.

Table 5. The average of 26 inter-correlation matrices. The standard errors of estimate are miniscule

921 normalized, ranked market capitalizations of large U.S. corporations	921 normalized, ranked IP particle wealth amounts best fitted to the normalized, ranked market capitalizations	921 particle parameters of the particles that best fit the 921 normalized, ranked market capitalizations
1.0	0.9952	-0.6337
0.9952	1.0	-0.6466
-0.6337	-0.6466	1.0

Because the vector of IP particle wealth is statistically interchangeable with the vector of market caps, the latter’s correlation with the vector of IP particle parameters is almost as large in absolute value, -0.6337, and is at the limit of perfectability of inference of resolving what is, by hypothesis, a mixture distribution of market cap distributions conditioned on particular IP parameter values into those conditional distributions. The distribution of market caps is a heavy-tailed distribution, but as Table 6 shows, some of its inferred conditional distributions have IP particle parameters that point to approximation by a gamma pdf (Angle, 2019).

Table 6. The estimated size distribution of IP particle parameters of the best fitting IP particle parameters averaged over 26 searches

IP parameter bin (note: the smaller the IP parameter, the greater the expectation of wealth	Mean count of corporations with a parameter in bin	Minimum estimated count	Maximum estimated count
0.01 to 0.099	68.65	56	85
0.10 to 0.199	38.04	11	64
0.20 to 0.299	60.54	23	100
0.30 to 0.399	123.2	73	162
0.40 to 0.499	208.8	145	262
0.50 to 0.599	215.5	157	310
0.60 to 0.699	131.9	82	194
0.70 to 0.799	55.73	27	95
0.80 to 0.899	16.23	1	30
0.90 to 0.999	2.346	0	7

The distribution of most of the estimated IP particle parameters is a unimodal distribution, roughly similar to a binomial distribution, with its mode at the IP particle parameter frequency bin 0.50 to 0.599. The fact that so many of the inferred IP particle parameters are clustered close to the mode explains why the vector of implied particle wealth amounts can be statistically interchangeable with the distribution of market caps, while only correlated -.6466 with the vector of IP particle parameters that generated it: the overlap of particle wealth of particles in particle parameter equivalence classes that have small differences in their parameters is extensive.

. While Table 6 shows a substantial central tendency of particle parameters, there is a small secondary mode in Table 6: that of particle parameters with quite small particle parameters, those in the parameter bin 0.01 to 0.099, and the bin next to it. In the IP these are the parameters of IP particles whose wealth circulates randomly over the largest wealth amounts. While their frequency is smaller than the frequencies at or near the main distribution mode, the particles with the smallest parameters have a disproportionately large influence on decreasing the harmonic mean of particle parameters in the whole population of particles. See equation 1 for what results from a smaller harmonic mean of parameters in the whole population for particles with parameters larger than that harmonic mean: a smaller expectation of particle wealth. It is the particles with very small particle parameters that are the way the IP accounts for the especially large market caps that dominate the capitalization-weighted indexes of stocks.

Appendices

This paper's Appendices contain material from many previous papers on the Inequality Process by the author, with occasional revisions and extensions

Appendix A.1 Inequality Process Basics

A1.1 The Inequality Process in words

The Inequality Process (IP) (Angle, 1983-2019) is a particle system, in which wealth, a positive quantity, is transferred between two particles according to the following rules:

1) All particles in a population are randomly paired.
2) Each pair flips and calls a fair coin.
3)The general pair is particle ψ and particle θ.
4) If particle ψ wins the toss, it takes an ω_θ share of particle θ's wealth.
5) If particle θ wins the toss, it takes an ω_ψ share of particle ψ's wealth.
6) Repeat.

Particle wealth changes at each encounter. Particles make no decisions. In the basic Inequality Process described here, the share of wealth a particle gives up when it loses does not change. That share is its parameter, omega, ω. Particles that lose less when they lose (i.e., with smaller omega) have a higher expectation of wealth than particles that lose more, since the probability of loss is 50/50 for all.

A.12 The Transition equations of the Inequality Process (IP)

The IP's transition equations determine which of two paired particles wins a competitive encounter and how much of the loser's wealth is transferred to the winner:

$$x_{it} = x_{i(t-1)} + d_t \omega_\theta x_{j(t-1)} - (1 - d_t)\omega_\psi x_{i(t-1)}$$
$$x_{jt} = x_{j(t-1)} - d_t \omega_\theta x_{j(t-1)} + (1 - d_t)\omega_\psi \quad {}_{i(t-1)}$$

A1.3

$x_{it} \equiv$ particle i's wealth at time − step t in multiples of μ , the unconditional mean of wealth

$x_{j(t-1)} \equiv$ particle j's wealth at time − step (t − 1)

$0 < \omega_{\theta j} < 1.0$, fraction lost in loss by particle j

$0 < \omega_{\psi i} < 1.0$, fraction lost in loss by particle i

$d_t =$ an i.i.d. 0,1 uniform discrete r.v. equal to 1 with probability .5 at time − step t (a Bernoulli variable)

Note that the only way to gain wealth is via winning an encounter but since winning/losing is 50/50 in the long run the only way to gain more wealth is by losing less in a loss. In the long run particles that lose less when they lose, i.e., those with smaller particle parameters, smaller omegas, ω's, have a higher expectation of wealth. The verbal theory from which the IP was abstracted asserts that the more productive of wealth are more sheltered in the competition for wealth. Smaller omegas is the way the IP operationalizes that assertion. Note, however, that the IP's transition equations neither create nor destroy wealth. The basic IP is dilation symmetric (gauge invariant) with respect to the grand total of particle wealth (as long as it is positive) or the size of the population of particles (as long as there are at least several dozen particles). Consequently, the basic IP applies to societies up and down the arc of techno-cultural evolution, societies with great differences in population size, technology, culture, and wealth.

A1,4 An odd provenance for a theory of corporate market capitalization

Angle (1983, 1986) describe the abstraction of the Inequality Process (IP) from an old anthropological theory, speculatively extended by a sociologist, Gerhard Lenski (1966). Old anthropology is an unusual source of new economics. The anthropological theory is the Surplus Theory of Social Stratification. It is viewed as an uninteresting truism in anthropology (Harris, 1959). The Surplus Theory explains why hunter/gatherer society, viewed as the most egalitarian societal form, turned into the chiefdom, the society of the "god king", the societal form anthropologists view as the most inegalitarian., when the hunter/gatherers acquired the ability to store more food than they consumed at one time for a later use, storeable surplus, an expanded wealth, and in a more fugitive form than, for example, human capital, when it appeared. Anthropologists were struck by how quickly and universally the transition from apparently egalitarian hunter/gather society to the inegalitarian chiefdom was: all cultures, all races, all times, all places.

The first evidence of storeable food surplus and the first evidence of substantial inequality of wealth often occurred in the same archeological strata. The Surplus Theory explains the simultaneity. The theory runs that humans, like most species, certainly of all mammals, compete with others of the species for resources, 'niche' in population biology, anything with positive utility for life and reproduction. The importance of intra-species competition for niche is the assumption on which the Lotka-Volterra equations rest, the basis of quantitative population biology. Unlike other species, humans have a unidimensional measure of the concept of niche, money.

The Surplus Theory has one obvious flaw: no explanation for why as societies evolved in techno-cultural terms beyond the chiefdom the concentration of wealth gradually decreased. Lenski (1966) provides a number of speculative reasons why such might be the case. The Inequality Process (IP) models one of them: workers who are more productive of wealth are more sheltered in the competition for wealth. So a worker's productivity of wealth (value of wealth produced in a unit of time) is the empirical referent of the IP's particle parameter, omega, the fraction of wealth a particle loses when it loses an encounter with another particle. Since winning and losing is 50/50 and the winner gains an omega share of the loser's wealth, particles with smaller omegas have a higher expectation of wealth. Evidence of the IP's appearance in stock market statistics led Angle

(2018) to conjecture that the IP might be an evolutionarily optimal form of competition, naturally selected to maximize wealth creation.

Appendix B: Confirmed Inequality Process (IP) Hypotheses

1. The universal pairing (all times, all places, all cultures, all races) of the appearance of extreme social inequality in the chiefdom, the society of the god-king, after egalitarian hunter/gatherers acquire a storeable food surplus (Angle, 1983, 1986).
2. The pattern of the Gini concentration ratio of personal wealth and income over the course of techno-cultural evolution beyond the chiefdom (Angle, 1983, 1986).
3. The right skew and gently tapering right tail of all distributions of income and wealth, a broad statement of the Pareto Law of income and wealth distribution. (Angle, 1983, 1986).
4. a) The sequence of shapes of the distribution of labor income by level of worker education, b) why this sequence of shapes changes little over decades, and c) why a gamma pdf model works well for fitting the distribution of labor income at each level of worker education (Angle, 1990, 2002, 2003, 2006, 2007b);
5. How the unconditional distribution of personal income appears to be gamma regardless of level of geographic aggregation although the gamma distribution is not closed under mixture (Angle, 996);
6. Why sequences of Gini concentration ratios of labor income by level of education from low to high recapitulates the sequence of Gini concentration ratios of labor income over the course of techno-cultural evolution (a social science analogue of "ontogeny repeats phylogeny" (Angle, 1983, 1986, 2002, 2003, 2006, 2007b);
7. Why the sequence of shapes of the distribution of labor income by level of education from low to high recapitulates the sequence of shapes of the distribution of labor income over the course techno-cultural evolution, a social science analogue of "ontogeny repeats phylogeny" (Angle, 1983, 1986, 2002, 2003, 2006, 2007b);
8. The dynamics of the distribution of labor income conditioned on education as a function of the unconditional mean of labor income and the distribution of education in the labor force (Angle, 2003a, 2006, 2007b);
9. The pattern of correlations of the relative frequency of an income smaller than the mean with relative frequencies of other income amounts (Angle, 2005; 2007a).
10. The surge in the relative frequency of large incomes in a business expansion (Angle, 2007b);
11. The "heaviness" of the right tail of income being heavy enough to account for total annual wage and salary income in the U.S. National Income and Product Accounts (Angle, 2002c; 2003a).
12. Why and how the distribution of labor income is different from the distribution of income from tangible assets; (Angle, 1997)
13. Why the IP's parameters estimated from a time-series of the labor incomes of individual workers are ordered as predicted by the IP's meta-theory and approximate estimates of the same parameters from cross-sectional data on the distribution of wage income conditioned on education; (Angle, 2002)
14. The Kuznets Curve in the Gini concentration ratio of labor income during the industrialization of an agrarian economy; (Angle, Nielsen, and Scalas, 2009)
15. An elaboration of the basic Inequality Process in which all particles have an equal probability of winning a competitive encounter for wealth. This elaboration allows a majority group of particles to rig the probability of one of its members winning a competitive encounter with a member particle at $.5 + \varepsilon$, which equals probability of the minority group particle losing that encounter. This elaboration of the IP yields the following features of the joint distribution of personal income to African-Americans and 'other Americans' (i.e., non-African-Americans): a) the smaller median personal income of African-Americans than 'other Americans';

b) the difference in shapes between the African-American distribution of personal income and that of 'other Americans'; this difference corresponds to a larger Gini concentration of the African American distribution;

c) the % minority effect on discrimination (the larger the minority, the more severe discrimination on a per capita basis, as reflected in a bigger difference between the median personal incomes of African-Americans and 'other Americans' in areas with a larger % African-American);

d) the relatively high ratio of median African-American personal income to the median of 'other Americans' in areas where the Gini concentration ratio of the personal income of 'other Americans' is low;

e) the relatively high ratio of median African-American to that of 'other Americans' in areas where the median income of 'other Americans' is high;

f) the fact that relationships in d) and e) can be reduced in magnitude by controlling for a measure of economic development of an area or % African-American;

g) the greater hostility of poorer 'other Americans' to African-Americans than wealthier 'other Americans' (Angle, 1992).

Appendix C: Seven Verbal Maxims of Mainstream Economics Jointly Implied by the Inequality Process[2]

Maxim of Economics:	Inequality Process' Implication:
1) All distributions of labor income are right skewed with tapering right tails; hence the impossibility of radical egalitarianism, the ideologically motivated findings of Vilfredo Pareto's study of income and wealth distribution.	The IP generates right skewed distributions shaped like empirical distributions of labor income or personal assets (depending on the particle parameter). The IP implies that the unconditional distribution of personal money income from labor is an exponential family pdf (probability density function) shape mixture. Such a mixture has a right tail approximately as heavy as empirical right tails of money income and the Pareto pdf, the model of those right tails preferred in economics.
2) Differences of wealth and income arise easily, naturally, and inevitably via a ubiquitous stochastic process; a general statement of Gibrat's Law; hence the impossibility of radical egalitarianism. Like Pareto, Robert Gibrat's interest in income distribution was motivated by the desire to deny the possibility of a radically egalitarian income distribution.	In the IP, differences of wealth arise easily, naturally, and inevitably, via an ubiquitous stochastic process.
3) A worker's earnings are tied to that worker's productivity [i.e., a central tenet of economics since Aesop's fable of the ant and the grasshopper was all there was to economics] but there is a wide variety of dissimilar returns to similarly productive workers.	An IP particle's expected wealth is determined by the ratio of mean productivity in the population to that particular particle's productivity (the ratio of the harmonic mean of particle parameters in the population to an individual particle's parameter). The IP implies a distribution around this expectation whose shape is determined by each particle's productivity.
4) Labor incomes small and large benefit from a business expansion strong enough to increase mean labor income, i.e., there is a community of interest between all workers regardless of their earnings in a business expansion. A conclusion encapsulated in a	In the IP's Macro Model, an increase in the unconditional mean of wealth increases all percentiles of the stationary distribution of wealth by an equal factor. In pithy statement form: "A rising tide lifts the logarithm of all boats equally.".

[2] Angle, 2006e, 2013a.

favorite saying of mainstream economists: "A rising tide lifts all boats."	
5) Competition transfers wealth to the more productive of wealth via transactions without central direction, i.e., via parallel processing.	In the IP, competition between particles causes wealth to flow via transactions from particles that are by hypothesis and empirical analogue less productive of wealth to those that are more productive of wealth, enabling the more productive to create more wealth, explaining economic growth without a) requiring knowledge of how wealth is produced or b) central direction, i.e., with a minimum of information, two reasons for hypothesizing that the IP would arise to allocate wealth in every economy. These features enable the IP to operate homogeneously over the entire course of techno-cultural evolution independently of wealth level.

Appendix D The similarity of the Inequality Process (IP) to the Particle System of the Kinetic Theory of Gases

D.1 Two Substitutions Transform The Inequality Process Into The Kinetic Theory of Gases

While the IP did not originate in tinkering with the best known particle system of statistical physics, that of the kinetic theory of gases (KTG), the two are closely related.

Two substitutions into the Inequality Process' transition equations for the exchange of a positive quantity, x, between two particles transform them into the transition equations of the interacting particle system model of the kinetic theory of gases (Angle, 1990), the best known statistical law of physics). The transition equations of the Inequality Process are:

$$\begin{aligned} x_{it} &= x_{i(t-1)} + d_t \omega_\theta x_{j(t-1)} - (1-d_t)\omega_\psi x_{i(t-1)} \\ x_{jt} &= x_{j(t-1)} - d_t \omega_\theta x_{j(t-1)} + (1-d_t)\omega_\psi x_{i(t-1)} \end{aligned} \tag{B.1a,b}$$

$$\begin{aligned} x_{it} &\equiv \text{particle } i\text{'s wealth at time} - \text{step } t \text{ in multiples of } \mu\text{, the unconditional mean of wealth} \\ x_{j(t-1)} &\equiv \text{particle } j\text{'s wealth at time} - \text{step } (t-1) \\ 0 &< \omega_{\theta j} < 1.0\text{, fraction lost in loss by particle j} \\ 0 &< \omega_{\psi i} < 1.0\text{, fraction lost in loss by particle i} \\ d_t &= \text{an i.i.d. } 0{,}1 \text{ uniform discrete r.v. equal to 1 with probability .5 at time} - \text{step } t \text{ (a Bernoulli variable)} \\ \mu &= \text{unconditional mean of wealth} \end{aligned}$$

If

1) d_t, a discrete 0,1 uniform random variable is replaced by a continuous [0,1] uniform random variable, ϵ_t , and,
2) the ω's are replaced by 1.0,

then (B.1a,b) has been transformed into the transition equations of the particle system of the kinetic theory of gases:

$$x_{it} = \varepsilon_t\left(x_{i(t-1)} + x_{j(t-1)}\right)$$
$$x_{jt} = (1-\varepsilon_t)\left(x_{i(t-1)} + x_{j(t-1)}\right)$$

(B.1c,d)

where:

$X_{i(t-1)}$ = particle i's kinetic energy at time-step (t-1)
X_{jt} = particle j's kinetic energy at time-step t
ϵ_t = a [0,1] continuous uniform random variable at time-step t

(B.1c,d) is Whitney's (1990:103) statement of the transition equations for the transfer of kinetic energy between two molecules in the kinetic theory of gases. So, in this narrow sense, it is certain that the Inequality Process is like an established model of statistical physics, part of Auguste Comte's 19th century vision of what sociology should become.

The transformation from (B.1.a,b) into (B.1.c,d) is perhaps more easily recognized if (B.1.a,b) is re-written as

$$x_{it} = (1-\omega_\psi)\, x_{i(t-1)} + d_t\left(\omega_\psi\, x_{i(t-1)} + \omega_\theta\, x_{j(t-1)}\right)$$
$$x_{jt} = (1-\omega_\theta)\, x_{j(t-1)} + (1-d_t)\left(\omega_\psi\, x_{i(t-1)} + \omega_\theta\, x_{j(t-1)}\right)$$

(B.1e,f)

with $d_t \to \epsilon_t$ and the ω's -> 1.0. Both particle systems are otherwise identical apart from the labels on the variables. In both particle systems, particles are collectively isolated. Since in both particle systems, random pairings of particles result in transfers of a positive quantity that is neither created nor destroyed, the sum of that quantity over all particles is constant. Figure 1 shows that in the IP there is more information in losing than winning, and that particles that lose less when they lose.

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References

Angle, John. 2019. "The Inequality Process' PDF Approximation Model For Heavy-Tail Financial Distributions. ***Proceedings of the 2019 Joint Statistical Meetings***, (American Statistical Association, Economic and Business Statistics Section, Joint Statistical Meetings). Pp. 227-244. CD-ROM. Alexandria, VA: American Statistical Association.

_____. 2018. "The Inequality Process' Footprint On Stock Market "Stylized Facts". ***Proceedings of the 2018 Joint Statistical Meetings***, (American Statistical Association, Economic and Business Statistics Section, Joint Statistical Meetings). Pp. 759-773. CD-ROM. Alexandria, VA: American Statistical Association.

_____. 2017, "Detecting Politically Motivated Tampering With Workers' Labor Income In Survey Data". ***Proceedings of the 2017 Joint Statistical Meetings*****,** (American Statistical Association, Social Statistics Section, Joint Statistical Meetings). Pp.1681-1686. CD-ROM. Alexandria, VA: American Statistical Association.

------. 2013a. "How To Win Acceptance of The Inequality Process as Economics?". ***Management and Society Review*** 2(#2): 117-134.

-----. 2013b. "The Inequality Process is "Demonic": Conditional Entropy Maximization and Minimization and Wage Incomes". ***Proceedings of the 2013 Joint Statistical Meetings*****,** (American Statistical Association, Social Statistics Section, Joint Statistical Meetings. Pp.1375-1389. CD-ROM. Alexandria, VA: American Statistical Association.

_____. 2012. "The Inequality Process v. The Saved Wealth Model: Which is the More Likely to Imply an Analogue of Thermodynamics in Social Science". **Journal of Mathematical Sociology**. An earlier version of this paper **"** The Inequality Process vs. The Saved Wealth Model. Two Particle Systems of Income Distribution; Which Does Better Empirically?" is on RePEc/MPRA. [*http://mpra.ub.uni-muenchen.de/20835/*], a revision of "A Test of Two Similar Particle System Models of Wage Income Distribution Conditioned on Education", 2009 Joint Statistical Meetings, in its proceedings volume.

_____. 2010. "The Inequality Process as an Evolutionary Algorithm". ***Proceedings of the 2010 Joint Statistical Meetings*****,** (American Statistical Association, Section on Statistical Learning and Data Mining), Joint Statistical Meetings. Pp. 2295-2309. CD-ROM. Alexandria, VA: American Statistical Association.

_____, François Nielsen, and Enrico Scalas. 2009. "The Kuznets Curve and the Inequality Process". Pp. 125-138. In Banasri Basu, Bikas K. Chakrabarti, Satya R.

Chakravarty, Kausik Gangopadhyay, editors, ***Econophysics and Economics of Games, Social Choices and Quantitative Techniques***. (Proceedings of the Econophys-Kolkata IV Conference, March 2009, Kolkata, India, jointly sponsored by the Indian Statistical Institute and the Saha Institute of Nuclear Physics.) [http://www.isical.ac.in/cmp/econophys4.com/]. Milan: Springer.
_____. 2007a. "The Macro Model of the Inequality Process and The Surging Relative Frequency of Large Wage Incomes". Pp. 171-196 in A. Chatterjee and B.K. Chakrabarti, (eds.), ***The Econophysics of Markets and Networks*** (Proceedings of the Econophys-Kolkata III Conference, March 2007 [http://www.saha.ac.in/cmp/econophys3.cmp/]). Milan: Springer [ISBN: 978-8847006645]. [on-line at: http://arxiv.org/abs/0705.3430].
_____. 2007b. "The Inequality Process is an evolutionary process". ***The Constructal Theory of Social Dynamics***. Adrian Bejan and Gilbert Merkx, eds. (Proceedings of the Conference on the Constructal Theory of Social Dynamics, Duke University, April 2006). New York: Springer [ISBN: 978-0387476803].
_____. 2006a. "Not a Hollowing Out, A Stretching: Trends in U.S. Nonmetro Wage Income Distribution, 1961-2003". Self published as a downloadable *.pdf file on RePEc/MPRA. [on-line at: http://mpra.ub.uni-muenchen.de/10111/].
_____. 2006b (received 8/05; electronic publication: 12/05; hardcopy publication 7/06). "The Inequality Process as a wealth maximizing algorithm"'. ***Physica A: Statistical Mechanics and Its Applications*** 367:388-414 (DOI information: 10.1016/j.physa.2005.11.017). A draft version of the paper can be downloaded from the Luxembourg Income Study website: [http://www.lisproject.org/publications].
_____. 2006c. "A measure of intergroup discrimination: color and wage income in the nonmetropolitan U.S.". ***Proceedings of the 2006 Joint Statistical Meetings*, (**American Statistical Association, Social Statistics Section), August, 2006. Pp, 1889-1894. [CD-ROM], Alexandria, VA: American Statistical Association.
_____. 2006d. "The Inequality Process as an evolutionary process". ***Proceedings (late breaking papers) of the Genetic and Evolutionary Computation Conference (GECCO), July, 2006***. Late breaking paper (lbp) #137.
_____. 2006e. "A comment on Gallegati et al.'s "Worrying Trends in Econophysics" ". Pp. 250-253 in A. Chatterjee and B.K. Chakrabarti, (eds.), ***The Econophysics of Stocks and Other Markets*** (Proceedings of the Econophys Kolkata II Conference, February, 2006 [http://www.saha.ac.in/cmp/econophys2.cmp/]). Milan: Springer. (ISBN: 978-8847005013).
_____. 2003a. "The dynamics of the distribution of wage and salary income in the nonmetropolitan U.S.". ***Estadistica***.55: 59-93.
_____. 2003b. "Inequality Process, The". An entry in T. Liao, et al., (eds.), ***The Encyclopedia of Social Science Research Methods***. Volume 2: 488-490. Thousand Oaks, CA: Sage.
_____. 2003c. "Imitating the salamander: a model of the right tail of the wage distribution truncated by topcoding". November, 2003 ***Conference of the Federal Committee on Statistical Methodology***, [http://www.fcsm.gov/events/papers2003.html].
_____. 2002a. "The statistical signature of pervasive competition on wages and salaries". ***Journal of Mathematical Sociology***. 26:217-270. [abstracted in Cambridge Science Abstracts and Sociological Abstracts].
_____. 2002c. "Contingent forecasting of bulges in the left and right tails of the nonmetro wage and salary income distribution". ***Proceedings of the 2002 Federal Forecasters' Conference***. Washington, DC: U.S. Government Printing Office.
_____. 1999. "Contingent forecasting of the size of a vulnerable nonmetro population". ***Proceedings of the 1999 Federal Forecasters' Conference***. Pp. 161-169. Washington, DC: U.S. Government Printing Office.

_____. 1997. "A theory of income distribution". ***1997 Proceedings of the American Statistical Association*****,** Social Statistics Section. Pp. 388-393. Alexandria, VA: American Statistical Association.
_____. 1996. "How the gamma law of income distribution appears invariant under aggregation". ***Journal of Mathematical Sociology***. 21:325-358.
_____, 1993a. "Deriving the size distribution of personal wealth from 'the rich get richer, the poor get poorer'". ***Journal of Mathematical Sociology*** 18:27-46.
_____, 1992. "The Inequality Process and the distribution of income to blacks and whites". ***Journal of Mathematical Sociology*** 17:77-98.
_____. 1990. "A stochastic interacting particle system model of the size distribution of wealth and income." ***1990 Proceedings of the American Statistical Association*****,** Social Statistics Section. Pp. 279-284. Alexandria, VA: American Statistical Association.
_____, 1986. "The surplus theory of social stratification and the size distribution of Personal Wealth." **Social Forces** 65:293-326.
_____. 1983. "The surplus theory of social stratification and the size distribution of Personal Wealth." ***1983 Proceedings of the American Statistical Association*****,** Social Statistics Section**.** Pp. 395-400. Alexandria, VA: American Statistical Association.
Aptech Systems, Inc. (2021). GAUSS (Version 21). Phoenix, AZ: Aptech Systems.

Byrro Ribeiro, Marcelo. 2020. ***Income Distribution Dynamics of Economic Systems: An Econophysics Approach.*** Cambridge, UK: Cambridge University Press.

Childe, V. Gordon. 1944. "Archeological Ages as Technological Stages." ***Journal of the Royal Anthropological Institute of Great Britain and Ireland***. 74:7-24.

Finlay, Richard. 2009. ***The Variance Gamma (VG) Model With Long Run Dependence: A model for financial data incorporating long range dependence in squared returns***. Saarbrücken, Germany: VDM Verlag Dr. Müller.
Fiorani, Filo. 2009. ***Option Pricing Under The Variance Gamma Process***. Saarbrücken, Germany: VDM Verlag Dr. Müller.

Georgakoupoulos, Harry. 2015. ***Quantitative Trading With R: Understanding Mathematical and Computational Tools from a Quant's Perspective.*** New York: Palgrave Macmillan.

Herskovits, Melville J. 1940. ***The Economic Life of Primitive Peoples***. New York: Knopf.
Harris, Marvin. 1959. "The Economy Has No Surplus?" ***The American Anthropologist***. 61 :185-99.

Kirkpatrick, S., C.D. Gelatt, M.P. Vecchi. 1983. "Optimization via simulated annealing". Science 220: 671-680.
Kleiber, Christian, and Samuel Kotz. 2003. ***Statistical Size Distributions in Economics and Actuarial Science***. Hoboken, New Jersey: Wiley.
Kotz, Samuel, Tomasz Kozubowski, Krzysztof Podgorski. 2001. ***The Laplace Distribution and Generalizations***. Boston, Massachusetts: Birkhäuser.

Lenski, Gerhard. 1966. ***Power and Privilege***. New York: McGraw-Hill.
Lux, Thomas. 2005. "Emergent statistical wealth distributions in simple monetary exchange models: a critical review". Pp. 51-60 in A. Chatterjee, S. Yarlagadda, and B.K. Chakrabarti, (eds.), ***Econophysics of Wealth Distributions***, (the proceedings volume of

the International Workshop on the Econophysics of Wealth Distributions, March, 2005, Kolkata, India.

Madan, Dilip, and Eugene Seneta. 1990. "The Variance Gamma (V.G.) Model for Share Market Returns". ***The Journal of Business***. Vol. 63, #4, pp. 511-524.
Malkiel, Burton. 2015[1973]. ***A Random Walk Down Wall Street.*** New York: Norton.
Morgan, Lewis Henry. (1877) 1963. ***Ancient Society*** (Eleanor B. Leacock, ed.). New York: World Publishing.

Resnick, Sidney. 2007. ***Heavy-Tail Phenomena: Probabilistic and Statistical Modeling.*** New York: Springer.
Ruppert, David and David Matteson. 2015. ***Statistics and Data Analysis for Financial Engineering. 2nd edition.*** New York: Springer.

Seneta, Eugene. 2004. "Fitting the Variance-Gamma Model to Financial Data". ***Journal of Applied Probability/Stochastic Methods and Their Application***. Vol. 41, pp. 177-187.

Tsay, Ruey. 2013. ***An Introduction to Analysis of Financial Data with R.*** New York: Wiley.

Weatherall, James. 2014. ***The Physics of Wall Street.*** Boston: Mariner.
Whitney, Charles. 1990. ***Random Processes in Physical Systems***. New York:Wiley.
Wilmott, Paul and David Orrell. 2017. ***The Money Formula***. Chichester, United Kingdom: John Wiley and Sons.