Option Pricing with Higher-order Stochastic Volatility Models

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Abstract
We study the performance of higher-order stochastic volatility [SV(p)] models in the valuation of options. This class of models provides more flexibility to represent volatility persistence and heavy tails and are natural extensions of the leading Hull and White (1987) model used in option pricing. A simulation-based option pricing algorithm is developed, which uses the winsorized ARMA-based estimator of Ahsan and Dufour (2021). The proposed algorithm is applied to S&P 500 European call options (2015-2019). We find that the SV(3) model provides the smallest pricing error among the competing models in all levels of moneyness. Our findings highlight the usefulness of higher-order SV models for option pricing.

Key Words: Option pricing, stochastic volatility, financial time series.

1. Introduction

Modelling time-varying volatility of asset returns is pivotal in option pricing. To deal with such features, two main classes of parametric models have been proposed: (1) ARCH [Engle (1982)] and GARCH models [Bollerslev (1986)], where volatility is modelled as a deterministic function of past shocks; (2) stochastic volatility (SV) models [Taylor (1986)], where volatility is a latent stochastic process. Several reviews of GARCH and SV literature are available; for GARCH, see Bollerslev (2010), and for SV, see Ghysels et al. (1996), Broto and Ruiz (2004), and Shephard (2005).

Several studies have documented the superior performance of SV models over GARCH-type models for several reasons. *First*, SV models constitute discrete versions of continuous-time diffusion processes, which are widely used in the option-pricing literature; see Hull and White (1987), Taylor (1994), Shephard and Andersen (2009). *Second*, SV models are flexible and relatively robust to model misspecification. GARCH models often require adding a random jump component or allowing for innovations with heavy-tailed distributions to tackle these problems. Such modifications substantially improve the performance of the standard GARCH, but do not appear to be required for SV models; see Carnero et al. (2004), Chan and Grant (2016). *Finally*, SV models perform better than GARCH-type models in volatility forecasting, which suggests that time-varying volatility is better modelled as a latent first-order autoregression; see Kim et al. (1998), Yu (2002), Poon and Granger (2003), Koopman et al. (2005).

Despite these attractive features, the estimation of SV models is much more complicated than it is for GARCH-type models. In particular, due to the presence of latent variables, likelihood-based methods are difficult to apply, and statistical inference (estimation and testing) for SV models is quite challenging. Consequently, a variety of methods have been proposed to estimate SV(1) model, where the latent volatility process is modelled as a first-order autoregression. These include: quasi-maximum likelihood (QML) [Nelson (1988), Harvey et al. (1994), Ruiz (1994)], the generalized method of moments (GMM)

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[Melino and Turnbull (1990), Andersen and Sørensen (1996)], the simulated method of moments (SMM) [Gallant and Tauchen (1996), Monfardini (1998), Andersen et al. (1999)], Monte Carlo likelihood (MCL) [Sandmann and Koopman (1998)], simulated maximum likelihood (SML) [Danielsson (1994), Durham (2006), Durham (2007), Richard and Zhang (2007)], the method based on linear representation [Francq and Zakoïan (2006)], closed-form moment-based estimators [Dufour and Valéry (2006), Dufour and Valéry (2009), Ahsan and Dufour (2019)], and Bayesian techniques based on Markov Chain Monte Carlo (MCMC) methods [Jacquier et al. (1994), Kim et al. (1998), Chib et al. (2002), Fiorentini et al. (2004), Flury and Shephard (2011)].

The vast majority of the above methods are either computer-intensive and/or inefficient. Apart from the closed-form moment-based estimators, the above estimation methods are based on simulation techniques and/or numerical optimization. Simulation-based methods such as SML, MCL, SMM, and Bayesian MCMC methods [via the Metropolis-Hastings algorithm or the Gibbs sampler] are computer-intensive, inflexible across models, hard to implement in practice, and may converge very slowly; see Broto and Ruiz (2004). Implementing these methods requires one to choose a sampling scheme, initial parameters, and an auxiliary model (which is largely conventional). The choice of initial parameter values for QML, GMM or MCMC plays a pivotal role in convergence. In particular, a poorly assigned prior may lead to a fragile Bayesian inference. In the context of GMM estimation, Broto and Ruiz (2004) pointed out that the criterion surface is highly irregular, so optimization often fails to converge in small samples, *e.g.*, Andersen and Sørensen (1996) have documented a large number of non-converging GMM estimations. Further, GMM usually produces imprecise estimates due to an ill-conditioned weighting matrix. By contrast, the closed-form moment-based estimators are analytically tractable, computationally simple, and very easy to implement.

In this paper, we consider higher-order stochastic volatility [SV(p)] models for option pricing. In an SV(p) model, the underlying latent volatility process follows an autoregressive process of order p. The estimation of SV(p) models is even more challenging than it is for an SV(1) model. Due to the intrinsic complexity of SV(p) models, the work on estimating this class of models remains scarce. Most of the proposed ones are inflexible, computationally costly, and limited to low orders [see Gallant et al. (1997), Asai (2008), Chan and Grant (2016)]. Recently, Ahsan and Dufour (2021) proposed simple estimation methods for SV(p) models by exploiting the non-Gaussian ARMA representation of these models. These ARMA-SV methods use the moment structure of the logarithm of squared residual returns. Furthermore, these estimators are analytically tractable and computationally inexpensive.

We develop an option pricing algorithm for SV(p) models, which uses the recently developed OLS-based winsorized version of the ARMA-SV (W-ARMA-SV) estimator. This estimator substantially increases the probability of getting acceptable parameter values and also improves efficiency. We apply the proposed algorithm to 5 years of S&P 500 European daily call options. In all levels of moneyness, we find that the SV(3) model provides the smallest pricing error compare to the GARCH$(1,1)$ model.

The paper proceeds as follows: Section 2 specifies SV(p) models and assumptions. Section 3 discusses simple estimators and the recursive prediction algorithm. Section 4 develops a option pricing algorithm for SV(p) models. Section 5 assesses the pricing performance. Section 6 concludes.

2. Framework

We consider a standard discrete-time SV process of order p, which is described below following Ghysels et al. (1996) and Gallant et al. (1997). Specifically, we say that a variable y_t follows a discrete-time stationary SV(p) process if it satisfies the following assumptions, where $t \in \mathbb{N}_0$, and $\mathbb{N}_0$ represents the non-negative integers.

Assumption 2.1 (Stochastic volatility of order p) The process $\{y_t : t \in \mathbb{N}_0\}$ satisfies the equations

$$y_t = \sigma_y \exp(w_t/2) z_t \,, \tag{1}$$

$$w_t = \sum_{j=1}^{p} \phi_j w_{t-j} + \sigma_v v_t \,, \tag{2}$$

where the vectors $(z_t, v_t)'$ are i.i.d. according to a $\mathcal{N}(0,\, I_2)$ distribution, while $(\phi_1,\, \ldots,\, \phi_p,\, \sigma_y,\, \sigma_v)'$ are fixed parameters.

Assumption 2.2 (Stationarity) The process $l_t = (y_t, w_t)'$ is strictly stationary.

The latter assumption entails that all the roots of the characteristic equation of the volatility process $[\phi(z) = 0]$ lie outside the unit circle [*i.e.*, $\phi(z) \neq 0$ for $|z| \leq 1$]. The SV(p) model consists of two stochastic processes, where y_t describes the dynamics of asset returns and $w_t := \log(\sigma_t^2)$ captures the dynamics of latent log volatilities.[1] The latent process w_t can be interpreted as a random flow of uncertainty shocks or new information in financial markets, while the ϕ_j's capture volatility persistence.

Let us now transform y_t by taking the logarithm of its squared value. We get in this way the following *measurement equation*:

$$\log(y_t^2) = \log(\sigma_y^2) + w_t + \log(z_t^2) = \mu + w_t + \epsilon_t \tag{3}$$

where $\mu := \mathbb{E}[\log(y_t^2)] = \log(\sigma_y^2) + \mathbb{E}[\log(z_t^2)]$ and $\epsilon_t := \log(z_t^2) - \mathbb{E}[\log(z_t^2)]$. Under the normality assumption for z_t, the errors ϵ_t are i.i.d. according to the distribution of a centered $\log(\chi_1^2)$ random variable [*i.e.*, ϵ_t has mean zero and variance $\mathbb{E}(\epsilon_t^2)$] with

$$\mathbb{E}[\log(z_t^2)] \simeq -1.27 \,, \quad \sigma_\epsilon^2 := \mathbb{E}(\epsilon_t^2) = \pi^2/2 \,, \tag{4}$$

$$\mathbb{E}(\epsilon_t^3) = \psi^{(2)}(1/2) \,, \quad \mathbb{E}(\epsilon_t^4) = \pi^4 + 3\sigma_\epsilon^2 \,, \tag{5}$$

where $\psi^{(2)}(z)$ is the *polygamma function* of order 2; see Abramowitz and Stegun (1970, Chapter 6).[2] On setting

$$y_t^* := \log(y_t^2) - \mu \,, \tag{6}$$

and by combining (2) and (3), the SV(p) model can be written in state-space form:

$$\text{State Transition Equation:} \quad w_t = \sum_{j=1}^{p} \phi_j w_{t-j} + v_t \,, \tag{7}$$

$$\text{Measurement Equation:} \quad y_t^* = w_t + \epsilon_t \,, \tag{8}$$

where v_t's are i.i.d. $\mathcal{N}(0, \sigma_v^2)$ and ϵ_t's are i.i.d. $\log(\chi_1^2)$; for further discussion of this representation, see Nelson (1988), Harvey et al. (1994), Ruiz (1994), Shephard (1994), Breidt and Carriquiry (1996), Harvey and Shephard (1996), Kim et al. (1998), Sandmann and Koopman (1998), Steel (1998), Chib et al. (2002), Knight et al. (2002), Francq and Zakoïan (2006), Omori et al. (2007).

[1] Usually the y_t's are residual returns, such that $y_t := r_t - \mu_r$ and $r_t := 100[\log(p_t) - \log(p_{t-1})]$, where μ_r is the mean of returns (r_t) and p_t is the raw prices of an asset. It is noteworthy to mention that y_t is ordinarily the error term of any time series regression model, see Jurado et al. (2015).

[2] The $\log(\chi_1^2)$ distribution is often approximated by a normal distribution with mean of -1.2704 and variance of $\pi^2/2$ [see Broto and Ruiz (2004)], or by a mixture distribution [see Kim et al. (1998)].

3. Simple ARMA-based estimation

Recently, Ahsan and Dufour (2021) proposed simple estimators for SV(p) models by exploiting the ARMA representation of the process y_t^*. They derived the following ARMA(p,p) representation for y_t^*:

$$y_t^* = \sum_{j=1}^{p} \phi_j y_{t-j}^* + \eta_t - \sum_{j=1}^{p} \theta_j \eta_{t-j} \tag{9}$$

with $\eta_t - \sum_{j=1}^{p} \theta_j \eta_{t-j} = v_t + \epsilon_t - \sum_{j=1}^{p} \phi_j \epsilon_{t-j}$, where the error processes $\{v_t\}$ and $\{\epsilon_t\}$ are mutually independent, the errors v_t are i.i.d. $N(0, \sigma_v^2)$, and the errors ϵ_t are i.i.d. according to the distribution of a $\log(\chi_1^2)$ random variable.

From the above expression, y_t^* has the following autocovariances:

$$\mathrm{cov}(y_t^*, y_{t-k}^*) := \gamma_{y^*}(k) = \begin{cases} \phi_1 \gamma_{y^*}(k-1) + \cdots + \phi_p \gamma_{y^*}(k-p) + \sigma_v^2 + \sigma_\epsilon^2; & \text{if } k = 0, \\ \phi_1 \gamma_{y^*}(k-1) + \cdots + \phi_p \gamma_{y^*}(k-p) - \phi_k \sigma_\epsilon^2; & \text{if } 1 \le k \le p, \\ \phi_1 \gamma_{y^*}(k-1) + \cdots + \phi_p \gamma_{y^*}(k-p); & \text{if } k > p. \end{cases} \tag{10}$$

The above autocovariances yield the following closed-form expressions for SV parameters:

$$\boldsymbol{\phi_p} = \boldsymbol{\Gamma_{(p+j-1)}^{-1}} \boldsymbol{\gamma_{(p+j)}}, \quad j \ge 1 \tag{11}$$

$$\sigma_y = [\exp(\mu + 1.27)]^{1/2}, \tag{12}$$

$$\sigma_v = [\gamma_{y^*}(0) - \boldsymbol{\phi_p'} \boldsymbol{\gamma_{(1)}} - \pi^2/2]^{1/2}, \tag{13}$$

where $\boldsymbol{\phi_p} := (\phi_1, \ldots, \phi_p)'$, $\boldsymbol{\gamma_{(p+j)}} := [\gamma_{y^*}(p+j), \ldots, \gamma_{y^*}(2p+j-1)]'$ are vectors and $\boldsymbol{\Gamma_{(p+j-1)}}$ is a p-dimensional Toeplitz matrices such that

$$\boldsymbol{\Gamma_{(p+j-1)}} := \begin{bmatrix} \gamma_{y^*}(p+j-1) & \gamma_{y^*}(p+j-2) & \cdots & \gamma_{y^*}(j) \\ \gamma_{y^*}(p+j) & \gamma_{y^*}(p+j-1) & \cdots & \gamma_{y^*}(j+1) \\ \vdots & \vdots & & \vdots \\ \gamma_{y^*}(2p+j-2) & \gamma_{y^*}(2p+j-3) & \cdots & \gamma_{y^*}(p+j-1) \end{bmatrix}.$$

where p is the SV order, $\gamma_{y^*}(k) = \mathrm{cov}(y_t^*, y_{t-k}^*)$, with y_t^* and μ defined in (3).

Now, it is natural to estimate $\gamma_{y^*}(k)$ and μ by the corresponding empirical moments:

$$\hat{\gamma}_{y^*}(k) = \frac{1}{T-k} \sum_{t=1}^{T-k} y_t^* y_{t+k}^*, \qquad \hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} \log(y_t^2), \tag{14}$$

where by construction y_t^* is a mean corrected process. Setting $j = 1$ in (11) and replacing theoretical moments by their corresponding empirical moments yield the following *simple ARMA-SV* estimator of the SV(p) coefficients:

$$\hat{\boldsymbol{\phi}}_{\boldsymbol{p}} = \hat{\boldsymbol{\Gamma}}_{\boldsymbol{(k,p)}}^{\boldsymbol{-1}} \hat{\boldsymbol{\gamma}}_{\boldsymbol{(k,p)}}, \tag{15}$$

$$\hat{\sigma}_y = [exp(\hat{\mu} + 1.27)]^{1/2}, \tag{16}$$

$$\hat{\sigma}_v = [\hat{\gamma}_{y^*}(0) - \hat{\boldsymbol{\phi}}_{\boldsymbol{p}}' \hat{\boldsymbol{\gamma}}_{\boldsymbol{(k,p)}} - \pi^2/2]^{1/2}. \tag{17}$$

3.1 Restricted estimation

These simple estimators may yield a solution outside the admissible area, *i.e.*, some of the eigenvalues of the latent volatility process [it is an AR(p) process] may lie outside the unit circle or equal to unity. This issue can arise especially in small samples or in the presence of outliers. When this happens, a simple fix is projecting the estimate on the space of acceptable parameter solutions by altering the eigenvalues that lie on or outside the unit circle. The characteristic equation of the latent AR(p) process is given by $C(\lambda) = \lambda^p - \phi_1 \lambda^{p-1} - \cdots - \phi_p = 0$, and the stationary condition requires all roots lie inside the unit circle, *i.e.*, $|\lambda_i| < 1,\ i = 1, \cdots, p$. If the estimated parameters fail to satisfy this condition, then the restricted estimation can be done in the following two steps:

1. Given the estimated unstable parameters, we calculate the roots of the characteristic equation and restrict their absolute values to less than unity.

2. Given these restricted roots, we calculate the constrained parameters which ensure stationarity.

For example, in case of an SV(2) model, the characteristic equation of the latent volatility process is $C(\lambda) = \lambda^2 - \phi_1 \lambda - \phi_2 = 0$. It may have two types of roots: (i) if $\phi_1^2 + 4\phi_2 \geq 0$, then $C(\lambda)$ has two real roots, and these are given by $\lambda_{1,2} = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$ and (ii) if $\phi_1^2 + 4\phi_2 < 0$ then $C(\lambda)$ has two complex roots, and these are given by $\lambda_{1,2} = \frac{\phi_1}{2} \pm i \frac{\sqrt{-(\phi_1^2 + 4\phi_2)}}{2}$. When the estimated polynomial coefficients produce an unstable solution, then we restrict the absolute value of the roots less than unity, *i.e.* $|\lambda_{1,2}| < 1$ or $|\lambda_{1,2}| = 1 - \Delta$ where Δ is a very small number. Given these restricted roots, we solve for restricted parameters which ensure the stationarity condition. These steps can be done very easily in MATLAB. In MATLAB, the **roots** function calculates the roots given the parameters, and the **poly** function calculates the parameters given the roots.

3.2 ARMA-based winsorized estimation

One can achieve better stability and efficiency of ARMA-SV estimator by using "winsorization" which exploits (11). Winsorization (censoring) substantially increases the probability of getting admissible values. From (11), it is easy to see that:

$$\boldsymbol{\phi_p} = \sum_{j=1}^{\infty} \omega_j \boldsymbol{\Gamma}^{-1}_{(p+j-1)} \boldsymbol{\gamma}_{(p+j)} \tag{18}$$

for any ω_j sequence with $\sum_{j=1}^{\infty} \omega_j = 1$. Using (18), we can define a more general class of estimators for $\boldsymbol{\phi_p}$ by taking a weighted average of several sample analogs of the ratio $\boldsymbol{\Gamma}^{-1}_{(p+j-1)} \boldsymbol{\gamma}_{(p+j)}$:

$$\tilde{\boldsymbol{\phi}}_{\boldsymbol{p}} = \sum_{j=1}^{J} \omega_j \hat{\boldsymbol{\Gamma}}^{-1}_{(p+j-1)} \hat{\boldsymbol{\gamma}}_{(p+j)}, \tag{19}$$

where $1 \leq J \leq T - p$ with $\sum_{j=1}^{J} \omega_j = 1$ and T is the length of time series. We can expect that a sufficiently general class of weights may improve the efficiency of the ARMA-SV estimators.

Using (19), Ahsan and Dufour (2021) proposed the OLS-based *winsorized ARMA-SV* (W-ARMA-SV) estimator of $\boldsymbol{\phi_p}$ as follows:

$$\hat{\boldsymbol{\phi}}_{\boldsymbol{p}}^{\,ols} = (\bar{a}'\bar{a})^{-1}\bar{a}'\bar{e}, \tag{20}$$

where $\bar{a} = (\hat{\Gamma}_{(p)}\omega_1^{1/2}, \ldots, \hat{\Gamma}_{(p+J-1)}\omega_J^{1/2})'$ and $\bar{e} = (\hat{\boldsymbol{\gamma}}_{(\boldsymbol{p+1})}\omega_1^{1/2}, \ldots, \hat{\boldsymbol{\gamma}}_{(\boldsymbol{p+J})}\omega_J^{1/2})'$. Clearly, different OLS-based W-ARMA-SV can be generated by considering different weights $w_1, \ldots, w_J$. In our empirical applications, we focus on the case where the weights are equal ($\omega_j = 1/J$). Note that, in case of an SV(2), the W-ARMA-SV-OLS (with equal weights) yields:

$$\hat{\phi}_1^{ols} = \frac{\sum_{j=1}^{J}[\hat{\gamma}_{y^*}(j+1)\hat{\gamma}_{y^*}(j+2) - \hat{\gamma}_{y^*}(j)\hat{\gamma}_{y^*}(j+3)][\hat{\gamma}_{y^*}(j+1)^2 - \hat{\gamma}_{y^*}(j)\hat{\gamma}_{y^*}(j+2)]}{\sum_{j=1}^{J}[\hat{\gamma}_{y^*}(j+1)^2 - \hat{\gamma}_{y^*}(j)\hat{\gamma}_{y^*}(j+2)]^2} \tag{21}$$

$$\hat{\phi}_2^{ols} = \frac{\sum_{j=1}^{J}[\hat{\gamma}_{y^*}(j+1)\hat{\gamma}_{y^*}(j+3) - \hat{\gamma}_{y^*}(j+2)^2][\hat{\gamma}_{y^*}(j+1)^2 - \hat{\gamma}_{y^*}(j)\hat{\gamma}_{y^*}(j+2)]}{\sum_{j=1}^{J}[\hat{\gamma}_{y^*}(j+1)^2 - \hat{\gamma}_{y^*}(j)\hat{\gamma}_{y^*}(j+2)]^2}. \tag{22}$$

The above simplification [simple regressions] follows from (11) with $p = 2$, which can be written as following:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \gamma_{y^*}(j+1) & \gamma_{y^*}(j) \\ \gamma_{y^*}(j+2) & \gamma_{y^*}(j+1) \end{bmatrix}^{-1} \begin{bmatrix} \gamma_{y^*}(j+2) \\ \gamma_{y^*}(j+3) \end{bmatrix} = \begin{bmatrix} \frac{\gamma_{y^*}(j+1)\gamma_{y^*}(j+2) - \gamma_{y^*}(j)\gamma_{y^*}(j+3)}{\gamma_{y^*}(j+1)^2 - \gamma_{y^*}(j)\gamma_{y^*}(j+2)} \\ \frac{\gamma_{y^*}(j+1)\gamma_{y^*}(j+3) - \gamma_{y^*}(j+2)^2}{\gamma_{y^*}(j+1)^2 - \gamma_{y^*}(j)\gamma_{y^*}(j+2)} \end{bmatrix}. \tag{23}$$

All these estimators are depend on J and for $J = 1$, they are equivalent to the simple ARMA-SV estimator which is given by (15).

3.3 Recursive estimation for SV(p) models

It is possible to estimate higher-order SV(p) models using a recursive Durbin-Levinson (DL) type estimation algorithm. For notational convenience, we use a different indexation for the autoregressive parameters of the volatility process [only for this section]. For example, the SV(p) parameters are now denoted by $\Theta_p^{SV} := (\{\phi_{p,j}\}_{j=1}^{p}, \sigma_{pv}, \sigma_y)'$.

The recursive estimation of the ARMA-SV estimator exploits extended Yule-Walker (EYW) equations of the observed process. When the MA order is fixed, the system of the EYW equations constitutes a nested Toeplitz system. A *Generalized Durbin-Levinson* algorithm for the ARMA-SV estimator for SV(p) model is useful when neither the AR order nor the MA order is known. We consider the case $i = p$, *i.e.*, the MA order is p, which also implies that the AR order is p.

For $i = 0$, use the Durbin-Levinson algorithm to calculate

$$\{\hat{\phi}_{p,j}^{(0)} \mid p \geq 1, \ j = 1, \ldots, p\}.$$

For $i \geq 1$, calculate

$$\hat{\phi}_{p,0}^{(i-1)} = -1,$$

and

$$\hat{\phi}_{p,j}^{(i)} = \hat{\phi}_{p+1,j}^{(i-1)} - \frac{\hat{\phi}_{p+1,p+1}^{(i-1)}}{\hat{\phi}_{p,p}^{(i-1)}}\hat{\phi}_{p,j-1}^{(i-1)}, \text{ where } p \geq 1, \ j = 1, \ldots, p,$$

$$\hat{\sigma}_y = [\exp(\hat{\mu} + 1.27)]^{1/2},$$

$$\hat{\sigma}_{pv} = [\hat{\gamma}_{y^*}(0) - \sum_{j=1}^{p}\hat{\phi}_{p,j}\hat{\gamma}_{y^*}(j) - \pi^2/2]^{1/2}.$$

This algorithm is the same as Tsay and Tiao (1984) algorithm [except for equations involving $\hat{\sigma}_y$ and $\hat{\sigma}_{pv}$] for calculating the extended sample autocorrelation function under the stationarity assumption.

4. Option pricing with SV(p) models

The main reason for estimating stochastic volatility models for option pricing is that we are interested in the time series of volatilities and/or the volatility estimates' accuracy. Volatilities can be estimated using a Kalman smoother conditional on the parameters assuming that the measurement error ϵ_t is Gaussian. Alternatively, by a simulation smoother (extended Kalman filter) conditional on the parameters using the mixture distribution of the measurement error density $f(\epsilon_t)$.

Option prices depend on the average expected volatility over the option contract's length, and this averaging should reduce standard errors. In the limit, the average volatility over a long horizon converges to the unconditional variance, which is known without error when conditioning the process's parameters.

Under a set of assumptions, Hull and White (1987) showed that the value of a European call option on stocks based on a general specification of stochastic volatility is the Black-Scholes price integrated over the distribution of the mean volatility. Using a characteristic function approach, Heston (1993) derived a closed-form solution for a European call option based on a square-root specification of volatility. For most other SV models, including SV(p) models, option prices have no closed form solution and hence have to be approximated. A flexible way for approximating option prices is via Monte Carlo simulations. Hull and White (1987) outlined an efficient procedure for conducting Monte Carlo simulations to calculate a European call option on stocks.

To examine the economic importance of SV(p) models on option pricing, we price options using both the SV(1) and SV(p) models. To price options, we follow an algorithm similar to Mahieu and Schotman (1998) and Yu et al. (2006). Table 1 summarizes some well-known SV models and shows their parameter relations. For the continuous-time SV models, their Euler discrete-time versions are considered. Some specifications in Table 1 may be different from the actual specifications given in the original papers. However, they are equivalent to each other via Ito's lemma. SV(p) models are natural extensions of Hull and White (1987) model used in the option pricing literature.

Consider a European call option on a stock with maturity τ (measured in the number of days). The value of the call option is given by

$$C(S_t, w_{0,t:t-p}) = e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}[\max(S_{t+\tau} - X,\ 0)].$$

where S_t is the spot price of the underlying index at time t, r is the risk-free rate (an annual rate, expressed in terms of continuous compounding), and X is the strike price. The expected payoff is a function of both the current and past lags of log-volatilities $w_{0,t:t-p} = (w_{0,t}, \ldots, w_{0,t-p})'$ and the current spot price S_t. The expectation is taken with respect to the risk-neutral density of the stock price.

Hull and White (1987) assume a continuous-time SV process and show that the value of the option depends on the expected average variance of the exchange rate over the remaining life of the option, conditional on the current volatility. Following Hull and White (1987) and assuming that volatility risk is not priced, the pricing formula can be written as follows:

$$C(S_t, w_{0,t:t-p}) = e^{-r\tau}\int_0^\infty \mathcal{BS}(\widetilde{w}_{t+\tau})p(\widetilde{w}_{t+\tau}|w_{0,t:t-p})\mathrm{d}\widetilde{w}_{t+\tau} = e^{-r\tau}\mathbb{E}_t[\mathcal{BS}(\widetilde{w}_{t+\tau})]\,,$$

where $\mathcal{BS}(\widetilde{w}_{t+\tau})$ is the Black-Scholes value

$$\mathcal{BS}(\widetilde{w}_{t+\tau}) = F_t\mathcal{N}(d_1) - X\mathcal{N}(d_2)$$

Table 1: Alternative specifications for the latent volatility process

Studies	Models
Taylor (1982), Wiggins (1987), Chesney and Scott (1989), Jacquier et al. (1994), Kim et al. (1998) and Scott (1987)	$\log \sigma_t^2 = \mu + \phi(\log \sigma_{t-1}^2 - \mu) + \sigma_v v_t$
Scott (1987), Andersen (1994), and Stein and Stein (1991)	$\sigma_t = \mu + \phi(\sigma_{t-1} - \mu) + \sigma_v v_t$
Heston (1993)	$\sigma_t = \phi\sigma_{t-1} + \sigma_v v_t$
Hull and White (1987) and Johnson and Shanno (1987)	$\log \sigma_t^2 = \mu + \phi \log \sigma_{t-1}^2 + \sigma_v v_t$
Andersen (1994)	$\sigma_t^2 = \mu + \phi(\sigma_{t-1}^2 - \mu) + \sigma_v v_t$
Clark (1973)	$\log \sigma_t^2 = \mu + \sigma_v v_t$
Yu et al. (2006)	$\frac{(\sigma_t^2)^\delta - 1}{\delta} = \mu + \phi\left(\frac{(\sigma_{t-1}^2)^\delta - 1}{\delta} - \mu\right) + \sigma_v v_t$
Ahsan and Dufour (2021)	$\log \sigma_t^2 = \mu + \phi_j \sum_{j=1}^{p} \log \sigma_{t-j}^2 + \sigma_v v_t$

Notes:

1. Yu et al. (2006) proposed a nonlinear SV specification with the inverse Box-Cox transformation and δ is a parameter of the smooth function.

in which F_t is the forward price applying to time $t + \tau$, and d_1 and d_2 are defined as

$$d_1 = \frac{\ln(F_t/X) + \frac{1}{2}\widetilde{w}_{t+\tau}^2}{\widetilde{w}_{t+\tau}}, \quad d_2 = d_1 - \widetilde{w}_{t+\tau}$$

$$\widetilde{w}_{t+\tau}^2 = \int_t^{t+\tau} \exp(w_s^2)\mathrm{ds}$$

The expectation is taken with respect to the conditional density $p(\widetilde{w}_{t+\tau}|w_{0,t:t-p})$ of the total lifetime volatility $\widetilde{w}_{t+\tau}$ given $w_{0,t:t-p}$. Since we assume that volatility risk is not priced, the density $p(\widetilde{w}_{t+\tau}|w_{0,t:t-p})$ coincides with the actual density. Note that the expectation is taken conditional on knowing the current and past $\log$-volatilities $w_{0,t:t-p}$, *i.e.* assuming that the market knows the volatility. The density $p(\widetilde{w}_{t+\tau}|w_{0,t:t-p})$ is a function of the parameters of the stochastic volatility process, but does not involve any data information. The econometric problem is to estimate the option value given time-series data of stock prices, but without directly observing log volatilities $w_{0,t:t-p}$ — these volatilities are only available through the stochastic volatility model. In discrete time we replace the integral $\int_t^{t+\tau} \exp(w_s^2)\mathrm{ds}$ by the summation $\widetilde{w}_{t+\tau}^2 = \sum_{i=1}^{\tau} \exp(w_{t+\tau})$ (see Amin and Ng (1993)). Further, assuming continuous yield dividends $F_t = S_t e^{(r-q)\tau}$, where q is the constant dividend yield. The value of the European call option can be computed by direct simulation given in Algorithm 1.

5. Option valuation performance

In this section, the SV(p) option pricing algorithm (proposed in Algorithm 1) is applied to daily S&P 500 index European call options. We consider five years of option prices from Jan. 2015 to Dec. 2019. The data is sourced from the OptionMetrics database. We also obtain daily S&P 500 prices from the Yahoo Finance website. All options were quoted on the CBOE. The midpoint between bid and ask prices is used as the option price. We filter zero volume quotes and apply the filtering rules suggested by Bakshi et al. (1997) and only keep options with 126 days to maturity. We fix the daily risk-free rates to $r = 0.0039\%$, $q = 0$ and $K = 100,000$.

Algorithm 1 SV(p) option pricing algorithm

Ensure: Initialize with drawing $\hat{w}_{0,t:t-p} = (\hat{w}_t, \ldots, \hat{w}_{t-p})'$ from the simulation smoother.

1: **for** $k \leftarrow 1$ to K **do**
2: **for** $i \leftarrow 1$ to τ **do**
3: Draw $v_i \sim \mathcal{N}(0, 1)$
4: Generate a sequence of $w_{t+i},\ i = 1, \ldots, \tau$ according to $w_{t+i} = \hat{\mu} + \sum_j^p \hat{\phi}_j w_{t+i-j} + \hat{\sigma}_v v_{t+i}$
5: Calculate $\widetilde{w}^2_{t+\tau} = \sum_{i=1}^{\tau} \exp(w_{t+i})$
6: Calculate $d_1 = \frac{\ln(F_t/X) + \frac{1}{2}\widetilde{w}^2_{t+\tau}}{\widetilde{w}_{t+\tau}}, \quad d_2 = d_1 - \widetilde{w}_{t+\tau}$
7: Calculate the Black-Scholes value $\mathcal{BS}_1(\widetilde{w}_{t+\tau}) = F_t \mathcal{N}(d_1) - X\mathcal{N}(d_2)$
8: Repeat Steps 4-7 using $\{-v_i\}$ and calculate the Black-Scholes value $\mathcal{BS}_2(\widetilde{w}_{t+\tau})$
9: Calculate the average value of $\mathcal{BS}_1(\widetilde{w}_{t+\tau})$ and $\mathcal{BS}_2(\widetilde{w}_{t+\tau})$ and define it as $\bar{\mathcal{BS}}(\widetilde{w}_{t+\tau})$
10: **end for**
11: Repeat Steps 2-10 for K times to give a sequence of $\bar{\mathcal{BS}}^{(k)}(\widetilde{w}_{t+\tau})$ values.
12: Compute the Hull-White-type option price as $\mathcal{HW}(\widetilde{w}_{t+\tau}) = e^{-r\tau}\frac{1}{K}\sum_{i=1}^{K} \bar{\mathcal{BS}}^{(k)}(\widetilde{w}_{t+\tau})$
13: **end for**

Require: Parameter estimates of the SV(p) model: $\hat{\theta} = \left(\hat{\mu}, \hat{\phi}_1, \ldots, \hat{\phi}_p, \hat{\sigma}_v\right)'$.
Require: Filtered latent log volatilities $\hat{w}_{0,t:t-p} = (\hat{w}_t, \ldots, \hat{w}_{t-p})'$.
Require: Observed values from option and price data: τ, r, S_t, X, q.

We consider only Out-of-The-Money (OTM) call options. Using X/S_t as a definition of moneyness, we filter out Deep-Out-The-Money (DOTM) options with moneyness larger than 1.25 for call options. A wider range of moneyness could be considered. However, for DOTM options, all models should require a correction to include the impact of jumps. We investigated what happens for moneyness ranging from 0.95 to 1.25. We refer to call as DOTM options if their moneyness is between 1.1 and 1.25 and as Out-The-Money (OTM) if $1.02 < X/S_t \leq 1.1$. Options are at-the-money (ATM) if $0.95 < X/S_t \leq 1.02$.

For each filtered call option, we recalibrate the model with new parameter estimates. So the models are estimated using asset returns from Jan. 1996 to the day the option was recorded. This removes the effect of measurement error, which may cause some instability in model parameters. We report the estimated parameter and standard deviation across samples in Table 2. From the estimated models, we get the parameter estimates and filtered current and past volatilities. Using these parameter estimates, the evaluation of option prices via Monte Carlo is straightforward.

We consider relative root mean squared error, %RMSE, which is computed as follows

$$\%RMSE = \frac{\$RMSE}{\bar{V}}, \quad \text{where} \quad \$RMSE = \sqrt{\frac{1}{N}\sum_{i}^{N}\left(V_i - V_i^{model}\right)^2},$$

where V_i is the observed price, V_i^{model} is the model price and $\bar{V}$ is the average price. We consider three lower-order SV(p) models and GARCH$(1,1)$ of Heston and Nandi (2000).

Table 3 reports the performance of the competing models. The main findings are the following. *First*, the SV(3) model provides the smallest pricing error among the competing models. This finding holds across different levels of moneyness. *Second*, in all levels of moneyness, the GARCH$(1,1)$ model of Heston and Nandi (2000) is outperformed by all SV models: pricing errors of the GARCH$(1,1)$ model are large.

Table 2: GARCH and SV estimates: Estimation Period (Jan 1996- Dec 2019)

	GARCH Parameters	λ	ω	α	β	γ
GARCH$(1,1)$	Estimates	0.0055	1.77E-08	5.11E-06	0.8201	139.7759
	SD	0.0003	4.51E-08	3.50E-07	0.0110	0.8385
	SV Parameters	ϕ_1	ϕ_2	ϕ_3	σ_y	σ_v
SV(1)	Estimates	0.9849			0.8881	0.7609
	SD	0.0035			0.0270	0.0367
SV(2)	Estimates	0.3385	0.6470		0.8881	0.5982
	SD	0.0148	0.0203		0.0270	0.0540
SV(3)	Estimates	0.1404	0.3951	0.4402	0.8881	0.5622
	SD	0.0541	0.1133	0.0637	0.0270	0.0825

Notes: The models are estimated using S&P returns from Jan. 1996 to the day the option was recorded. This removes the effect of measurement error, which may cause some instability in model parameters. We report the mean estimated parameter (Estimates) and standard deviation (SD) across samples.

Table 3: Option-Pricing Results: Estimation Period (Jan 2015- Dec 2019)

%RMSE relative to GARCH$(1,1)$ of Heston and Nandi (2000)				
	Monyness	SV(1)	SV(2)	SV(3)
All	$(0.95 \leq X/S_t \leq 1.25)$	0.3760	0.3946	0.3535
ATM	$(0.95 \leq X/S_t \leq 1.05)$	0.2188	0.2534	0.2022
OTM	$(1.05 < X/S_t \leq 1.15)$	0.4044	0.4210	0.3782
DOTM	$(1.15 < X/S_t \leq 1.25)$	0.4452	0.4556	0.4238

Notes: These %RMSE are relative to the reference model GARCH$(1,1)$ of Heston and Nandi (2000) and values smaller than unity indicate better pricing performance than the GARCH$(1,1)$ model.

6. Conclusion

We consider higher-order SV models for forecasting volatility and pricing options, in which we depart from the existing literature that uses only the first-order persistence in latent volatility. We estimate this class of models by the simple W-ARMA estimator proposed by Ahsan and Dufour (2021). Given the simple estimates with the procedures developed in Section 4, pricing derivatives are straightforward. Empirically, we find that the SV(3) model provides the smallest pricing error among the competing models in all levels of moneyness. This finding highlights the usefulness of higher-order SV models for option pricing.

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