Arithmetically and/or Geometrically Progressed Discrete Probability Distributions, Stochastic Processes and the Generalized Linear Models

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Abstract
Various arithmetically and/or geometrically distributed discrete probability distributions viz arithmetically and/or geometrically progressed Binomial, Multinomial, Poisson, Geometric, Negative Binomial Distributions have been developed. Several features like moments, estimators of the parameters have been unfolded. Some arithmetically and/or geometrically distributed Stochastic Processes like arithmetically and/or geometrically progressed Poisson Processes, etc have been developed along with their features. The arithmetically and/or geometrically progressed generalized linear models like arithmetically and/or geometrically progressed Binomial, Multinomial, Poisson, Geometric, Negative Binomial Regressions have been developed. Their related properties have also been developed.

Key Words: Arithmetically and/or geometrically distributed Binomial; Arithmetically and/or geometrically progressed Poisson Processes; Arithmetically and/or geometrically progressed generalized linear models, slope of success(es), amplification of success(es).

1. Introduction

During a single trial in a discrete probability distribution, either 0, or 1 or more than one successes may occur. If the occurrences of these successes follow a pattern or a combination of patterns, new classes of discrete probabilities should be developed. As for example, for the study of twins, each of successive trails may give no success or two successes, or four successes, etc.

The number of successes of the proposed distributions is represented by a combination of arithmetic and geometric progression $a + bn^{d}$, where, a is non-negative minimum number successes or intercept of success, b is positive integer representing the slope of successes, d is the amplification or exponentiation of successes and n is a non-negative integer indicating the total number of trails.

Adnan *et al* (2017, 2018) developed a class of discrete probability distributions and called it Generalized Discrete Probability Distributions. The current paper demonstrates more general form of the discrete probability distributions for both Arithmetically and Geometrically Progressed success(es). Based on these Arithmetically and Geometrically Progressed success(es) discrete probability distributions, various Stochastic Processes as well as Generalized Linear Models can be developed.

2. Arithmetically and or Geometrically Progressed Discrete Probability Distributions

The discrete distributions are widely used in the diversified field and among them the distribution like Binomial, Multinomial, Poisson and Geometric are the most commonly used discrete distributions. The binomial distribution was first studied in connection with the games of pure chance, but it is not limited within that area only, where the Multinomial distribution is considered as the generalization of the binomial distribution. The number of mutually exclusive outcomes from a single trial are k in multinomial distribution compared to two outcomes namely success or failure of Binomial distribution.

2.1 Arithmetically and or Geometrically Progressed Binomial Distribution

A random variable X is said to be an arithmetically and or geometrically progressed binomial distribution if it has the following probability mass function

$$P(x;a,n,d,b,p)=\frac{\binom{a+bn^d}{x}p^x q^{a+bn^d-x}}{\sum_{x=a}^{a+bn^d}\binom{a+bn^d}{x}p^x q^{a+bn^d-x}};\quad x=a,a+b,a+b2^d,\dots,a+bn^d$$

$$\text{Moment Generation Function }=\frac{\sum_{x=a}^{a+bn^d}\binom{a+bn^d}{x}(pe^t)^x q^{a+bn^d-x}}{\sum_{x=a}^{a+bn^d}\binom{a+bn^d}{x}p^x q^{a+bn^d-x}},$$

$$\text{Mean}=(a+bn^d)p,\ \text{Variance}=(a+bn^d)pq,$$

$$\mu_3=(a+bn^d)pq(1-2p),\ \mu_4=(a+bn^d)pq\left[1+3(a+bn^d-2)pq\right]$$

For $a=0,b=d=1$, all the moments are like those of the traditional Binomial distribution where n is the total number of trials and p is the probability of a success.

Theorem 2.1.1
The maximum likelihood estimator of the parameter p is

$$\hat{p}=\frac{x}{(a+bn^d)},$$

where x is the total number of success from maximum of $(a+bn^d)$ success in n trials.

Theorem 2.1.2
The MLE estimator of total number of trials n of the distribution is

$$\hat{n}=\sqrt[d]{\frac{\left(\frac{\sum_{i=1}^{n}x_i}{\tilde{p}}-a\right)}{b}}$$

Theorem 2.1.3
Normal distribution is a limiting form of arithmetically and or geometrically progressed binomial distribution.

For $a=0,b=d=1$, all estimators become like those of the traditional Binomial distribution.

2.2 Arithmetically and/or Geometrically Progressed Multinomial Distribution

A random variable X is said to be an arithmetically and or geometrically progressed multinomial distribution if it has the following probability mass function

$$P(x_1,\ x_2,\dots,x_k\ ;a,n,d,b,p_1,\ p_2,\dots,p_k\ )=\frac{\frac{a+bn^d!}{x_1!\,x_2!\dots x_k!}p_1^{x_1}p_2^{x_2}\dots p_k^{x_k}}{\sum_{x_1,\ x_2,\dots,x_k=a}^{a+bn^d}\frac{a+bn^d!}{x_1!\,x_2!\dots x_k!}p_1^{x_1}p_2^{x_2}\dots p_k^{x_k}};$$

$$x_i=a,a+b,a+b2^d,\dots,a+bn^d,\sum_{i=1}^k x_i=a+bn^d,$$

$$M_{x_1,\ x_2,\dots,x_k}(t_1,\ t_2,\dots,t_k)=\frac{\sum_{x_1,x_2,\dots,x_k=a}^{a+bn^d}\frac{a+bn^d!}{\prod_{i=1}^k x_i!}\prod_{i=1}^k(p_ie^{t_i})^{x_i}}{\sum_{x_1,x_2,\dots,x_k=a}^{a+bn^d}\frac{a+bn^d!}{\prod_{i=1}^k x_i!}\prod_{i=1}^k(p_i)^{x_i}}=\left(a+bn^d\right)p_i,$$

$$E(X_i)=\left(a+bn^d\right)p_i,V(X_i)=\left(a+bn^d\right)p_i(1-p_i),$$

$$\mu_3=\left(a+bn^d\right)p_i(1-p_i)(1-2p_i),$$

$$\mu_4=\left(a+bn^d\right)p_i(1-p_i)\left[1+3\left(a+bn^d-2\right)p_i(1-p_i)\right].$$

For a = 0, b = d = 1, all the moments are like those of the traditional Multinomial distribution where n is the total number of trials and p_i is the probability of a success from category j.

Theorem 2.2.1

The maximum likelihood estimator of the parameter $\mathrm{p_i}$ is

$$\hat{\mathrm{p}}_\mathrm{i}=\frac{\mathrm{x_i}}{(\mathrm{a+bn^d})},$$

where x is the total number of success from maximum of $\left(\mathrm{a+bn^d}\right)$ success in n trials.

Theorem 2.2.2

Arithmetically and/or Geometrically Progressed Binomial Distribution is a special case of the Arithmetically and/or Geometrically Progressed Multinomial Distribution. In the probability mass function of the Multinomial Distribution,

$$P(x_1,\ x_2,\dots,x_k\ ;a,n,d,b,p_1,\ p_2,\dots,p_k\ )$$

$$=\frac{\frac{a+bn^d!}{x_1!\,x_2!\dots x_k!}}{\sum_{x_1,\ x_2,\dots,x_k=a}^{a+bn^d}\frac{a+bn^d!}{x_1!\,x_2!\dots x_k!}p_1^{x_1}p_2^{x_2}\dots p_k^{x_k}}p_1^{x_1}p_2^{x_2}\dots p_k^{x_k}$$

considering $k=2\ such\ that\ x_1+x_2=x+\left(a+bn^d-x\right)and\ p_1+p_2=1$, we obtain the probability mass function of the arithmetically and or geometrically progressed Binomial distribution along with the probability mass function,

$$P(x;a,n,d,b,p)=\frac{\binom{a+bn^d}{x}p^x q^{a+bn^d-x}}{\sum_{x=a}^{a+bn^d}\binom{a+bn^d}{x}p^x q^{a+bn^d-x}};\quad x=a,a+b,a+b2^d,\dots,a+bn^d.$$

For $a=0, b=d=1$, MLE estimator of p_i become like that of the traditional Multinomial distribution.

2.3 Arithmetically and or Geometrically Progressed Poisson Distribution

A random variable X is said to be an arithmetically and or geometrically progressed Poisson distribution if it has the following probability mass function

$$P(x;\lambda,a,n,b,d)=\frac{\lambda^{x-a}}{x!\left[\frac{1}{a!}+\frac{\lambda^b}{a+b!}+\frac{\lambda^{b2^d}}{a+b2^d!}+\frac{\lambda^{b3^d}}{a+b3^d!}+\cdots\infty\right]};$$
$$x=a,a+b,a+b2^d,\dots,a+bn^d\dots\infty$$

$$\text{MGF}=\frac{\sum_{n=0}^{\infty}\frac{(\lambda e^t)^{a+bn^d}}{(a+bn^d)!}}{\left[\frac{1}{a!}+\frac{\lambda^b}{a+b!}+\frac{\lambda^{b2^d}}{a+b2^d!}+\frac{\lambda^{b3^d}}{a+b3^d!}+\cdots\infty\right]},\ \text{Mean}=\frac{\sum_{n=1}^{\infty}\frac{\lambda^{bn^d}(a+bn^d)}{(a+bn^d)!}}{\left[\frac{1}{a!}+\frac{\lambda^b}{a+b!}+\frac{\lambda^{b2^d}}{a+b2^d!}+\frac{\lambda^{b3^d}}{a+b3^d!}+\cdots\infty\right]},$$

$$\mu_2'=\left[\frac{\sum_{n=2}^{\infty}\frac{\lambda^{bn^d}(a+bn^d)^2}{(a+bn^d)!}}{\left(\frac{1}{a!}+\frac{\lambda^b}{a+b!}+\frac{\lambda^{b2^d}}{a+b2^d!}+\frac{\lambda^{b3^d}}{a+b3^d!}+\cdots\infty\right)}\right],\ \mu_3'=\left[\frac{\sum_{n=3}^{\infty}\frac{\lambda^{bn^d}(a+bn^d)^3}{(a+bn^d)!}}{\left(\frac{1}{a!}+\frac{\lambda^b}{a+b!}+\frac{\lambda^{b2^d}}{a+b2^d!}+\frac{\lambda^{b3^d}}{a+b3^d!}+\cdots\infty\right)}\right],$$
$$\mu_4'=\left[\frac{\sum_{n=4}^{\infty}\frac{\lambda^{bn^d}(a+bn^d)^4}{(a+bn^d)!}}{\left(\frac{1}{a!}+\frac{\lambda^b}{a+b!}+\frac{\lambda^{b2^d}}{a+b2^d!}+\frac{\lambda^{b3^d}}{a+b3^d!}+\cdots\infty\right)}\right],$$

For $a=0, b=d=1$, all the moments are like those of the traditional Poisson distribution.

Theorem 2.3.1

The maximum likelihood estimator of the parameter λ can be obtained by solving the following equation by Newton Raphson Method where $\bar{x}$ is the average number of occurrences.

$$\lambda\frac{\frac{d}{d\lambda}\left(\frac{1}{a!}+\frac{\lambda^b}{a+b!}+\frac{\lambda^{b2^d}}{a+b2^d!}+\frac{\lambda^{b3^d}}{a+b3^d!}+\cdots\infty\right)}{\left(\frac{1}{a!}+\frac{\lambda^b}{a+b!}+\frac{\lambda^{b2^d}}{a+b2^d!}+\frac{\lambda^{b3^d}}{a+b3^d!}+\cdots\infty\right)}-(\bar{x}-a)=0,\qquad(1)$$

Theorem 2.3.2

The AGP Poisson distribution can be derived from AGP Binomial distribution such that

$$p=\frac{\lambda}{a+bn^d},\ q=1-\frac{\lambda}{a+bn^d}$$

under the assumptions i) the probability of success in a Bernoulli trail is very small. i.e. $p\rightarrow 0$, ii) the number of trails is very large. i.e. $n\rightarrow\infty$, iii) $(a+bn^d)p=\lambda$ is constant, that is average number of success is finite.

For $a = 0, b = d = 1$, from equation (1) we get, $\lambda \frac{\frac{d}{d\lambda}(e^{\lambda})}{e^{\lambda}} - (\bar{x} - 0) = 0$ or, $\lambda = \bar{x}$ which is like MLE estimator of the traditional Poisson distribution.

2.4. Arithmetically and or Geometrically Progressed Geometric Distribution
A random variable X is said to be an arithmetically and or geometrically progressed geometric distribution if it has the following probability mass function

$$P(x; a, n, b, d) = \frac{q^{x-a}}{\left[1 + q^b + (q^b)^{2^d} + (q^b)^{3^d} + \cdots\right]};\ x = a, a + b, \ldots, a + bn^d, \ldots, \infty$$

$$\text{MGF}= \frac{\sum_{x=a}^{a+bn^d}(qe^t)^x}{\left[1+q^b+\left(q^b\right)^{2^d}+\left(q^b\right)^{3^d}+\cdots\right]q^a},$$

$$\text{Mean}= \frac{\left[a+(a+b)q^b+\left(a+b2^d\right)q^{2^d}+\left(a+b3^d\right)q^{3^d}+\cdots+\left(a+bn^d\right)q^{bn^d}\right]}{\left[1+q^b+\left(q^b\right)^{2^d}+\left(q^b\right)^{3^d}+\cdots\right]},$$

$$\mu_2' = \frac{\left[a^2+(a+b)^2q^b+\left(a+b2^d\right)^2q^{2^d}+\left(a+b3^d\right)^2q^{3^d}+\cdots+\left(a+bn^d\right)^2q^{bn^d}\right]}{\left[1+q^b+\left(q^b\right)^{2^d}+\left(q^b\right)^{3^d}+\cdots\right]},$$

$$\mu_3' = \frac{\left[a^3+(a+b)^3q^b+\left(a+b2^d\right)^3q^{2^d}+\left(a+b3^d\right)^3q^{3^d}+\cdots+\left(a+bn^d\right)^3q^{bn^d}\right]}{\left[1+q^b+\left(q^b\right)^{2^d}+\left(q^b\right)^{3^d}+\cdots\right]},$$

$$\mu_4' = \frac{\left[a^4 + (a+b)^4q^b + \left(a+b2^d\right)^4q^{2^d} + \left(a+b3^d\right)^4q^{3^d} + \cdots + \left(a+bn^d\right)^4q^{bn^d}\right]}{\left[1 + q^b + (q^b)^{2^d} + (q^b)^{3^d} + \cdots\right]}$$

For $a = 0, b = d = 1$, all the raw moments are like those of the traditional Geometric distribution (Devore, 2016).

Theorem 2.4.1
The maximum likelihood estimator of the parameter q can be obtained by solving the following equation by Newton Raphson Method where $\bar{x}$ is the average discrete waiting time preceding first success

$$q\frac{\frac{d}{dq}\left[1+q^b+\left(q^b\right)^{2^d}+\left(q^b\right)^{3^d}+\cdots\right]}{\left[1+q^b+\left(q^b\right)^{2^d}+\left(q^b\right)^{3^d}+\cdots\right]} - (\bar{x} - a) = 0, \tag{2}$$

Theorem 2.4.2
The new Geometric distribution is a special case of AGP Negative Binomial Distribution for K=1 or single number of success.

For $a = 0, b = d = 1$, we get from equation (2), $q\,\frac{\frac{d}{dq}(1-q)^{-1}}{(1-q)^{-1}} - (\bar{x} - a) = 0,\ \hat{q} = \frac{\bar{x}}{\bar{x}+1}, \hat{p} = \frac{1}{\bar{x}+1}$ which are similar to the MLE estimators of the traditional Geometric distribution.

3. Arithmetically and or Geometrically Progressed Stochastic Processes

During sequences of trials in a discrete Stochastic Process, either 0, or 1 or more than one successes may occur at each random trial. If the occurrences of the random sequence of these successes follow a class of patterns or a combination of patterns, new classes of discrete probabilities of the family of discrete random variables should be developed. As for example, for the study of twins, each of successive trails of a random sequence may give no success or two successes, or four successes, etc.

The number of successes of each trial of the random sequence is represented by a combination of arithmetic and geometric progression $a + bn^d$, where, a is non-negative minimum number successes or intercept of success, b is positive integer representing the slope of successes, d is the amplification or exponentiation of successes and n is a non-negative integer indicating the total number of trails.

3.1 Arithmetically and or Geometrically Progressed Poisson Processes

A family of random variables or counting process$N(t)$is said to be an arithmetically and or geometrically progressed Poisson process if it has the following probability mass function

$$P[N(t) = x; \lambda, a, n, b, d] = \frac{(\lambda t)^{x-a}}{x!\left[\frac{1}{a!} + \frac{(\lambda t)^b}{a+b!} + \frac{(\lambda t)^{b2^d}}{a+b2^d!} + \frac{(\lambda t)^{b3^d}}{a+b3^d!} + \cdots \infty\right]}$$

$$\forall\, x = a, a+b, a+b2^d, \ldots, a+bn^d \ldots \infty$$

$$\text{MGF} = \frac{\sum_{n=0}^{\infty} \frac{(\lambda e^t)^{a+bn^d}}{(a+bn^d)!}}{\left[\frac{1}{a!} + \frac{\lambda^b}{a+b!} + \frac{\lambda^{b2^d}}{a+b2^d!} + \frac{\lambda^{b3^d}}{a+b3^d!} + \cdots \infty\right]}, \quad \text{Mean} = \frac{\sum_{n=1}^{\infty} \frac{(\lambda t)^{bn^d}(a+bn^d)}{(a+bn^d)!}}{\left\{\frac{1}{a!} + \frac{(\lambda t)^b}{a+b!} + \frac{(\lambda t)^{b2^d}}{a+b2^d!} + \frac{(\lambda t)^{b3^d}}{a+b3^d!} + \cdots \infty\right\}},$$

$$\mu_2' = \frac{\sum_{n=2}^{\infty} \frac{(\lambda t)^{bn^d}(a+bn^d)^2}{(a+bn^d)!}}{\left\{\frac{1}{a!} + \frac{(\lambda t)^b}{a+b!} + \frac{(\lambda t)^{b2^d}}{a+b2^d!} + \frac{(\lambda t)^{b3^d}}{a+b3^d!} + \cdots \infty\right\}},$$

For $t = 1$, Arithmetically and Geometrically progressed Poisson Process is the Arithmetically and Geometrically progressed Poisson Distribution. For $a = 0,\ b = d = 1$, all the moments of the Arithmetically and Geometrically progressed Poisson Process are like those of the traditional Poisson Process.

4. Arithmetically and or Geometrically Progressed Regressions

Based on these Arithmetically and Geometrically Progressed success(es) discrete probability distributions, various Stochastic Processes as well as Generalized Linear Models can be developed.

4.1 Arithmetically and or Geometrically Progressed Binomial Regression

The Arithmetically and or Geometrically Progressed Binomial Regression Model will be

$$log\left[\frac{\pi_i}{\frac{a}{n_i} + bn_i^{d-1} - \pi_i}\right] = \sum_{j=1}^{p} x_{ij}\, \beta_j, \qquad \forall\, i = 1,2, \ldots, n_i \qquad (3)$$

$$\text{where, } \pi_i = \left(\frac{a}{n_i} + bn_i^{d-1}\right)\frac{e^{\sum_{j=1}^{p} x_{ij}\beta_j}}{1+e^{\sum_{j=1}^{p} x_{ij}\beta_j}}, \quad (4)$$

and y_i is distributed as an Arithmetically and Geometrically progressed Binomial distribution with probability mass function

$$P(y_i; \pi_i, a, n, b, d) = \frac{\binom{a+bn_i^d}{n_i y_i}\pi_i^{n_i y_i}(1-\pi_i)^{a+bn_i^d - n_i y_i}}{\sum_{x=a}^{a+bn_i^d}\binom{a+bn_i^d}{n_i y_i}\pi_i^{n_i y_i}(1-\pi_i)^{a+bn_i^d - n_i y_i}}; \forall n_i y_i$$
$$= a, a+b, a+b2^d, \dots, a+bn_i^d$$

For $a = 0, b = d = 1$, equations (3) and (4) give similar results for traditional Binomial Regression since

$$log\left[\frac{\pi_i}{\frac{a}{n_i}+bn_i^{d-1} - \pi_i}\right] = log\left[\frac{\pi_i}{1-\pi_i}\right] = \sum_{j=1}^{p} x_{ij}\,\beta_j$$

$$\text{and } \pi_i = (0+1)\frac{e^{\sum_{j=1}^{p} x_{ij}\beta_j}}{1+e^{\sum_{j=1}^{p} x_{ij}\beta_j}} = \frac{e^{\sum_{j=1}^{p} x_{ij}\beta_j}}{1+e^{\sum_{j=1}^{p} x_{ij}\beta_j}}.$$

4.2 Arithmetically and or Geometrically Progressed Multinomial Regression

The Arithmetically and or Geometrically Progressed Multinomial Regression Model will be

$$\pi_{ik} = \left(\frac{a}{n_i} + bn_i^{d-1}\right)\frac{e^{\sum_{j=1}^{p} x_{ij}\beta_{kj}}}{1+\sum_{h=1}^{c-1} e^{\sum_{j=1}^{p} x_{ij}\beta_{hj}}}, \ \forall\, k = 1,2,\dots,c-1 \quad (5)$$

where c is the baseline category and y_i is distributed as an Arithmetically and or Geometrically Progressed Multinomial with probability mass function

$$P(y_{i1}, y_{i2}, \dots, y_{ic}; \pi_{i1}, \pi_{i2}, \dots, \pi_{ic}, a, n_i, b, d) = \frac{a+bn_i^d!}{y_{i1}!y_{i2}!\dots y_{ic}!}\pi_{i1}^{y_{i1}}\pi_{i2}^{y_{i2}}\dots\pi_{ic}^{y_{ic}};$$

$$\forall n_i y_i = a, a+b, a+b2^d, \dots, a+bn_i^d$$

For $a = 0, b = d = 1$, equation (5) gives the traditional Multinomial Regression Model since

$$\pi_{ik} = (0+1)\frac{e^{\sum_{j=1}^{p} x_{ij}\beta_{kj}}}{1+\sum_{h=1}^{c-1} e^{\sum_{j=1}^{p} x_{ij}\beta_{hj}}} = \frac{e^{\sum_{j=1}^{p} x_{ij}\beta_{kj}}}{1+\sum_{h=1}^{c-1} e^{\sum_{j=1}^{p} x_{ij}\beta_{hj}}}.$$

4.3 Arithmetically and or Geometrically Progressed Poisson Regression

The Arithmetically and or Geometrically Progressed Poisson Regression Model will be

$$log\left[alog\lambda_i + log\left\{\frac{1}{a!} + \frac{\lambda_i^{b}}{a+b!} + \frac{\lambda_i^{b2^d}}{a+b2^d!} + \frac{\lambda_i^{b3^d}}{a+b3^d!} + \cdots\infty\right\}\right] = \sum_{j=1}^{p} x_{ij}\,\beta_j, \quad (6)$$
$$\forall\, i = 1,2,\dots,n$$

where, y_i is distributed as a Arithmetically and or Geometrically Progressed Poisson distribution with mass function

$$P(y_i;\lambda_i,a,n,b,d)=\frac{{\lambda_i}^{y-a}}{y!\left[\frac{1}{a!}+\frac{{\lambda_i}^{b}}{a+b!}+\frac{{\lambda_i}^{b2^d}}{a+b2^d!}+\frac{{\lambda_i}^{b3^d}}{a+b3^d!}+\cdots\infty\right]};y_i=a,a+b,a+b2^d,\dots,a+bn^d\dots\infty$$

For $a=0,b=d=1$, equation (6) gives traditional Poisson Regression model since

$$log\left[0+log\left\{1+\frac{{\lambda_i}^1}{1!}+\frac{{\lambda_i}^2}{2!}+\frac{{\lambda_i}^3}{3!}+\cdots\infty\right\}\right]=log[\lambda_i]=\sum_{j=1}^{p}x_{ij}\,\beta_j.$$

4.4 Arithmetically and or Geometrically Progressed Geometric Regression

The Arithmetically and or Geometrically Progressed Geometric Regression Model will be

$$log\left[alogq_i+log\left\{1+q^b+\left(q^b\right)^{2^d}+\left(q^b\right)^{3^d}+\cdots\right\}\right]=\sum_{j=1}^{p}x_{ij}\,\beta_j, \qquad (7)$$

$$\forall\, i=1,2,\dots,n$$

where, q is the probability of obtaining a failure and y_i is distributed as an Arithmetically and or Geometrically Progressed Geometric Distribution with mass function

$$P(y_i;q_i,a,n,b,d)=\frac{{q_i}^{y-a}}{\left[1+q^b+\left(q^b\right)^{2^d}+\left(q^b\right)^{3^d}+\cdots\right]};y_i=a,a+b,a+b2^d,\dots,a+bn^d\dots\infty.$$

For $a=0,b=d=1$, equation (7) gives traditional Geometric Regression model since

$$log[0+log\{1+q+q^2+q^3+\cdots\}]=log\left[log\frac{1}{1-q}\right]=\sum_{j=1}^{p}x_{ij}\,\beta_j.$$

Conclusion

If sampling is done in the usual manner, the new Arithmetically and or Geometrically Progressed Discrete Probability Distributions reduce to the traditional discrete probability distributions and thus Arithmetically and or Geometrically Progressed Stochastic Processes and Arithmetically and or Geometrically Progressed Generalized Linear Models also convert to traditional stochastic processes and generalized regression models respectively.

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