A comparison of Bayesian, frequentist and fiducial intervals for the difference between two binomial proportions

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Abstract

Estimating the difference between two binomial proportions will be investigated, where Bayesian, frequentist and fiducial (BFF) methods will be considered. Three vague priors will be used, the Jeffreys prior, a divergence prior and the probability matching prior. A probability matching prior is a prior distribution under which the posterior probabilities of certain regions coincide with their coverage probabilities. Fiducial inference can be viewed as a procedure that obtains a measure on a parameter space while assuming less than what Bayesian inference does, i.e. no prior. Fisher introduced the idea of fiducial probability and fiducial inference. In some cases the fiducial distribution is equivalent to the Jeffreys posterior. The performance of the Jeffreys prior, divergence prior and the probability matching prior will be compared to a fiducial method and other classical methods of constructing confidence intervals for the difference between two independent binomial parameters. These intervals will be compared and evaluated by looking at their coverage rates and average interval lengths. The probability matching and divergence priors perform better than the Jeffreys prior.

Key Words: Binomial proportions, Coverage, Divergence prior, Fiducial, Jeffreys prior, Probability matching prior

1. Introduction

Bayesian, frequentist and fiducial (BFF) methods will be considerd. Confidence intervals for the difference between two binomial proportions will be constructed. The well-known Wald interval and the Agresti-Caffo interval will be considered for the frequentist methods. For the Bayesian methods, three priors will be considered, the Jeffreys' prior, a matching prior and a divergence prior. A fiducial quantity will also be considered, where a fiducial confidence interval for the difference between two proportions will be constructed.

We assume that X_1, X_2 are independent binomial random variables with $X_i \sim Bin(n_i, p_i)$ for $i = 1, 2$. And we are interested in CIs for $\delta = p_1 - p_2$, the difference between two binomial proportions.

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2. Different Intervals

2.1 Frequentist

The well-known Wald interval will be considered,

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}.$$

As well as the Agresti-Caffo interval,

$$(\hat{p}_1^* - \hat{p}_2^*) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1^*\left(1-\hat{p}_1^*\right)}{n_1^*} + \frac{\hat{p}_2^*\left(1-\hat{p}_2^*\right)}{n_2^*}}.$$

where $n_i^* = n_i + 2$, $\hat{p}_i^* = {}^{(x_i+1)}/_{n_i^*}$, and $z_{\alpha/2}$ is the upper ${}^{\alpha}/_{2}$ quantile of the standard normal. These intervals were considered in Roths & Tebbs (2006).

2.2 Fiducial

The concept of fiducial probability was introduced by Fisher (1930). The following are needed for the fiducial approach:

- A sufficient statistic for the parameter in question.
- A pivot, function of both sufficient statistic and true value of the parameter.
- The fiducial argument - from the distribution of the pivot, a distribution for the parameter can be derived on the basis of the sampled sufficient statistic.

Let $X \sim Bin(n, p)$ and let $B_{a,b}$ denote the beta RV with parameters a and b. For an observed value x of X, $P(X \geq x|n, p) = P(B_{x,n-x+1} \leq p)$ and $P(X \leq x|n, p) = P(B_{x+1,n-x} \geq p)$.

From this we can see that there is a pair of fiducial distributions for p:

- $B_{x,n-x+1}$
- $B_{x+1,n-x}$

As stated in Krishnamoorthy & Zhang (2015), instead of having two fiducial variables, a random quantity that is between $B_{x,n-x+1}$ and $B_{x+1,n-x}$ can be used as a single approximate fiducial variable for p. From Cai (2005), a simple choice is $B_{x+0.5,n-x+0.5}$.

We are interested in a fiducial quantity for $p_1 - p_2$. Let x_1 be an observed value of X_1 and x_2 be an observed value of X_2.

The fiducial quantity for p_i is given by $B_{x_i+0.5, n_i-x_i+0.5}$ for $i = 1, 2$.

The fiducial quantity for the difference $\delta = p_1 - p_2$ is given

$$Q_\delta = B_{x_1+0.5, n_1-x_1+0.5} - B_{x_2+0.5, n_2-x_2+0.5.}$$

The $1-\alpha$ fiducial CI is given by

$$\left(Q_{\delta,\frac{\alpha}{2}}, Q_{\delta,1-\frac{\alpha}{2}}\right).$$

Note: This fiducial CI gives the same result as a Bayesian CI when using the Jeffreys prior.

2.3 Bayesian intervals

- **Jeffreys' Prior:**

$$\pi_J(p_1, p_2) \propto \prod_{i=1}^{2} p_i^{-\frac{1}{2}} (1-p_i)^{-\frac{1}{2}}.$$

The resulting joint posterior distribution will then be

$$\begin{aligned}\pi_J(p_1, p_2 | x_1, x_2) = & \frac{1}{B\left(x_1+\frac{1}{2}, n_1-x_1+\frac{1}{2}\right)} p_1^{x_1-\frac{1}{2}} (1-p_1)^{n_1-x_1-\frac{1}{2}} \\ & \times \frac{1}{B\left(x_2+\frac{1}{2}, n_2-x_2+\frac{1}{2}\right)} p_2^{x_2-\frac{1}{2}} (1-p_2)^{n_2-x_2-\frac{1}{2}}.\end{aligned}$$

- **Divergence Prior:**

$$\pi_D(p_1, p_2) \propto \prod_{i=1}^{2} p_i^{-\frac{1}{4}} (1-p_i)^{-\frac{1}{4}}.$$

The resulting joint posterior distribution will then be

$$\pi_D(p_1, p_2 | x_1, x_2) = \frac{1}{B\left(x_1 + \frac{3}{4}, n_1 - x_1 + \frac{3}{4}\right)} p_1^{x_1 - \frac{1}{4}} (1 - p_1)^{n_1 - x_1 - \frac{1}{4}} \times \frac{1}{B\left(x_2 + \frac{3}{4}, n_2 - x_2 + \frac{3}{4}\right)} p_2^{x_2 - \frac{1}{4}} (1 - p_2)^{n_2 - x_2 - \frac{1}{4}}.$$

- **Probability Matching Prior:**

  Datta & Ghosh (1995) derived the differential equation which a prior must satisfy if the posterior probability of a one sided credibility interval for a parametric function and its frequentist probability agree up to a certain order. They proved that the agreement between the posterior probability and the frequentist probability holds if and only if $\sum_{i=1}^{k} \frac{\partial}{\partial p_i} \left\{ \eta_i\left(\underline{p}\right) \pi\left(\underline{p}\right) \right\} = 0$.

  Theorem 1 Assume that X_1, X_2 are independent Binomial random variables with $X_i \sim Bin(n_i, p_i)$ for $i = 1, 2$. The probability matching prior for $\delta = p_1 - p_2$, the difference between two Binomial proportions, is given by

$$\pi_M(p_1, p_2) \propto \left\{ \sum_{i=1}^{2} p_i (1 - p_i) \right\}^{\frac{1}{2}} \prod_{i=1}^{2} p_i^{-1} (1 - p_i)^{-1}.$$

  The resulting joint posterior distribution will then be

$$\pi_M(p_1, p_2 | x_1, x_2) \propto \left\{ \sum_{i=1}^{2} p_i (1 - p_i) \right\}^{\frac{1}{2}} \prod_{i=1}^{2} p_i^{x_i - 1} (1 - p_i)^{n_i - x_i - 1}.$$

  It was shown in Raubenheimer & Van der Merwe (2011) that this posterior distribution is proper.

3. Simulation Study

In this section a simulation study will be done, where coverage rates and average interval lengths for $p_1 - p_2$ will be obtained. Raubenheimer & Van der Merwe (2011) compared the performance of the Jeffreys and probability matching priors with that of the frequentist results.

Table 1: Exact coverage rates (CR) and average interval lengths (LE), the nominal level is 0.95.

	WAL	AGC	π_J/ FID	π_D	π_M
$n_1 = n_2 = 10$					
CR	0.917	0.963	0.945	0.954	0.948
LE	0.659	0.680	0.650	0.667	0.649
$n_1 = n_2 = 20$					
CR	0.931	0.958	0.945	0.949	0.949
LE	0.489	0.494	0.482	0.497	0.481

Table 2: Exact coverage rates (CR) and average interval lengths (LE), the nominal level is 0.95, $n_1 = n_2 = 10$.

p_1	p_2		WAL	AGC	π_J/ FID	π_D	π_M
0.1	0.1	CR	0.950	0.991	0.967	0.991	0.988
		LE	0.456	0.578	0.542	0.559	0.568
0.1	0.7	CR	0.915	0.945	0.945	0.947	0.937
		LE	0.619	0.656	0.617	0.619	0.617
0.3	0.3	CR	0.905	0.963	0.937	0.962	0.968
		LE	0.756	0.727	0.717	0.708	0.699
0.3	0.7	CR	0.932	0.955	0.935	0.948	0.943
		LE	0.754	0.727	0.707	0.698	0.690
0.5	0.5	CR	0.912	0.958	0.956	0.956	0.960
		LE	0.830	0.771	0.763	0.751	0.741

Table 3: Exact coverage rates (CR) and average interval lengths (LE), the nominal level is 0.95, $n_1 = n_2 = 20$**.**

p_1	p_2		WAL	AGC	π_J/ FID	π_D	π_M
0.1	0.1	CR	0.960	0.988	0.942	0.961	0.963
		LE	0.352	0.396	0.378	0.387	0.394
0.1	0.7	CR	0.913	0.955	0.948	0.953	0.941
		LE	0.464	0.473	0.456	0.456	0.459
0.3	0.3	CR	0.931	0.950	0.940	0.950	0.957
		LE	0.552	0.538	0.534	0.531	0.528
0.3	0.7	CR	0.928	0.944	0.942	0.944	0.956
		LE	0.552	0.538	0.499	0.528	0.522
0.5	0.5	CR	0.919	0.957	0.950	0.957	0.958
		LE	0.604	0.578	0.576	0.571	0.564

4. Illustrative Example

Consider an example from Ornaghi et al. (1999). The goal of this experiment was to assess if male and female insects transmit the Mal de Rio Cuarto virus to susceptible maize plants at similar rates.

Assume that, at a specific stage, the researchers want to estimate the difference $p_1 - p_2$, where p_1 is equal to the proportion infected plants for male insects and p_2 is the proportion infected plants for female insects. Stage 1 will be considered here, where $\hat{p}_1 = 9/29$ and $\hat{p}_2 = 5/31$.

Table 4: 95% Credibility Intervals for the Difference in Disease Transmission Probabilities among Male and Female insects.

	Interval	WAL	AGC	π_J/ FID	π_D	π_M
Stage 1	Lower limit	-0.063	-0.070	-0.060	-0.063	-0.062
	Upper limit	0.631	0.351	0.352	0.348	0.345
	Length	0.425	0.421	0.412	0.411	0.406

5. Conclusion

In this paper the probability matching prior for the difference between two Binomial rates were derived. Limited simulation studies have shown that the probability matching prior achieves its sample frequentist coverage results somewhat better than in the case of the Jeffreys' prior. The matching prior and the divergence prior yielded similar results. The fiducial CI is the same as the Bayesian CI when using the Jeffreys' prior.

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