A Reparametrized Linear Model that Parses Qualitative and Quantitative Characteristics of a Set of Predictors

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Abstract
The following study shows how a reparameterization of the general linear model for regular regression can serve to quantify qualitative aspects of predictors. It uses data from the High School Longitudinal Study of 2009 on mathematics course-taking and achievement as an example. Results show that all mathematics courses are not equally predictive of math achievement. Thus, taking into account qualitative aspects of mathematics courses is useful. The study ends with a justification of quantifying qualitative aspects of predictors relative to a criterion with extensions to other linear models.

Key words: Course-taking, achievement, profile analysis

1. Introduction

Davison and Davenport (2002) show a way to reparametrize a regression equation into two variables with inherent meaning. They called these two variables level and pattern. *Level* can be indexed by the mean $\bar{X}_p = \frac{1}{V}\sum_v X_{pv}$ or total $\mathrm{X}_p = \sum_v X_{pv}$ of the predictors for an observation. Pattern is the covariance, $\sum_v \frac{(b_v-\bar{b})(X_v-\bar{X})}{V}$, for the individual relative to the optimal pattern of regression weights for predicting performance on the criterion. *Level* is the quantitative information inherent in a sum or average of a set of predictors (it acts as if all predictors are the same relative to predictability). In contrast, *Pattern* is the covariance of an observation's predictor scores to that of the optimal regression weights leading to higher expected scores on the criterion. It is our conjecture that *Pattern* allows one to quantify qualitative features of a set of predictors as they relate to a criterion. Thus, the approach forwarded in this paper allows one to consider both quantitative and qualitative aspects of a set of predictors as they relate to a criterion simultaneously.

The goals of the current study are to: 1) show that qualitative aspects of predictors can be quantified, 2) show that *Level* and *Pattern* can be related to quantitative and qualitative aspects of the predictors relative to a criterion, respectively, 3) show that qualitative aspects of mathematics course-taking (type of courses) is more predictive of mathematics achievement than the quantitative aspects (amount of courses), and 4) generalize this idea to other linear models.

2. Literature Review

The goal of science is parsimony; to explain the world in which we live as simply as possible. While both qualitative and quantitative approaches to research serve this goal; they do so from vastly differing philosophical bases (Smith, 1983; Smith & Heshusius, 1986). While the histories of these two approaches have been contentious, the current view is to allow for the "two different, equally legitimate approaches to inquiry" (Smith & Heshusius, 1986, p.4). Moreover, there is some effort to present both approaches simultaneously (Leman, House, & Hoegh, 2015). While, the present manuscript presents both qualitative and quantitative aspects of data, it does not attempt to simultaneously combine qualitative and quantitative research. This paper is more in the framework of Young (1981) which attempts to quantify qualitative aspects of data so that qualitative data is more amenable to quantitative analyses.

Mathematics is a unique field of study because courses form a quasi-hierarchy relative to content. Knowledge of the content of lower-level math courses is prerequisite for success in higher level math courses. Finkelstein et al. (2012) observed that students in the advanced math track in 7th grade are likely to take more advanced courses in high school (e.g. pre-calculus, calculus, AP Statistics). Students typically start high school (9th grade) taking geometry or algebra 1. Being in geometry, which is further along in the course-taking sequence than algebra, is advantageous for high school freshmen seeking to go to college, since more advanced placement at the beginning of high school allows students to more readily take high-level courses such as AP Calculus and AP Statistics during high school. Taking these courses in high school is advantageous when applying for college admission (Lawyers' Committee for Civil Rights of the San Francisco Bay Area, 2013). Few students from the general track catch up to students in the advanced math track because it involves completing extra math coursework and for students who are behind in mathematics course-work, the chances of them taking relatively more mathematics than the students with more math courses is slim. Moreover, some students experience continued difficulties in lower-level math courses. For example, students who were required by their school to repeat an algebra course still struggled to achieve proficiency during their second time through the course (Finkelstein et al., 2012).

Ma and Wilkins (2007) observed that, while all math courses contribute towards increased math achievement, advanced math courses such as trigonometry, pre-calculus, and calculus "demonstrated the greatest regulatory power" for improving math achievement. The disparity in math performance relative to math courses taken was particularly apparent on the 2005 *NAEP* mathematics assessment. High school students who graduated with geometry or lower as their highest mathematics course had an average score below the *Basic* achievement level on the 2005 *NAEP* mathematics assessment, whereas, high school students who graduated with calculus or greater as their highest math course averaged scores at the *Proficient* level on the same assessment (Shettle et al., 2007). Implications of advanced math coursework extend beyond high school graduation. Researchers have shown that students who took more advanced math courses had a greater likelihood of attending college (Spielhagen, 2006; Byun et al., 2015) and graduating from college (Trusty & Niles, 2003). Thus, there appear to be qualitative differences between different mathematics courses relative to several criteria.

The use of profiles to determine disease states, apprehend criminals, identify successful candidates for school or work, etc. has been a staple in many clinical and applied fields (e.g. medicine, psychology, college admissions, human resources, etc.). Regardless of discipline, multiple measures are taken and the pattern of scores on these measures is compared to a prototypical pattern, where the prototypical pattern shows the pattern of scores indicative of a particular classification or diagnosis. An individual profile pattern is the arrangement of scores in a respondent's vector of scores. Cronbach and Gleser (1953) discuss assessing the similarity between profiles and note three defining characteristics of a profile: elevation, scatter, and shape. They state that "Elevation is the mean of all scores for a given person. Scatter is the square root of the sum of squares of the individual's deviation scores about his own mean; that is, it is the standard deviation within the profile. Shape is the residual information in the score set after equating profiles for both elevation and scatter." There has been much activity since the 1950's to quantify profiles (Meehl, 1950, Cronbach and Gleser, 1953). Most past methods yield profiles that may not have criterion-related validity as many rely only on the subtest variables within the profile with no link to an external criterion (e.g. cluster analysis [Glutting, McGrath, Kamphaus, & McDermott 1992; Glutting & McDermott, 1994; Konold, Glutting, McDermott, Kush, & Watkins, 1999; McDermott, Glutting, Jones, Watkins, & Kush, 1989], latent profile analysis [Gibson, 1959], modal profile analysis, MFA [Pritchard, Livingston, Reynolds, &, Moses, 2000; Skinner, 1977, 1979], profile analysis via multidimensional scaling, PAMS [Davison, 1994; Davison, Gasser, & Ding, 1996; Davison, Kuang, & Kim, 1999)], and configural frequency analysis, CFA [Stanton & Reynolds, 2000; von Eye, 1990, 2002]).

Davison and Davenport (2002) suggest a reparameterization of the normal regression equation that makes use of Cronbach and Gleser's (1953) decomposition of an individual's set of scores. Specifically, Davison and Davenport (2002) show that any regression equation can be reparametrized into two variables. The first representing *Level* in Cronbach and Gelser's approach and the second representing a combination of scatter and shape (*Pattern*). Moreover, one can parse variance in the criterion related to *Level* or *Pattern* (individually, collectively, and incrementally). We maintain that *Level* is the quantitative information in the predictors, while *Pattern* is the qualitative information. Also, due to how it is defined (covariance of the optimal regression weights for predicting the criterion with the predictors with the observation's scores on the predictors), *Pattern* has criterion validity in that those whose predictor profiles are more similar to the criterion-related pattern will have higher expected criterion scores.

Level of a profile refers to the height of that person's scores described by the mean or sum of the scores in the profile (the mean and sum are equivalent for prediction purposes as one is just a linear combination of the other; the mean is the sum times a constant). For our purposes here we will use the mean: $\bar{X}_p = \frac{1}{V}\sum_v X_v$ as our *Level* value. *Pattern* can be described as the covariance of a vector of contrast coefficients (one for each variable), $\boldsymbol{X}_p^* = \{(X_{pv} - \bar{X}_p)\}$ which represents the deviation of each score from the *Level*, $\bar{X}_p$ with the corresponding deviation of the optimal regression coefficients for the corresponding predictors, *COVp* $= \sum_v \frac{(b_v - \bar{b})(X_v - \bar{X})}{V}$. This variable, computed for each student, is a criterion match statistic, representing the degree to which a student's values on the predictors matches the optimal pattern leading to higher predicted scores on the criterion. The optimal pattern differentiates high performers from low performers. The higher an individual's scores on the predictors with the higher regression weights, the higher their *Pattern* scores and the higher the predicted criterion score. Note, that the least squares regression weights constitute the "optimal criterion pattern;" one that maximizes the variance accounted for in the criterion variable. Together, *Level* and *Pattern* predict the same amount of variation in the criterion as the original set of predictors. Thus, $\hat{Y} = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + \ldots + b_VX_V$ can be represented by the two variables *Pattern* and *Level* (as we have defined them here): $\hat{Y} = b_0 + b_1\frac{\sum_{k=1}^{K}(b_v - \overline{b})(X_v - \overline{X})}{V} + b_2\overline{X}$. The proof of this assertion is given in the appendix of Davison and Davenport (2002).

The degree to which *Pattern* will aid in the prediction is the degree to which the predictors are differentially related to the criterion. *Pattern* is useful to the extent that a one unit increase in one variable leads to differential predictability than a one unit increase in another. In other words, there are qualitative differences in the predictors relative to the criterion. One can see that one's predicted score will be higher to the degree to which participants have higher scores on predictors with higher regression weights. *Level* represents the quantitative information of a one unit increase in the predictors, regardless of which predictor (strictly quantitative).

3. Methods

3.1 Data

Data for this study come from the National Center for Education Statistics' (NCES) High School Longitudinal Study of 2009 (HSLS:09) which contains mathematics test scores at 9th and 11th grades and high school transcript data from a nationally representative cohort of students in ninth grade in 2009. There are over 23,000 students from 940 schools. Almost 22,000 students had partial or full transcripts (Ingels, Pratt, Herget, Bryan, Fritch, Ottem, Rogers, and Wilson, 2015). Students were chosen who have valid transcript data and both a mathematics test score at 9th and 11th grades. This leads to a sample of 15,750 students. Analyses used transcript weights suitable for generalizing to the national population of ninth graders in 2009. We also used a design effect (Kish, 1965) to account for dependencies in the data caused by the cluster sample design. For *HSLS: 09*, the average design effects for students' ethnicities are 4.0, 4.9, 3.7, 2.7, and 3.1 for Hispanic, Asian, Black, White, and more than one race, respectively (Ingels, et al. 2014, p.126), so 4.0 was used as a design effect for this study. Teitelbaum (2003) used a design effect when analyzing data similar to ours. Given our design effect, our effective sample size was approximately 3,940.

3.2 Variables

Information from high school transcripts was keyed and coded (Ingels et al., 2015). Cases with duplicate records in the transcript were deleted. Only courses taken in mathematics were selected; courses having a School Code for the Exchange of Data (SCED) from 02001 to 02999. Moreover, only courses completed with a grade of D- or higher (passed) were selected. Since the data file for high school courses and transcripts is organized by course, credits earned for courses having the same SCED codes were summed to make course variables for each student. For instance, the 'sum02001' variable is calculated by summing all credits completed in courses having the SCED code of '02001' for each student. Therefore, variables representing credits completed in each of 67 SCED math courses were obtained.

In addition to the 67 course-taking variables, NCES created X3THIMATH with 14 categories to represent highest mathematics course taken. This variable was created based on the hierarchical nature of mathematics courses with higher numbers corresponding to more advanced mathematics content. This variable has 14 categories from No math to

AP/AB Calculus (0=No math, 1=Basic Math, 2=Other Math, 3=Pre-Algebra, 4=Algebra I, 5= Geometry, 6=Algebra II, 7=Trigonometry, 8=Other advanced math, 9=Probability and statistics, 10=Other AP/IB math, 11=Precalculus, 12=Calculus, 13=AP/IB Calculus). Following this latter representation of mathematics course-taking, we created thirteen mutually exclusive course sequences using the sum of credits completed in the courses fitting into the above named course categories. For instance, to make the variable, Probability and statistics, the credits completed in the courses SCED 02201 (Probability and Statistics), 02202 (Inferential Probability and Statistics), 02204 (Particular Topics in Probability and Statistics), 02207 (Probability and Statistics—Independent Study), and 02209 (Probability and Statistics—Other) were summed together.

The remaining two independent variables in this study, *Level* and *Pattern*, were obtained from the 13 sequences of mathematics course-taking we created. Here, *Level* is the mean of credits completed in the 13 course sequences ($\frac{1}{V}\sum_v X_{pv}$) and *Pattern* is the covariance of the regression weights for the 13 course sequences with the number of credits taken in each sequence, $\sum_v \frac{(b_v-\bar{b})(X_v-\bar{X})}{V}$. The regression weights necessary to calculate *Pattern* were obtained by regressing the 11th grade mathematics test score on the 13 course sequences.

3.3 Analysis

We first ran a regression of the 13 math course sequence variables predicting 11th grade mathematics test score. This returned the total variance explained by the courses and the regression weights necessary to calculate the *Pattern* scores. We next ran a correlation analysis for *Level, Pattern*, and the 11th grade mathematics test score. This gave us information on the inter-relationship of the variables as well as predictability of *Level* and *Pattern* (individually) to the 11th grade math test. We also ran a regression analysis predicting the 11th grade mathematics test score from *Level* and *Pattern* to show equivalency of the results in terms of variance accounted for.

4. Results

The first analysis is a regression using the 13 course-taking variables predicting 11th grade mathematics test score. The results are given in Table 1. One of the first things to notice is that the regression weights differ. If all weights were the same, then a one course increase (decrease) would have the same effect on predicted 11th grade mathematics test score irrespective of the course in which the one unit increase (decrease) occurred. The fact that they differ suggests that there are qualitative differences between the courses as they predict mathematics performance. The relationship between the regression weights and the courses are easily seen in Figure 1. Figure 1 contains a plot of the raw regression weights for the 13 course-taking variables as ordered by X3THIMATH. This plot gives credence to the fact that the courses differ qualitatively. Moreover, it helps justify the hierarchical ordering of the courses by NCES; as there is in general a monotonic relationship between the regression weights and the NCES ordering of the courses (Other Advanced being the only substantial exception to monotonicity). Given the regression weights, one can see that taking courses with a higher number in NCES's hierarchy (more advanced course) in general leads to higher expected scores for the criterion (11th grade mathematics test).

Table 1

Raw Regression Weights for Predicting 11th Grade Test Score with the 13 Course Sequences

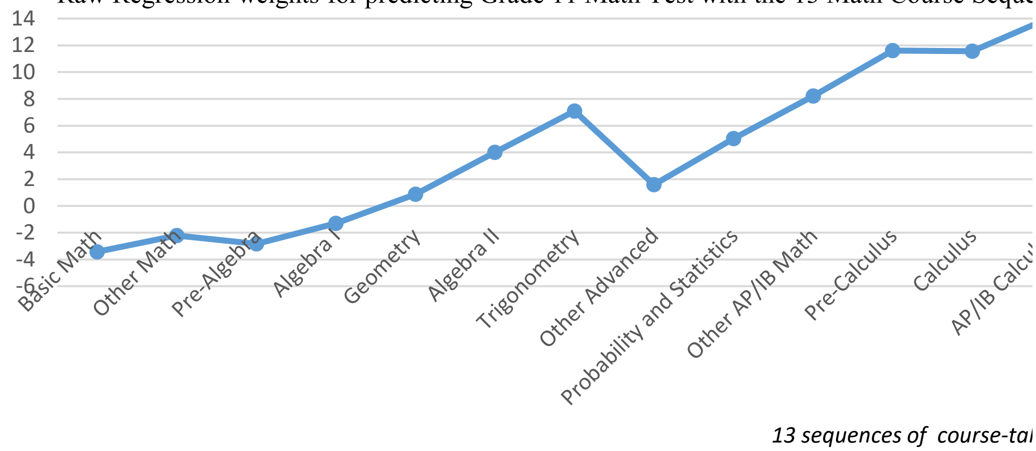
	Estimate	SE*	t*	p*
(Intercept)	56.75	0.84	67.45	0.00
Basic Math	-3.42	0.58	-5.93	0.00
Other Math	-2.21	0.47	-4.69	0.00
Pre-Algebra	-2.83	0.73	-3.89	0.00
Algebra I	-1.31	0.43	-3.04	0.00
Geometry	0.86	0.57	1.53	0.13
Algebra II	4.00	0.48	8.39	0.00
Trigonometry	7.10	0.65	10.95	0.00
Other Advanced	1.58	0.41	3.82	0.00
Probability and Statistics	5.02	1.01	4.99	0.00
Other AP/IB Math	8.23	1.03	8.00	0.00
Pre-Calculus	11.61	0.57	20.27	0.00
Calculus	11.56	1.06	10.87	0.00
AP/IB Calculus	14.13	0.70	20.29	0.00
R^2	0.3994			
Adjusted R^2*	0.3974			
AIC*	47253.889			
BIC*	47341.783			

Note. *after accounting for weights and the design effect of 4
SOURCE: U.S. Department of Education, National Center for Education Statistics, High School Longitudinal Study of 2009 (HSLS:09) Base Year (2009), First Follow-up (2012, 2013), Transcript (2013, 2014).

Table 2 provides information on the predictability of *Level* and *Pattern* when each is the sole predictor of the 11th grade math test. For this we simply take the squared correlation between each of these variables and the 11th grade mathematics score (see Table 4 which has this information in terms of variance accounted for). Results from this table answers one of the research goals of this study; to show that qualitative aspects of course taking (type of course) is more predictive of mathematics performance than the quantitative aspect of course-taking (amount of courses). The correlation of *Pattern* with the math test score is higher than the corresponding correlation of *Level*. Thus, simply counting courses without attention to the type (qualitative difference in the course) is not as predictive. This result concurs with Davenport et al. (2013) where they found little difference in number of mathematics courses taken by Black versus White students, but substantial difference in the types of courses taken as well as mathematics achievement.

Figure 1

Raw Regression weights for predicting Grade 11 Math Test with the 13 Math Course Seque



SOURCE: U.S. Department of Education, National Center for Education Statistics, High School Longitudinal Study of 2009 (HSLS:09) Base Year (2009), First Follow-up (2012, 2013), Transcript (2013, 2014).

Table 2

Correlations among mathematics achievement, Level, and Pattern

	11^{th} Grade Achievement	*Level*	*Pattern*
11^{th} Grade Achievement	1.00		
Level	0.30	1.00	
Pattern	**0.58**	0.07	1.00

SOURCE: U.S. Department of Education, National Center for Education Statistics, High School Longitudinal Study of 2009 (HSLS:09) Base Year (2009), First Follow-up (2012, 2013), Transcript (2013, 2014).

Table 3 is included to illustrate the qualitative quantification of courses taken via *Pattern*. If the predictive power of courses differ, one can see that treating all courses the same (and simply adding them) will not be as predictive as finding a way to quantify the differences in types of course-work. We maintain that this is done via the *Pattern* variable as it is a function of the covariance between the actual courses taken and relationship to the criterion (as indexed by the regression weights). Students who take more courses with higher regression weights should have higher expected test score values. We believe that both *Level* and *Pattern* will be predictive, but that *Pattern* will be more so (as seen in Table 2). We also believe that *Level* and *Pattern* ought to be correlated because as students take more math courses, they should take higher level courses, but this is not always the case. The correlation of 0.07 given in Table 2 for Level and Pattern is significant at $P < 0.0001$ (with the adjustment for the design effect).

Table 3

Number of Carnegie Units in Each Sequence and Other Scores for Five Students

Course Sequence	Student1	Student2	Student3	Student4	Student5
Basic Math	-	-	-	2.00	-
Other Math	2.00	-	-	-	-
Pre-Algebra	-	1.00	-	1.00	-

Algebra I	1.00	-	-	1.00	-
Geometry	1.00	1.00	1.00	1.00	-
Algebra II	-	1.00	1.00	1.00	-
Trigonometry	-	-	-	-	2.00
Other Advanced	-	-	-	-	-
Probability and Statistics	-	-	-	-	-
Other AP/IB Math	-	-	-	-	-
Pre-Calculus	-	1.00	1.00	-	-
Calculus	-	-	1.00	-	-
AP/IB Calculus	-	-	-	-	1.00
	-	-	-	-	-
Level	0.31	0.31	0.31	0.46	0.23
Pattern*	-1.66	-0.24	0.87	-2.40	1.22
11^{th} Grade Test	56.46	78.19	82.99	48.44	99.51

* Pattern values here are 13*COV = Corrected Sums of Squares and Cross-Products.
SOURCE: U.S. Department of Education, National Center for Education Statistics, High School Longitudinal Study of 2009 (HSLS:09) Base Year (2009), First Follow-up (2012, 2013), Transcript (2013, 2014).

Table 3 shows scores for 5 students on the variables of interest for this study. The first three students have the same number of math credits. Each has the same *Level* score, 0.31 (4/13). Thus, they are quantitatively the same relative to their amount of mathematics course-work. Note, however, that the students take different courses. As the Student number increases (from 1-3), the student takes more advanced courses. Note, too, that the students *Pattern* scores rise as does their 11^{th} grade mathematics test score as we go from Student1 to Student3. Our *Pattern* score appears to align itself well with course content (and achievement). Student4 has the most course credits, but most of their coursework is in lower-level classes. Their *Pattern* score is lowest of the 5 students listed here as is their 11^{th} grade math test score. Finally, Student5 has the fewest number of math credits for courses, but has credits in the most advanced math course listed (AP/IB Calculus). Student5 has the highest *Pattern* score and the highest test score of the 5 students listed here.

Table 4 shows the variance in 11^{th} grade math test score accounted for by *Level* and *Pattern*, solo, jointly, and incrementally. As a sole predictor, *Level* accounts for 9% of the variance of the test score, while *Pattern* by itself accounts for 33.6%. Both variables together (as does the 13 predictors in Table 1) account for 39.9% of the variance in the 11^{th} grade math score (39.94%). Finally, if *Level* is already in the model *Pattern* accounts for an additional 30.9%. In contrast, if *Pattern* is already in the model *Level* accounts for an additional 6.3%.

Table 4
Variance Accounted for by *Level* and *Pattern* Relative to the 11^{th} Grade Math Test

	As Sole Predictor	Unique Variance
Level	9.0%	6.3%
Pattern	33.6%	30.9%

Total Variance 39.9%

SOURCE: U.S. Department of Education, National Center for Education Statistics, High School Longitudinal Study of 2009 (HSLS:09) Base Year (2009), First Follow-up (2012, 2013), Transcript (2013, 2014).

5. Discussion

Davenport and Davison (2002) give a procedure that allows one to quantify qualitative aspects of a set of predictors relative to a criterion. This works because *Pattern* is a profile match statistic that accesses the match (covariance) of the profile of the individual's scores to the optimal regression weights predicting success for the criterion. It quantifies the relationship of the pattern of scores in the predictors with the optimal pattern of the predictors as determined by their regression weights. If a student's profile of scores is similar to what is optimal (e.g. advanced mathematics courses), their *Pattern* score is higher. Moreover, we showed in Table 3 the relationship between quantification of the types of math courses taken (as represented by *Pattern*) and the actual courses taken by students.

We also used *Level* and *Pattern* to show that type of mathematics course taken was more predictive than amount of mathematics courses taken. Note that if there is no additional qualitative information in the predictors *Level* will be a sufficient predictor of the criterion. To the extent that the predictors are differentially predictive, *Pattern* will add predictability incrementally over *Level* (see Table 4).

The procedure given is general and will work for any set of predictors. Thus, we can always parse degree of predictability due to quantitative versus qualitative aspects of the predictors. While Davison & Davenport (2002) discussed only regression models, the approach can be extended. The analysis has been extended to generalized linear probit and logit analyses with maximum likelihood estimation (Booth, Murray, Overdun, Matthews, & Furnham, 2015; Davison, Jew, & Davenport, 2014; Morse, Daegling, McGraw, & Pompish, 2013), to latent variable regression models in structural equations modeling (Davison, Chang, & Davenport, 2014), to canonical regression analysis (Aragon, Culpepper, McKee, & Perkins, 2014), and to meta-analyses (Wiernik, Wilmot, Davison, & Ones, in press).

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