A Generalized Feedback Asymmetric Conditional Autoregressive Range Model

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Abstract

The Conditional Autoregressive Range (CARR) model is an alternative to the Generalized Autoregressive Conditionally Heteroscedastic (GARCH) approach of modeling volatility. The former models the price range while the latter focuses on modeling the price returns. The Asymmetric CARR (ACARR) model was introduced to allow for separate modeling of upward and downward ranges observed within each day, with the actual range expressed as the sum of these two components. This formulation, however, ignores any feedback from one type of range to another. The Feedback Asymmetric Conditional Autoregressive Range (FACARR) was introduced in 2017 to remedy this drawback. The FACARR, however, limits this cross feedback to past ranges and do not include past conditional means. The proposed Generalized Feedback Asymmetric Conditional Autoregressive Range Model (GFACARR) removes this limitation and allows the upward range model to include both past upward and past downward ranges along with their respective conditional means. A similar model is defined for modeling downward range as well. The proposed model is more in line with the multivariate CARR model. The use of the GFACARR model is illustrated by its application to several price series, including the S&P 500.

Key Words: Volatility Modeling, CARR Models, ACARR, Price Range, Time Series.

1. Introduction

Financial volatility is an essential factor that policy makers and investors should consider prior to any form of financial decision making. Modelling volatility is crucial in understanding the nature of the dynamics of the financial market. Financial volatility of asset prices has been discussed extensively in the financial and econometric literature over the years. One of the most successful volatility models used by researchers to model volatilities in a time series setting is the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model introduced by Bollerslev (1986). Engle (1982) proposed the Autoregressive Conditional Heteroscedasticity (ARCH) model in order to address the complexities of time varying volatility and volatility clustering in the financial time series. In the ARCH formulation, the conditional volatility is modeled as a function of past returns. The GARCH model is an extension of ARCH formulation and models the conditional volatility as a function of lagged squared returns as well as past conditional variances. Since all models afore mentioned focus on modeling price returns, they can be identified as examples of return-based volatility models.

In many financial time series applications, standard deviation is the most common measures of stock return volatility since it not only calculates the dispersion of returns but also summarizes the probability of seeing extreme values in returns. Since the time the concept of volatility was introduced, researchers have sought alternative measures of financial volatility. One such alternative is the range. Range measures the variability of a random variable. Parkinson (1980) argued that volatility measure could be calculated by considering daily high, daily low, and opening price of a stock in addition to the traditional closing prices. He also compared traditional measures of volatility that are calculated simply by using closing prices, with extreme value methods by taking high and low prices of an asset. He concluded that the range-based method is far superior to the standard methods based on returns. Beckers (1983) tested the validity of different volatility estimators. In his paper, he mentioned that the range of a stock price contains far important and fresh information. He also mentions that using the range of a stock price is better than using close-to-close changes. Kunitomo (1992), improved the Parkinson's original result and proposed a new range-based estimator which is 10 more times efficient than the standard volatility estimator. In another study, Alizadeh , Brandt and Diebold (2002) proved that the range-based volatility estimators are highly efficient compared to the classical volatility proxies based on log absolute returns or squared returns and showed that log range is approximately normal. Hence range of an asset price for a given period can be used as a more informative proxy variable to measure an asset's volatility for a well-defined period such as a day.

According to the results of Alizadeh *et al*. (2002) and many others, both the GARCH family of models and stochastic volatility models (Tylor, 1986) ignore the price fluctuations of the reference period, making them relatively inaccurate and inefficient. Therefore, some researchers focused on the alternative approach to volatility modeling and developed the theoretical framework for range-based models with comprehensive empirical examples. For example, the reader is referred to Chou (2005), Chou (2006), Brandt and Jones (2006), and Chou and Liu (2010). Chou (2005) introduced the Conditional Auto Regressive Range (CARR) model as a special case of Autoregressive Conditional Duration (ACD) model of Engle (1998). CARR is employed to model price volatility of an asset by considering range of the log prices for a given fixed time interval. The CARR model is similar to the standard volatility models such as the GARCH formulation. However, one distinct difference between the two models is that the GARCH model uses rate of return as its volatility measure while CARR model uses the range as its volatility measure. The CARR model proposed by Chou is a simple but an efficient tool to analyze the volatility clustering property compared to the GARCH models. This was shown empirically by Chou via out of sample forecasting of S&P500 data. Chou showed that the effectiveness of volatility estimates produced by CARR models is higher than the estimates of standard return-based models such as the GARCH. Brandt and Jones (2006) integrated the properties of exponential GARCH (Nelson (1991)) with daily log range data and proposed a ranged based EGARCH model. This model has a simple framework but is an effective tool for capturing the important characteristics that are present in stock return data such as clustering, negative correlation, and log normality. The range based EGARCH model is different from the CARR model in many ways. For example, it utilizes the lagged log range rather than lagged range as in the CARR model. Moreover, the ranged based EGARCH model formulates conditional return volatility while CARR explains the conditional mean of the range data.

Extensive modifications had been done to the original CARR model. Chiang, Chou and Wang (2016), suggested the lognormal log CARR model in an outlier detection process and showed that the proposed method can effectively detect outliers. One major advantage of using a Log CARR model would be that these models relax positivity restrictions on the parameters of the conditional expectation function. Xie and Wu (2017), explained the disturbance term in the CARR model by using the gamma distribution (GCARR) and showed through empirical data that GCARR outperformed Weibull CARR (WCARR) model in the forecasting ability. The multivariate extension to the CARR (MCARR) model was proposed by Fernandes, Mota and Rocha (2005), and they derived conditions for stationarity, geometric ergodicity, and beta-mixing with exponential decay. Chou and Liu (2009), incorporated return based Dynamic Conditional Correlation (DCC) model of Engle (2002) with the CARR model and introduced the new class of range based DCC models. They concluded that the range based DCC model outperforms other return-based models (MA100, EWMA, CCC, return-based DCC, and diagonal BEKK) using RMSE and MAE, the accepted benchmarks of implied and realized covariance. Different types of range-based volatility models such as Liu *el at*. (2017), Chou and Liu (2010), Miao, Wu and Su (2012), and Xie and Wang (2013) are some of the variations that are found in the published literature. For additional details, the reader is referred to Chou, R., Chou, H., and Liu (2015), which provides a comprehensive review of range-based models.

The asymmetric volatility, which is a key phenomenon in financial data, suggests that conditional volatilities show high fluctuations during downward trends than during upward trends. Traditional methods of modelling return series such as ARCH and GARCH models used standard deviation which treat price returns symmetrically. Hence, they are not effective tools for capturing the asymmetric behavior present in the financial data. In order to model the asymmetry in stock returns, several econometric models have been introduced in the literature. Asymmetric ARCH model of Nelson (1991), EGARCH by Nelson and Cao (1992), GJR-GARCH model by Glosten, Jagannathan and Runkle (1993) and QGARCH by Sentana (1995) had been developed and these models overcome the drawbacks of GARCH models. In their paper Engle and Ng (1993), analyzed how the news effect on the conditional volatility and concluded that both EGARCH and GJR-GARCH capture the asymmetry but latter is the better model.

All the above models capture the asymmetry in return data. The CARR model proposed by Chou (2005), used range as the measure of price volatility. The study treated maximum and minimum price symmetrically. However, in the same study, he suggested the CARRX models (CARRX-a, and CARRX-b) by including exogenous variables such as (a) lagged return and (b) lagged absolute returns in the conditional mean equation. The purpose of this incorporation is to model one form of asymmetry, the leverage effect of Black and Nelson (1991). Chou (2006), presented Asymmetric CARR (ACARR) model in which both upward and downward price ranges were treated separately. The upward range is defined as the difference between the maximum price and the opening price and the downward range is defined as the difference between the opening price and the minimum price, all observed within a trading day. These definition can be extended to periods beyond a day in a similar fashion. Instead of treating high and low prices for a given fixed period symmetrically as in the CARR, the ACARR model incorporates a form of asymmetry by allowing the dynamic structure of the upward price movements to be different from that of the downward price movements. The ACARR model was extended to the ACARRX model by including exogenous variables such as trading volume (Lamourex and Lastrapes, 1990), lag return to count leverage effect (Black, 1976; Nelson, 1990), or a seasonal factor. It assumed independence between upward ranges and downward ranges and therefore parameters were estimated separately for each movement by using the QMLE method. An empirical study showed that the volatility forecasting ability of the ACARR model is superior to that of the CARR model. Chou and Wang (2014), combined the ACARR model, to capture current asymmetric volatility, with extreme value theory (EVT) to estimate the tail of the residual distribution. This methodology gave better Value at Risk (VaR) estimates than the GARCH model used by McNeil and Frey (2000).

Motivated by the independence between upward swing and downward plunge assumption made by Chou (2006), Xie (2018) proposed Feedback Asymmetric CARR (FACARR) model. By providing satisfactory evidence, he questioned the validity of the independence assumption and found cross-interdependence between upward movement and downward movement. Hence, the FACARR model was proposed as a more practical extension of the ACARR model. In other words, both upward and downward movements of asset prices were not only modeled asymmetrically, the conditional mean upward (downward) range is modeled by incorporating lagged downward (upward) ranges into each sub-model. Extensive empirical studies showed that the proposed FACARR performs significantly better than ACARR in both in sample and out of sample forecasting.

It is reasonable to assume that the dynamic movement of the upward (downward) range does not depend only on the lagged downward (upward) price range but also on the conditional mean of downward (upward) ranges. By consolidating on this fact, we decided to generalize the previous class of asymmetric CARR models and introduced Generalized Feedback Asymmetric CARR (GFCARR) model. The proposed model attempts to overcome the limitation of previous models by incorporating the cross-feedback term to account for the past conditional means. Since the proposed GFACARR model treat both upward and downward price range separately, this approach also allows the modeling of the asymmetry found in financial data.

The paper is summarized as follows. In Section 2a brief introduction to the CARR, ACARR and FACARR models are given. The proposed GFACARR model is introduced in addition to its statistical properties in section 3. Econometric methodology was presented in Section 4 and the results of a simulation study is presented in Section 5. Empirical study based on three different stock market indices namely S&P500, CAC40 and NIKKEI225arediscussed in Section 6 and conclusion are given in Section 7.

2. Review of CARR, ACARR and FACARR Models

2.1. The Conditional Autoregressive Range (CARR) Model

Chou (2005), proposed the CARR which is primarily a range-based model. The CARR formulation is used to model the price volatility of an asset by considering range as a measure of this volatility. Let R_t be the price range defined over the fixed time period *t*, where R_t is the difference between the highest (P_t^{high}) and lowest (P_t^{low}) logarithmic price of an asset during the time period *t*. That is,

$$R_t = P_t^{high} - P_t^{low}.$$

The CARR model of order (p, q) is presented as CARR (p, q) and defined as follows:

$$R_t = \lambda_t \varepsilon_t,$$

$$\mathrm{E}\left(R_t \mid \mathbb{F}_{t-1}\right) = \lambda_t = \omega + \sum_{i=1}^{p} \alpha_i . R_{t-i} + \sum_{j=1}^{q} \beta_j . \lambda_{t-j},$$

$$\varepsilon_t \sim i.i.d.\, f(.), E\left(\varepsilon_t\right) = 1, \text{ and}$$

$$0 < \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1, \alpha_i > 0, \beta_j > 0.$$

Here λ_t is the conditional expectation of the price range based on all information up to time$t$-1. The non-negative disturbance term, also known as the standardized range is denoted by ε_t which is independent and identically distributed with probability density function $f(.)$ with a non-negative support and a unit mean.

2.2 The Asymmetric Conditional Autoregressive Range (ACARR) Model

The ACARR model presented by Chou (2006), decomposed the range (R_t) series into two components namely upward range (R_t^u) and downward range (R_t^d). Upward and downward ranges are defined using the differences between the daily high (P_t^{high}), daily low (P_t^{low}), and the opening (P_t^{open}) logarithmic price of an asset over the time interval associated with t as follows:

$$\begin{aligned} R_t^u &= P_t^{high} - P_t^{open}, \\ R_t^d &= P_t^{open} - P_t^{low}, \\ R_t &= R_t^u + R_t^d = P_t^{high} - P_t^{open} + P_t^{open} - P_t^{low} = P_t^{high} - P_t^{low}. \end{aligned} \tag{1}$$

Here the upward range measures the maximum gain or the positive shock to the stock while downward range calculates the minimum gain or the negative impact to the stock price for the time period t.

The CARR model is symmetric because it treats the high and low price in a symmetric way. However, it is possible to assume that the upward and downward movements are different in their dynamics of how the shocks propagate. To allow the asymmetric behavior in price range data, Chou (2006) proposed and developed ACARR model.

The ACARR model of order (p, q) is as follows:

$$
\begin{aligned}
&R_t = R_t^u + R_t^d, \\
&R_t^u = \lambda_t^u \varepsilon_t^u, \\
&R_t^d = \lambda_t^d \varepsilon_t^d, \\
&\lambda_t^u = \omega^u + \sum_{i=1}^{p} \alpha_i^u R_{t-i}^u + \sum_{j=1}^{q} \beta_j^u \lambda_{t-j}^u, \\
&\lambda_t^d = \omega^d + \sum_{i=1}^{p} \alpha_i^d R_{t-i}^d + \sum_{j=1}^{q} \beta_j^d \lambda_{t-j}^d, \\
&\varepsilon_t^u \sim i.i.d.\, f^u(.), E\left(\varepsilon_t^u\right) = 1, \\
&\varepsilon_t^d \sim i.i.d.\, f^d(.), E\left(\varepsilon_t^d\right) = 1, \\
&0 < \sum_{i=1}^{p} \alpha_i^u + \sum_{j=1}^{q} \beta_j^u < 1, \alpha_i^u > 0, \beta_j^u > 0, \text{ and} \\
&0 < \sum_{i=1}^{p} \alpha_i^d + \sum_{j=1}^{q} \beta_j^d < 1, \alpha_i^d > 0, \beta_j^d > 0.
\end{aligned}
\tag{2}
$$

Here $\lambda_t^u \left(= E\left(R_t^u \mid \mathbb{F}_{t-1}\right)\right)$ is the conditional mean of the upward range and $\lambda_t^d \left(= E\left(R_t^d \mid \mathbb{F}_{t-1}\right)\right)$ is the conditional mean of the downward range, both conditional on all information up to time period t-1. The disturbance term of the upward (downward) range model $\varepsilon_t^u \left(\varepsilon_t^d\right)$ are independently and identically distributed with the density function $f^u(.)\left(f^d(.)\right)$ with unit mean. Moreover, the pairs of parameters, $\left(\omega^u, \omega^d\right), \left(\alpha_i^u, \alpha_i^d\right), \left(\beta_j^u, \beta_j^d\right)$ identify the asymmetric behavior between the upward range and downward range.

2.3. The Feedback Asymmetric Conditional Autoregressive Range (FACARR) Model

The ACARR model assumes that there is independence between the upward and downward shocks and Xie (2018), argued against this assumption and presented the FACARR model. This model include the cross-interdependence terms on top of the ACARR setting. Following the same definitions and notations, the FACARR model is defined as follows:

$$
\begin{aligned}
&R_t = R_t^u + R_t^d, \\
&R_t^u = \lambda_t^u \varepsilon_t^u, \\
&R_t^d = \lambda_t^d \varepsilon_t^d, \\
&\lambda_t^u = \omega^u + \sum_{i=1}^{p} \alpha_i^u R_{t-i}^u + \sum_{j=1}^{q} \beta_j^u \lambda_{t-j}^u + \sum_{k=1}^{l} \gamma_k^u R_{t-k}^d, \\
&\lambda_t^d = \omega^d + \sum_{i=1}^{p} \alpha_i^d R_{t-i}^d + \sum_{j=1}^{q} \beta_j^d \lambda_{t-j}^d + \sum_{k=1}^{l} \gamma_k^d R_{t-k}^u, \\
&\varepsilon_t^u \sim i.i.d.\, f^u(.), E\left(\varepsilon_t^u\right) = 1, \\
&\varepsilon_t^d \sim i.i.d.\, f^d(.), E\left(\varepsilon_t^d\right) = 1, \\
&0 < \sum_{i=1}^{p} \alpha_i^u + \sum_{j=1}^{q} \beta_j^u < 1, \alpha_i^u > 0, \beta_j^u > 0, \text{ and} \\
&0 < \sum_{i=1}^{p} \alpha_i^d + \sum_{j=1}^{q} \beta_j^d < 1, \alpha_i^d > 0, \beta_j^d > 0.
\end{aligned} \tag{3}
$$

In addition to the previous parameter set discussed in the model (2), FACARR has a new pair of parameters, namely $\left(\gamma^u, \gamma^d\right)$, which measures the magnitude and the direction of the lagged upward (downward) range on conditional mean range.

3. Generalized Feedback Asymmetric Conditional Autoregressive Range Model (GFACARR) and Statistical properties.

Let P_t^{open}, P_t^{high} and P_t^{low} be the opening, high and low logarithmic prices of the speculative asset respectively, at a given time period *t* (i.e. day). The observed price range for the time period *t*is denoted as $R_t$and it is defined as the sum of the upward range (R_t^u) and downward range (R_t^d):

$$
R_t = P_t^{high} - P_t^{low} = \left[P_t^{high} - P_t^{open}\right] + \left[P_t^{open} - P_t^{low}\right] = R_t^u + R_t^d .
$$

Here, upward and the downward range is defined exactly the same as in the ACARR model.

The proposed GFACARR model is as follows:

$$
\begin{aligned}
& R_t = R_t^u + R_t^d, \\
& R_t^u = \lambda_t^u \varepsilon_t^u, \\
& R_t^d = \lambda_t^d \varepsilon_t^d, \\
& E\left(R_t^u \mid \mathbb{F}_{t-1}\right) = \lambda_t^u = \omega^u + \sum_{i=1}^{p^u} \alpha_i^u R_{t-i}^u + \sum_{j=1}^{q^u} \beta_j^u \lambda_{t-j}^u + \sum_{k=1}^{r^u} \gamma_k^u R_{t-k}^d + \sum_{l=1}^{s^u} \delta_l^u \lambda_{t-l}^d, \\
& E\left(R_t^d \mid \mathbb{F}_{t-1}\right) = \lambda_t^d = \omega^d + \sum_{i=1}^{p^d} \alpha_i^d R_{t-i}^d + \sum_{j=1}^{q^d} \beta_j^d \lambda_{t-j}^d + \sum_{k=1}^{r^d} \gamma_k^d R_{t-k}^u + \sum_{l=1}^{s^u} \delta_l^d \lambda_{t-l}^u, \\
& \varepsilon_t^u \sim i.i.d\ f^u(.), E\left(\varepsilon_t^u\right) = 1, \\
& \varepsilon_t^d \sim i.i.d\ f^d(.), E\left(\varepsilon_t^d\right) = 1, \text{ and} \\
& \omega^u > 0, \alpha_i^u > 0, \beta_j^u > 0; \omega^d > 0, \alpha_i^d > 0, \beta_j^d > 0.
\end{aligned}
\tag{4}
$$

Here $\lambda_t^u \left(= E\left(R_t^u \mid \mathbb{F}_{t-1}\right)\right)$ is the conditional mean of the upward range based on all information up to time period t-1, and $\lambda_t^d \left(= E\left(R_t^d \mid \mathbb{F}_{t-1}\right)\right)$ is the conditional mean of the downward range on all information up to time period t-1. Note that the sigma field generated using information from setup to time period t-1, is denoted by $\mathbb{F}_{t-1}$. The upward (downward) range disturbance term is denoted by ε_t^u $\left(\varepsilon_t^d\right)$ and it is independently and identically distributed with unit mean.

In contrast to the FACARR model introduced by Xie (2018), the significance of the proposed formulation is that GFACARR model is capable of modelling the conditional expected upward (downward) range at time t based on not only the lagged downward (upward) ranges but also on the previous conditional volatilities of downward (upward) ranges.

3.1 The GFACARR Model.

Here, the mean conditional upward (downward) range at time period t, is modeled by considering both downward (upward) range and mean conditional downward (upward) range at time$t$-1, in addition to the existing terms.

The GFACARR model given in the equation (3) can be re-written as a bivariate CARR (1, 1) model as follows:

$$
\vec{R}_t = \Lambda_t\left(\Phi\right)\vec{\varepsilon}_t, \tag{5}
$$

where $\Lambda_t(\Phi) = diag\{\lambda_t^u, \lambda_t^d\}$ and $\lambda_t^i\,(i = u,d)$, is the conditional mean of $R_t^i\,(i = u,d)$ given $\mathbb{F}_{t-1}$. Here $\Phi = (\omega^u, \alpha^u, \beta^u, \gamma^u, \delta^u, \omega^d, \alpha^d, \beta^d, \gamma^d, \delta^d)'$ is the parameter vector and $\vec{\varepsilon_t} = (\varepsilon_t^u \quad \varepsilon_t^u)'$ has following conditions imposed on it:

1. $\operatorname{cov}(\varepsilon_t^u, \varepsilon_t^d) = 0.$
2. $\{\vec{\varepsilon_t}, t \in \mathbb{Z}^+\}$ is a sequence of independent and identically distributed $\mathbb{R}^2$-valued random variables with $E(\vec{\varepsilon_t}) = \begin{pmatrix} E(\varepsilon_t^u) \\ E(\varepsilon_t^d) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For the illustrative purpose we assume $\varepsilon_t^i \sim \exp(1), \forall i = u,d.$ Then the covariance matrix becomes $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2} = I_2.$
3. $\operatorname{cov}(\varepsilon_j^i, \varepsilon_k^i) = 0, \forall j \neq k, \forall i = u,d.$
4. From conditions1 and 2, conditional covariance matrix of $\vec{R_t}$ follows and is given by $H_t(\Phi)$ such that,
$$H_t(\Phi) = \begin{pmatrix} (\lambda_t^u)^2 & 0 \\ 0 & (\lambda_t^d)^2 \end{pmatrix}_{2*2} = \Lambda^2{}_t(\Phi).$$

This representation of model (4), is coincide with bivariate GARCH (1, 1) process with constant correlation (see. Bollerslev (1990), Jeantheau (1998)).

The GFACARR process can be formulated as a bivariate CARR (1, 1) model as follows:

$$\begin{pmatrix} \lambda_t^u \\ \lambda_t^d \end{pmatrix} = \begin{pmatrix} \omega^u \\ \omega^d \end{pmatrix} + \begin{pmatrix} \alpha^u & \gamma^u \\ \gamma^d & \alpha^d \end{pmatrix} \begin{pmatrix} R_{t-1}^u \\ R_{t-1}^d \end{pmatrix} + \begin{pmatrix} \beta^u & \delta^u \\ \delta^d & \beta^d \end{pmatrix} \begin{pmatrix} \lambda_{t-1}^u \\ \lambda_{t-1}^d \end{pmatrix},$$
$$\vec{\lambda_t} = \vec{\omega} + A\vec{R_t} + B\overrightarrow{\lambda_{t-1}}. \tag{6}$$

Here $\vec{\lambda_t} = \begin{pmatrix} \lambda_t^u \\ \lambda_t^d \end{pmatrix}$, $\overrightarrow{R_{t-1}} = \begin{pmatrix} R_{t-1}^u \\ R_{t-1}^d \end{pmatrix}$, $\vec{\omega} = \begin{pmatrix} \omega^u \\ \omega^d \end{pmatrix}$, $A = \begin{pmatrix} \alpha^u & \gamma^u \\ \gamma^d & \alpha^d \end{pmatrix}$ and $B = \begin{pmatrix} \beta^u & \delta^u \\ \delta^d & \beta^d \end{pmatrix}$.

If the vector $\vec{\omega} \in \mathbb{R}_{>0}^2$, and all the coefficients in $A \in [a_{i,j}], B \in [b_{i,j}] \forall i, j;$ and $i = 1,2;\ j = 1,2.$ are non-negative then this is sufficient for the non-negativity of the $\vec{\lambda_t} \in \mathbb{R}_{\geq 0}^2$. However, in this proposed model we allow negative values for the coefficients of the newly introduced lagged conditional expected upward (downward) term. Since both range and conditional mean range are positive variables it is important to preserve the positivity of the model. We studied closely the conditions for the non-negativity and positivity imposed in the Dynamic Conditional Correlation Multivariate GARCH models (Engle and Sheppard, 2001). Nelson and Cao (1992) introduced non-negativity constraints for the GARCH (p, q) models by relaxing the above mentioned sufficient condition. Since these conditions were not readily applicable to our model, we modified the conditions to suit our model. Our conditions are that $\vec{\omega} \in \mathbb{R}^2_{>0}$, with positive coefficients, and that $\left[a_{i,j}\right] \in \mathbb{R}$ and $\left[b_{i,j}\right] \in \mathbb{R}$ such that $\left[a_{i,j} + b_{i,j}\right] \in \mathbb{R}, \forall i, j.$ with eigenvalues of $\left(A+B\right)_{2\times 2}$, expressed as Δ_1 and Δ_2, follow the restrictions $\left|\Delta_1\right| < 1$ and $\left|\Delta_2\right| < 1$.

3.2 Statistical Properties of the GFACARR Model.

3.2.1. Weak Stationarity of GFACARR Model.

Since GFACARR can be presented as the bivariate CARR (1, 1) model and it can also be reparametrized as a bivariate ARMA (1, 1) model. Derivation is given bellow:

$$\begin{aligned}
&\vec{R_t} - E\left(\vec{R_t} \mid \mathbb{F}_{t-1}\right) = \vec{R_t} - \vec{\lambda_t} = \vec{\eta_t}, \\
&\vec{\lambda_t} = \vec{R_t} - \vec{\eta_t} = \vec{\omega} + A\overrightarrow{R_{t-1}} + B\left(\overrightarrow{R_{t-1}} - \overrightarrow{\eta_{t-1}}\right), \\
&\vec{R_t} = \vec{\omega} + A\overrightarrow{R_{t-1}} + B\overrightarrow{R_{t-1}} + \vec{\eta_t} + \left(-B\right)\overrightarrow{\eta_{t-1}}.
\end{aligned}$$

Let $\vec{\eta}_t$ be the difference vector and $\vec{R_t} = \vec{\omega} + \left(A+B\right).\overrightarrow{R_{t-1}} + \vec{\eta_t} - B.\overrightarrow{\eta_{t-1}}$ be a Bivariate ARMA (1, 1) model. If all the eigenvalues of the matrix $\left(A+B\right)_{2\times 2}$ are positive, but less than one, then the bivariate ARMA (1, 1) model for $\vec{R_t}$ is weakly stationary (Tsay, 2002). By following this claim we propose the weakly stationarity conditions for the GFACARR model.

Theorem 1: Let $\vec{\lambda_t} = \vec{\omega} + A.\overrightarrow{R_{t-1}} + B.\overrightarrow{\lambda_{t-1}}$ be a GFACARR process defined in (4)-(6). If all the eigenvalues of $\left(A+B\right)_{2\times 2}$, namely Δ_1 and Δ_2 are such that $\left|\Delta_i\right| < 1\ \forall i$, then the GFACARR model for $\vec{R_t}$ is weakly stationary.

The Proof of this theorem will be presented in the separate paper.

3.2.2 Unconditional Expectation of GFACARR Model.

Under the weak stationarity assumption $E\left(\vec{R_t}\right) = E\left(\overrightarrow{R_{t-1}}\right)$, and $E\left(\vec{\eta_t}\right) = E\left(\vec{R_t} - \vec{\lambda_t}\right) = 0$ so that:

$$\overrightarrow{R_t} = \vec{\omega} + (A+B)\overrightarrow{R_{t-1}} + \overrightarrow{\eta_t} + (-B)\overrightarrow{\eta_{t-1}},$$
$$E\left(\overrightarrow{R_t}\right) = \vec{\omega} + (A+B)E\left(\overrightarrow{R_{t-1}}\right) + E\left(\overrightarrow{\eta_t}\right) + E\left[(-B)\overrightarrow{\eta_{t-1}}\right],$$
$$E\left(\overrightarrow{R_t}\right) = \vec{\omega} + (A+B)E\left(\overrightarrow{R_{t-1}}\right) = \vec{\omega} + (A+B)E\left(\overrightarrow{R_t}\right),$$
$$\left[I-(A+B)\right]E\left(\overrightarrow{R_t}\right) = \vec{\omega}.$$

Since the $\overrightarrow{R_t}$ is a weakly stationary and $\det\left[I-(A+B)\right] \neq 0$, hence $E\left(\overrightarrow{R_t}\right)$ exists. Thus,

$$E\left(\overrightarrow{R_t}\right) = \left[I-(A+B)\right]^{-1}\vec{\omega},$$
$$\left[I-(A+B)\right]^{-1} = \begin{pmatrix} 1-\left(\alpha^u+\beta^u\right) & -\left(\gamma^u+\delta^u\right) \\ -\left(\gamma^d+\delta^d\right) & 1-\left(\alpha^d+\beta^d\right) \end{pmatrix}^{-1},$$

$$\left[I-(A+B)\right]^{-1} = \frac{1}{\left\{\left[1-\left(\alpha^u+\beta^u\right)\right]\left[1-\left(\alpha^d+\beta^d\right)\right]-\left[\gamma^u+\delta^u\right]\left[\gamma^d+\delta^d\right]\right\}}\begin{bmatrix} 1-\left(\alpha^d+\beta^d\right) & +\left(\gamma^u+\delta^u\right) \\ +\left(\gamma^d+\delta^d\right) & 1-\left(\alpha^u+\beta^u\right) \end{bmatrix},$$
$$E\left(\overrightarrow{R_t}\right) = \frac{1}{\left\{\left[1-\left(\alpha^u+\beta^u\right)\right]\left[1-\left(\alpha^d+\beta^d\right)\right]-\left[\gamma^u+\delta^u\right]\left[\gamma^d+\delta^d\right]\right\}}\begin{bmatrix} 1-\left(\alpha^d+\beta^d\right) & +\left(\gamma^u+\delta^u\right) \\ +\left(\gamma^d+\delta^d\right) & 1-\left(\alpha^u+\beta^u\right) \end{bmatrix}\begin{bmatrix} \omega^u \\ \omega^d \end{bmatrix},$$
$$E\left(\overrightarrow{R_t}\right) = \frac{1}{\left\{\left[1-\left(\alpha^u+\beta^u\right)\right]\left[1-\left(\alpha^d+\beta^d\right)\right]-\left[\gamma^u+\delta^u\right]\left[\gamma^d+\delta^d\right]\right\}}\begin{bmatrix} \left(1-\left(\alpha^d+\beta^d\right)\right)\omega^u+\left(\gamma^u+\delta^u\right)\omega^d \\ \left(1-\left(\alpha^u+\beta^u\right)\right)\omega^d+\left(\gamma^d+\delta^d\right)\omega^u \end{bmatrix}.$$

The unconditional mean of upward range $E\left(R_t^u\right)$ and unconditional mean of downward range $E\left(R_t^d\right)$, can be expressed as follows:

$$E\left(R_t^u\right) = \frac{\left[1-\left(\alpha^d+\beta^d\right)\right]\omega^u+\left(\gamma^u+\delta^u\right)\omega^d}{\left\{\left[1-\left(\alpha^u+\beta^u\right)\right]\left[1-\left(\alpha^d+\beta^d\right)\right]-\left(\gamma^u+\delta^u\right)\left(\gamma^d+\delta^d\right)\right\}},$$
$$E\left(R_t^d\right) = \frac{\left[1-\left(\alpha^u+\beta^u\right)\right]\omega^d+\left(\gamma^d+\delta^d\right)\omega^u}{\left\{\left[1-\left(\alpha^u+\beta^u\right)\right]\left[1-\left(\alpha^d+\beta^d\right)\right]-\left(\gamma^u+\delta^u\right)\left(\gamma^d+\delta^d\right)\right\}}.$$

Finally the unconditional mean range $E\left(R_t\right)$ is calculated as:

$$E\left(R_t\right) = E\left(R_t^u\right)+E\left(R_t^d\right),$$
$$E\left(R_t\right) = \frac{\left[1-\left(\alpha^d+\beta^d\right)\right]\omega^u+\left(\gamma^u+\delta^u\right)\omega^d+\left[1-\left(\alpha^u+\beta^u\right)\right]\omega^d+\left(\gamma^d+\delta^d\right)\omega^u}{\left[1-\left(\alpha^u+\beta^u\right)\right]\left[1-\left(\alpha^d+\beta^d\right)\right]-\left(\gamma^u+\delta^u\right)\left(\gamma^d+\delta^d\right)}.$$

4. Estimation of GFACARR Model.

Let $\left\{\varepsilon_t^u\right\}$ $\left(\left\{\varepsilon_t^d\right\}\right)$ be the sequence of independent and identically distributed exponential disturbance term with $E\left(\varepsilon_t^u\right)=1$ $\left(E\left(\varepsilon_t^d\right)=1\right)$, and $\left\{R_t^u\right\}_{t=1}^n=\{R_1^u,R_2^u,...,R_n^u\}$ $\left(\left\{R_t^d\right\}_{t=1}^n=\{R_1^d,R_2^d,R_3^d,...,R_n^d\}\right)$ be the realization of the model $R_t^u=\lambda_t^u\varepsilon_t^u$ $\left(R_t^d=\lambda_t^d\varepsilon_t^d\right)$. The parameter vector $\Phi=\left(\omega^u,\alpha^u,\beta^u,\gamma^u,\delta^u,\omega^d,\alpha^d,\beta^d,\gamma^d,\delta^d\right)'$ can be estimated by using the conditional likelihood method. In this section we are going to derive the log likelihood function for the proposed GFACARR model.

The conditional distribution of R_t^u and R_t^d given the information up to *t*-1, can be expressed as follows:

$$f\left(R_t^u\mid\mathbb{F}_{t-1},\Phi\right)\sim\frac{1}{\lambda_t^u}\exp\left(-\frac{R_t^u}{\lambda_t^u}\right),$$
$$f\left(R_t^d\mid\mathbb{F}_{t-1},\Phi\right)\sim\frac{1}{\lambda_t^d}\exp\left(-\frac{R_t^d}{\lambda_t^d}\right).$$

Since $\operatorname{cov}\left(\varepsilon_t^u,\varepsilon_t^d\right)=0$, conditional distributions of $f\left(R_t^u\mid\mathbb{F}_{t-1},\Phi\right)$ and $f\left(R_t^d\mid\mathbb{F}_{t-1},\Phi\right)$ are conditionally independent. Then the conditional joint distribution of the realized range data at time *t*, given the information set up to time *t*-1 is given by:

$$f\left(R_t^u,R_t^d\mid\mathbb{F}_{t-1},\Phi\right)=\left(f\left(R_t^u\mid\mathbb{F}_{t-1},\Phi\right)\right)\left(f\left(R_t^d\mid\mathbb{F}_{t-1},\Phi\right)\right),$$
$$f\left(R_t^u,R_t^d\mid\mathbb{F}_{t-1},\Phi\right)=\left(\frac{1}{\lambda_t^u}\exp\left(-\frac{R_t^u}{\lambda_t^u}\right)\right)\left(\frac{1}{\lambda_t^d}\exp\left(-\frac{R_t^d}{\lambda_t^d}\right)\right).$$

Therefore the conditional likelihood function $L\left(\Phi\mid\left\{R_t^u,R_t^d\right\}_{t=1}^n\right)$ and the log likelihood function of the data $l\left(\Phi\mid\left\{R_t^u,R_t^d\right\}_{t=1}^n\right)$ can be derived as follows:

$$L\left(\Phi\mid\left\{R_t^u,R_t^d\right\}_{t=1}^n\right)=\prod_{t=2}^n f\left(R_t^u,R_t^d\mid\mathbb{F}_{t-1},\Phi\right),$$
$$l\left(\Phi\mid\left\{R_t^u,R_t^d\right\}_{t=1}^n\right)=\ln\left(L\left(\Phi\mid\left\{R_t^u,R_t^d\right\}_{t=1}^n\right)\right)=\sum_{t=2}^n\ln\left(f\left(R_t^u,R_t^d\mid\mathbb{F}_{t-1},\Phi\right)\right),\qquad(7)$$
$$l\left(\Phi\mid\left\{R_t^u,R_t^d\right\}_{t=1}^n\right)=-\sum_{t=2}^n\left(\left(\ln\left(\lambda_t^u\right)+\frac{R_t^u}{\lambda_t^u}\right)+\left(\ln\left(\lambda_t^d\right)+\frac{R_t^d}{\lambda_t^d}\right)\right).$$

5. Simulation Study

We investigated the finite sample performance of estimators using a simulation study. We used *nloptr*, a nonlinear optimization function of R software to generate the relevant data. Length of the time series studied is set to *n*=1000 and *n*=3000, and $m = 1000$.Simulation runs were carried out for each parameter sample size combination. We carried out this simulation study for two different cases of the GFACARR model by excluding or including correlation between the upward and downward range disturbance terms. First, we selected the parameters for the GFACARR model, generated data from this model, and then, maximized the profile likelihood function (7) using the constrained nonlinear optimization function *nloptr* in R. The Mean Absolute Deviation Error (MADE) is utilized as the evaluation criterion. The MADE is defined as, $\frac{1}{m}\sum_{i=1}^{m}\left|\hat{\phi}_j - \phi_j\right|$ where *m* is the number of replications. Simulation results are reported in Table 1 and Table 2.

Table 1: Means of MLE estimates and MADE (within parentheses), for Upward Range component in GFACARR model.

Model	**Sample Size**	ω^u	α^u	β^u	γ^u	∂^u
True Coefficients		**0.01**	**0.20**	**0.40**	**0.10**	**0.20**
UPR_M1	n=1000	0.0113 (0.0103)	0.1968 (0.0320)	0.3871 (0.1111)	0.0973 (0.0256)	0.2079 (0.1203)
	n=3000	0.0101 (0.0068)	0.1992 (0.0183)	0.3937 (0.0620)	0.0989 (0.0139)	0.2063 (0.0775)
True Coefficients		**0.01**	**0.30**	**0.50**	**0.10**	**-0.02**
UPR_M2	n=1000	0.0117 (0.0049)	0.2960 (0.0315)	0.5081 (0.0662)	0.0991 (0.0106)	-0.0235 (0.0216)
	n=3000	0.0105 (0.0026)	0.2984 (0.0183)	0.5028 (0.0363)	0.0996 (0.0057)	-0.0211 (0.0116)
True Coefficients		**0.15**	**0.20**	**0.60**	**0.10**	**-0.10**
UPR_M3	n=1000	0.1548 (0.0255)	0.2047 (0.0279)	0.5428 (0.0943)	0.0968 (0.0153)	-0.0724 (0.0276)
	n=3000	0.1522 (0.0136)	0.2026 (0.0164)	0.5714 (0.0500)	0.0986 (0.0080)	-0.0863 (0.0137)

Table 2: Means of MLE estimates and MADE (within parentheses), for Downward Range component in GFACARR model.

Model	**Sample Size**	ω^d	α^d	β^d	γ^d	∂^d
True Coefficients		**0.02**	**0.10**	**0.80**	**0.02**	**-0.05**
DWNR_M1	n=1000	0.0235 (0.0089)	0.1011 (0.0221)	0.7714 (0.1115)	0.0183 (0.0261)	-0.0430 (0.0989)
	n=3000	0.0210 (0.0040)	0.1006 (0.0224)	0.7929 (0.0118)	0.0201 (0.0510)	-0.0500 (0.0143)
True Coefficients		**0.04**	**0.10**	**0.60**	**0.03**	**0.60**
DWNR_M2	n=1000	0.0438 (0.0142)	0.0965 (0.0257)	0.5919 (0.0642)	0.0322 (0.0612)	0.6157 (0.1788)
	n=3000	0.0415 (0.0077)	0.0990 (0.0144)	0.5935 (0.0374)	0.0295 (0.0354)	0.6154 (0.1084)
True Coefficients		**0.10**	**0.20**	**0.40**	**0.10**	**0.50**
DWNR_M3	n=1000	0.1004 (0.0593)	0.1975 (0.0308)	0.3934 (0.1171)	0.0951 (0.0490)	0.5213 (0.2376)
	n=3000	0.0934 (0.0355)	0.1991 (0.0172)	0.3954 (0.0643)	0.0967 (0.0278)	0.5242 (0.1440)

Table 1 shows the simulation results related to the upward range component parameters of the GFACARR model while Table 2 presents that of the downward range component of the GFACARR model. The results show that the estimates are very close to the true parameters values in most cases and that the MADE values are reasonably small, with an improvement seen in the 3000 sample size case. The conclusion that can be arrived based on this simulation study is that the maximum likelihood method provides reliable estimates of the model parameters, in spite of the complex nature of the model when compared to CARR or even the FACARR models.

6. Empirical Study

6.1 The Data Set

In this study, three stock indices from different markets are used to gauge the performance of the proposed GFACARR model. Daily data of Standard and Poor's 500 (S&P500) index of United States, CAC 40, which is a benchmark index of the French stock market, and Japan's NIKKEI225 index are considered. The sample periods for S&P500, CAC40 and NIKKEI225 are January 01, 1990 to May 5, 2017, January 01, 1990 to May 31, 2018, and January 01, 1990 to December 31, 2019,respectively. Daily values for opening price, closing price, high price, low price and adjusted price are reported over the span of the study period. The data set is downloaded from the "Yahoo Finance" web page (https://finance.yahoo.com/) by using *quantmod* package in R software. The data set is divided in to two sub-samples where the first sub-sample, also known as in-sample period, is used to estimate the model parameters and in-sample predictions. In-sample periods for S&P500 spans from January 01, 1990 to December 31, 2016, CAC40 spans from January

01, 1990 to December 31, 2017 and NIKKEI225 spans from January 01, 1990 to December 31, 2018. The second-sub sample, which is also called the out-of-sample period, is used for out-of-sample forecasting. Out-of-sample periods for S&P500, CAC40 and NIKKEI225 are from January 1, 2017 to May 05, 2017,January 1, 2018 to May 31, 2018 and January 1, 2019 to December 31, 2019,respectively. In general, Table 3 presents the summarization of the three stock indices, more specifically Table 3A, Table 3B and Table 3C present the summary statistics of S&P500, CAC40 and NIKKEI225 daily stock indices respectively. Daily price range, daily upward range and daily downward range values are calculating as discussed in equation (1).

Table 3: Summary Statistics of the Daily Range, Upward Range and Downward Range of S&P500, CAC 40 and NIKKEI 225 Indices.

Table 3A: Summary Statistics of S&P 500 : 01/01/1990 – 12/31/2016			
Summary Statistics	**Upward Range Component**	**Downward Range Component**	**Range**
Minimum	0.0000	0.0000	0.1774
Mean	0.6179	0.6618	1.2798
Maximum	10.2457	9.5522	10.9041
Standard Deviation	0.7007	0.8146	0.9227
Skewness	3.0574	3.1128	3.2090
Q (12)	2155.3***	4481.6***	24813***
Correlation (UPR, DWNR)	-0.2655***		
Table 3B: Summary Statistics of CAC 40 : 01/01/1990 – 12/31/2017			
Summary Statistics	**Upward Range Component**	**Downward Range Component**	**Range**
Minimum	0.0000	0.0000	0.0000
Mean	0.7315	0.8156	1.5471
Maximum	8.4229	7.7503	9.2607
Standard Deviation	0.7366	0.8437	0.9543
Skewness	2.5123	2.2870	2.2686
Q (12)	1273.6***	4750.6***	20893***
Correlation (UPR, DWNR)	-0.2765***		
Table 3C: Summary Statistics of NIKKEI 225 : 01/01/1990 – 12/31/2018			
Summary Statistics	**Upward Range Component**	**Downward Range Component**	**Range**
Minimum	0.0000	0.0000	0.0000
Mean	0.7180	0.7899	1.5079
Maximum	12.4347	13.7634	13.7634
Standard Deviation	0.8194	0.9016	0.9876
Skewness	2.9810	3.1395	2.9068
Q (12)	1209.5***	2042.1***	14538***
Correlation (UPR, DWNR)	-0.2765***		

Note: *** indicate significance at 1% level. Q (12) is the Ljung-Box statistics of lag 12.

According to the summary statistic results both upward and downward price range series for all three stock indices have large positive skewness and these values are suggestive that a positively skewed density functions should be used to model the disturbance terms. The Ljung-Box test the null hypothesis is that the time series data are independently distributed. After 12 lags of sample autocorrelations are examined, the large test statistic values and very small p-values (<0.0000) conclude that the data exhibit a strong persistence in daily price range data. Downward range component has higher Ljung-Box statistic than that for the upward range component which means that downward range component is more persistent than the upward range component. Furthermore, higher values for the mean and standard deviation of the downward range component compared to the upward range component is a primary indication of the difference between the upward and downward range components' volatility structures. The correlation coefficient between upward range and downward range components for all the three stock indices are significant at 0.001 significance level and these negative correlation values suggest that periods of higher downward range volatility are related to lower upward range volatility periods.

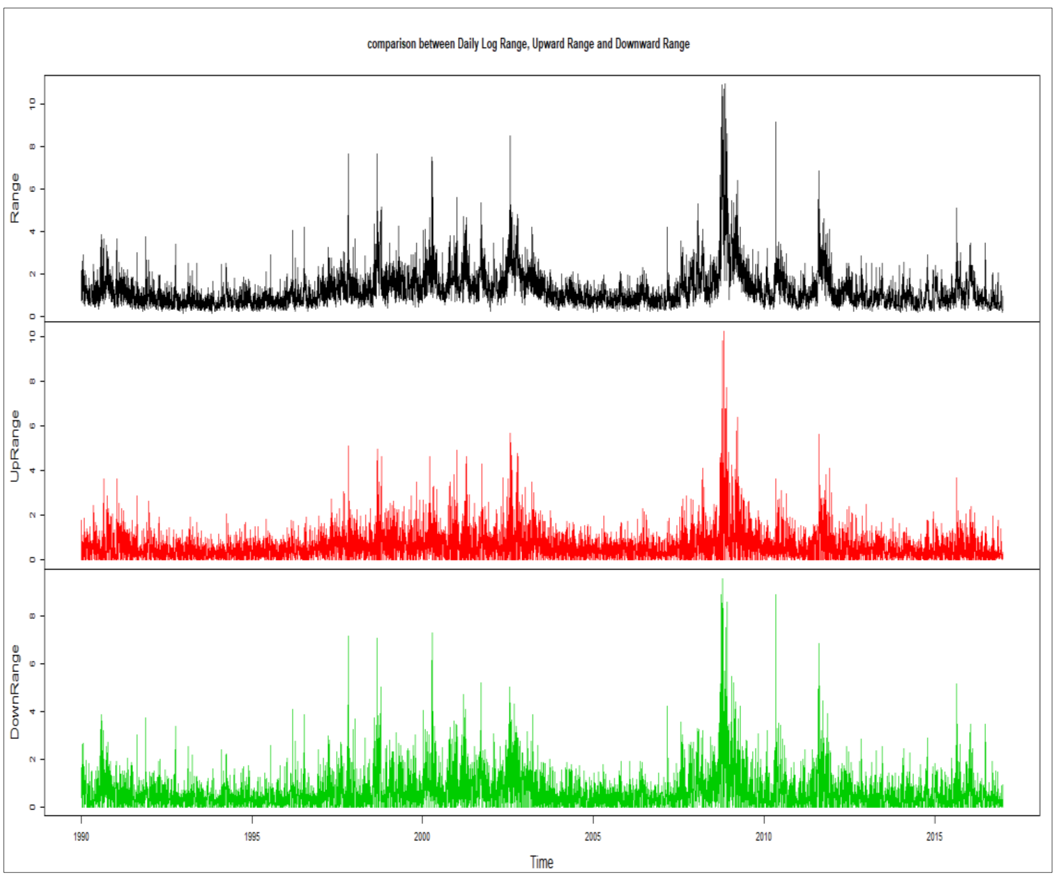

Figure 1:S&P500daily price range (black), daily upward range (red) and daily downward range (green) for the period of 01/02/1990: 12/31/2016.

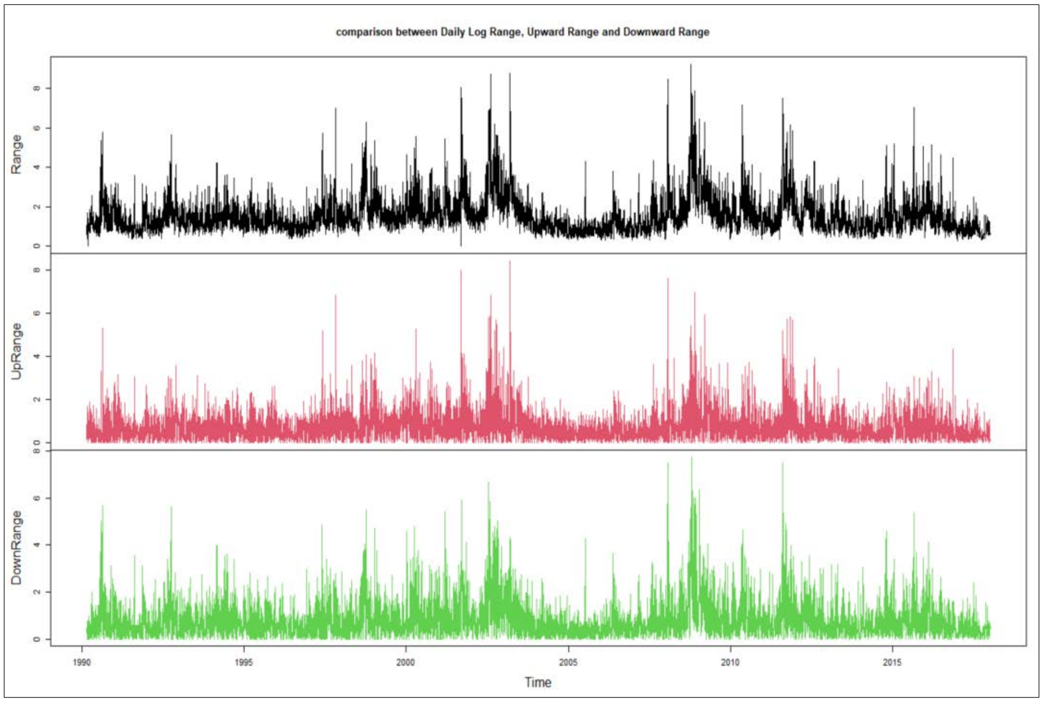

Figure 2: CAC40 daily price range (black), daily upward range (red) and daily downward range (green) for the period of 01/02/1990: 12/31/2017.

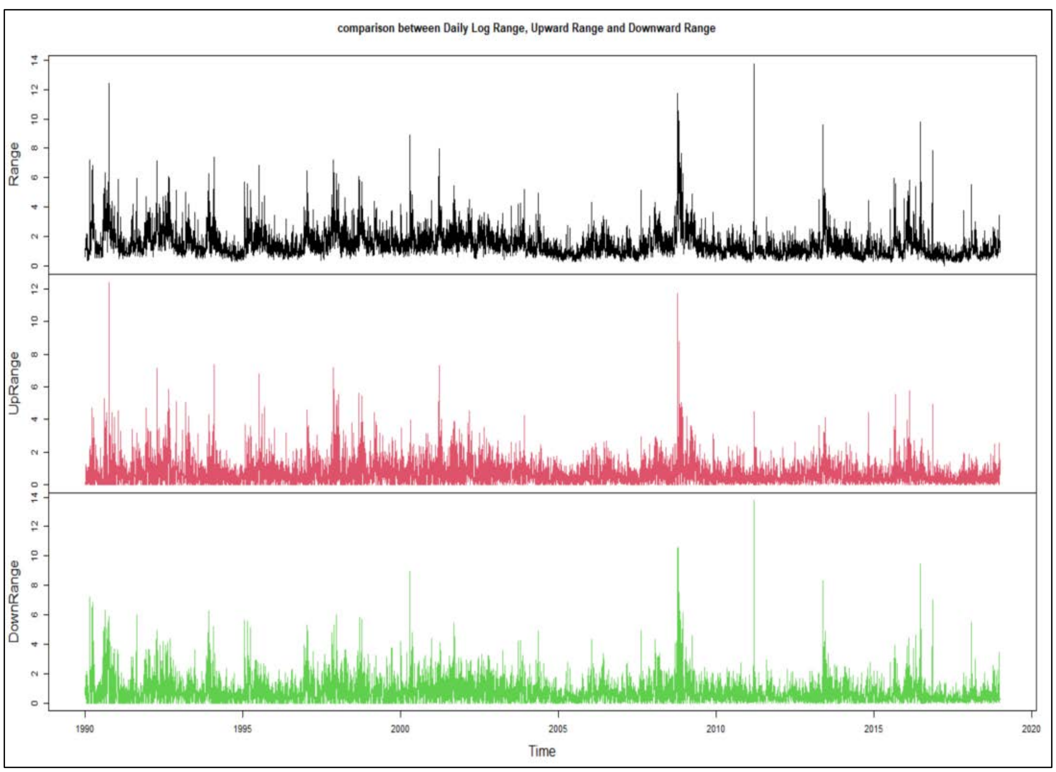

Figure 3: NIKKEI225 daily price range (black), daily upward range (red) and daily downward range (green) for the period of 01/01/1990-12/31/2018.

Time series plots for the daily price range, daily upward range and daily downward range series of S&P500 are present in Figure 1 through Figure 3. According the Figures, both upward and downward price range data have zeros. This is an important fact that needs to be considered when selecting the appropriate distributions to model the price series separately. In this study we use exponential distribution to model both the upward and downward price ranges, because the support of this distribution includes zero.

6.2. In-Sample Estimating Results

In this section we are going to discuss the parameter estimation for the both FACARR and GFACARR models. Model parameters are estimated by using the MLE method as discussed in Section 4 and results are presented in Table 4. According to the AIC value for the overall range period, the proposed GFACARR model performs slightly better than the FACARR model. However, in some situations FACARR has lower AIC value than that of the GFACARR model for upward or downward range models. Moreover, when compared to the FACARR model, GFACARR process can capture the negative relationship between current conditional mean of upward (downward) range and previous price range data or conditional mean of downward (upward) range. Table 5 and Table 6 summarize the comparison of results between FACARR and GFACARR models, including their performance during the 2009 recession period. Based on the results in Table 5, the proposed GFACARR model has lower RMSE and MAE when compared to the FACARR model during the recession period. This suggests that the GFACARR model fits the data from the recession periods better than the FACARR model. In general, GFACARR model has lower MAE and RMSE values when modeling the range than that of the FACARR model for all the three stock indices.

Table 4: Parameter Estimated Values for FACARR and GFACARR and GGFACARR Model.

Model	**S&P500**		**CAC40**		**NIKKEI225**	
	FACARR	**GFACARR**	**FACARR**	**GFACARR**	**FACARR**	**GFACARR**
ω^U	0.0141	0.0142	0.0249	0.0463	0.0171	0.0173
α^U	0.0327	0.0314	0.0546	0.0692	0.0748	0.0565
β^U	0.8292	0.9572	0.7886	0.3424	0.7721	0.9323
γ^U	0.1075	0.0989	0.1107	0.1447	0.1171	0.1018
δ^U		-0.1102		0.3290		-0.1131
ω^D	0.0155	0.0369	0.0140	0.0140	0.0218	0.0314
α^D	0.1007	0.1207	0.1025	0.0944	0.1114	0.1319
β^D	0.8469	0.2204	0.8303	0.8964	0.8204	0.7110
γ^D	0.0301	-0.0314	0.0553	0.0452	0.0454	0.0485
δ^D		0.6729		-0.0544		0.0826
AIC-UPR	**5717.44**	5721.62	8766.88	**8755.37**	8393.54	**8388.31**
AIC-DWNR	6681.03	**6648.76**	**10128.84**	10132.34	**9968.73**	9969.99
AIC-RANGE	12398.47	**12370.38**	18895.71	**18887.71**	18362.26	**18358.3**

Table 5: In-sample Comparison between FACARR, and GFACARR models for S&P500, CAC40 and NIKKEI225

Index	Model	Upward Range		Downward Range		Range	
		MAE	RMSE	MAE	RMSE	MAE	RMSE
S&P500	FACARR	**0.4314**	**0.6138**	0.5034	0.7336	0.4157	0.6112
	GFACARR	0.4317	0.6148	**0.5016**	**0.7323**	**0.4128**	**0.6071**
CAC40	FACARR	0.4858	0.6766	0.5428	**0.7562**	0.4767	0.6845
	GFACARR	**0.4847**	**0.6745**	**0.5426**	0.7566	**0.4758**	**0.6829**
NIKKEI225	FACARR	0.5234	**0.7487**	**0.5823**	**0.8514**	0.5080	0.7610
	GFACARR	**0.5225**	0.7496	0.5831	0.8520	**0.5066**	**0.7598**

Table 6: In-sample recession period Comparison between FACARR, and GFACARR models for S&P500, CAC40 and NIKKEI225

Index	Recession Period	
	MAE	RMSE
S&P500	0.7659	1.1488
	0.7650	**1.1456**
CAC40	1.2376	1.7728
	0.7506	**1.0828**
NIKKEI225	0.3778	0.4747
	0.3758	**0.4741**

In-sample prediction by GFACARR model for the S&P500, CAC40 and NIKKEI225 are given in the Figure 4, Figure 5 and Figure 6.

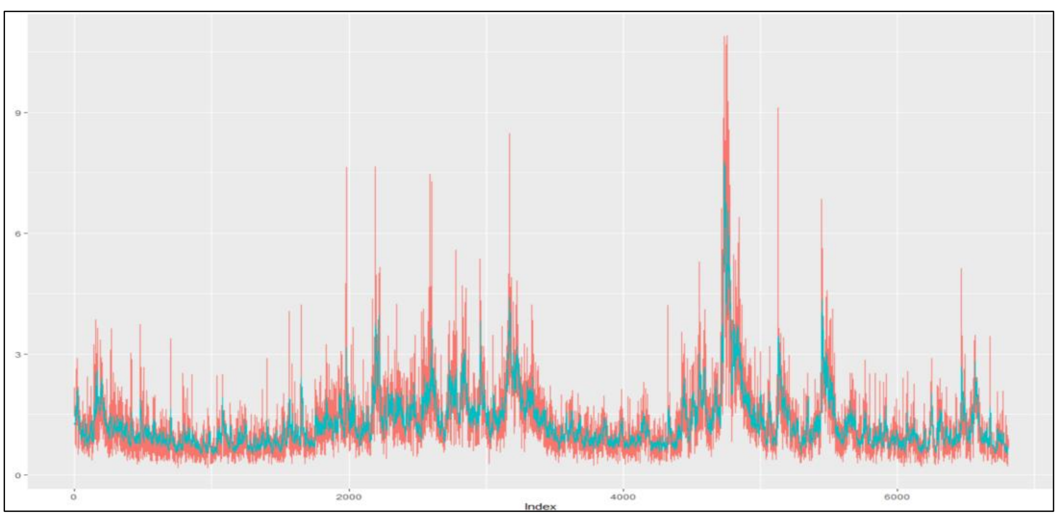

Figure 4: In-sample perdition of fitted GFACARR model (green) for the S&P500 (red) index.

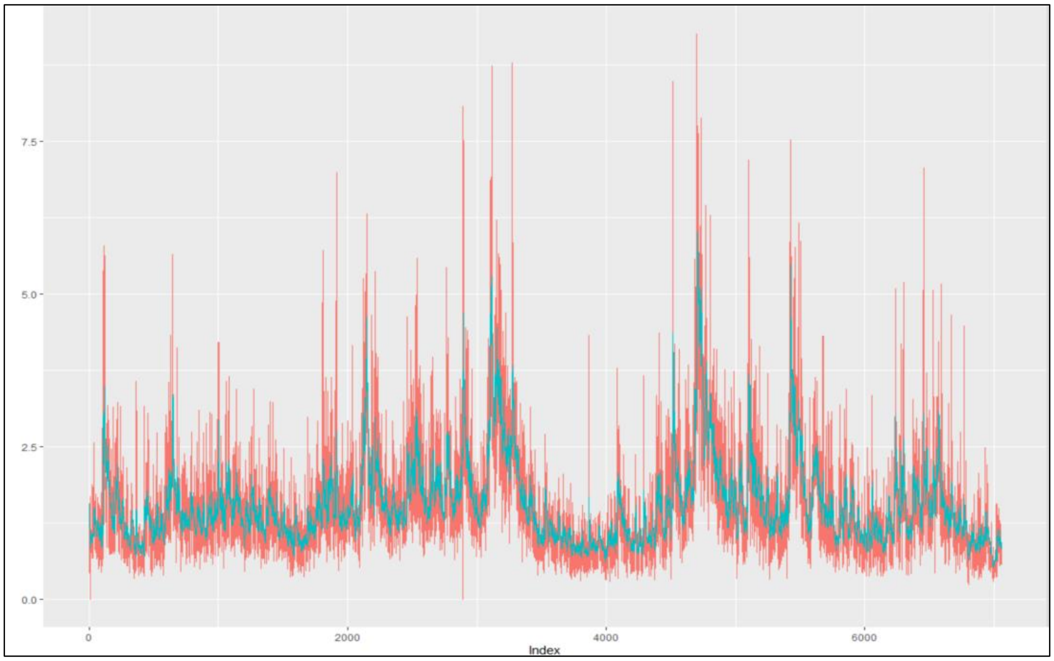

Figure 5: In-sample perdition of fitted GFACARR model (green) for the CAC40 (red) index.

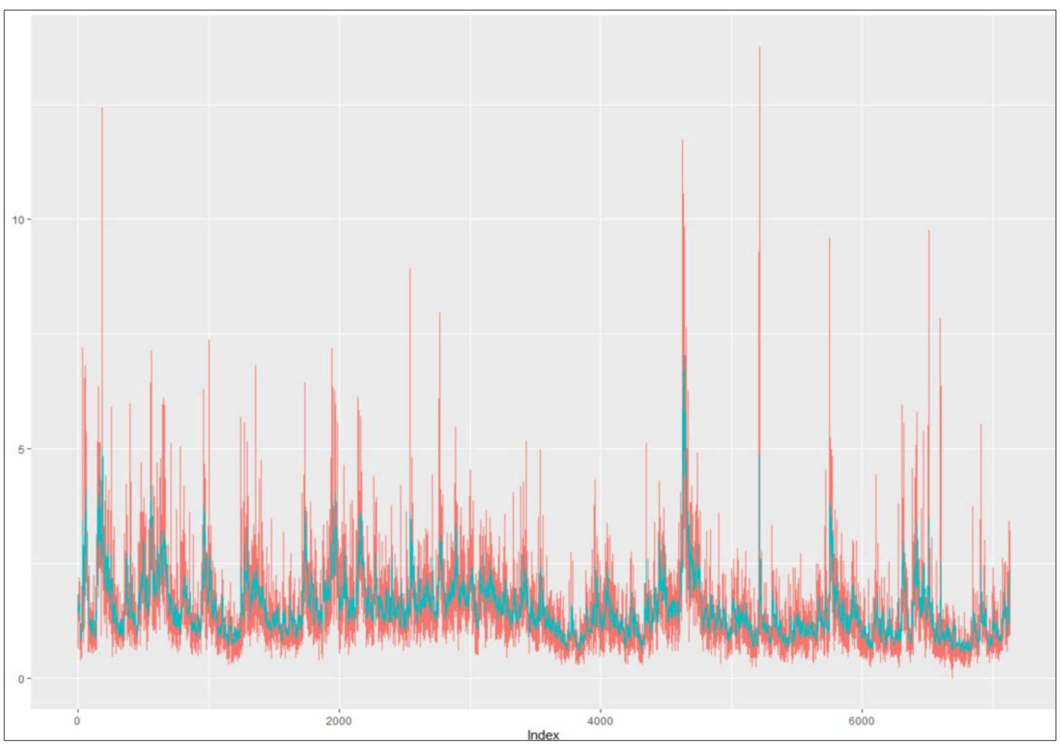

Figure 6: In-sample perdition of fitted GFACARR model (green) for the NIKKEI225 (red) index.

6.3. Out-of-Sample Forecasting

Out of Sample performance of the proposed GFACARR model is compared with the FACARR model with MAE and RMSE used as the forecasting performance evaluation indicators. Model with smaller forecasting error values indicates that it is relatively better than the other models. For out of sample predicting a recursive window estimation method was carried out. Moreover, Diebold& Marino (1995) test is used to check whether there is a significant difference between the GFACARR model forecasting accuracy and that of the FACARR model. If there a significant difference exists, then we checked whether GFACARR model is more accurate than FACARR model for forecasting future price range data. Table 7 presents the out of sample forecasting results. Based on the forecasting errors, GFACARR model have lower MAE and RMSE values for all the three stock indices than that of the FACARR model. DM test statistics and corresponding *p*-values suggested with 95% confidence that the proposed GFACARR model is more accurate than FACARR model in forecasting future values for S&P 500 and NIKKEEI225 indices. However, for the CAC40 stock index there is no significant difference between GFACARR and FACARR forecasting accuracy. Out of sample data and the forecasted values by GFACARR model are presented in the Figure 7through 9.

Table 6: Out of sample Comparison between FACARR and GFACARR models for S&P500, CAC40 and NIKKEI225

Index	**Model**	**Range**		
		MAE	**RMSE**	**DM test statistics (p value)**
S&P500	**FACARR**	0.2033	0.2517	-2.5013 (0.0062)
	GFACARR	**0.1994**	**0.2470**	
CAC40	**FACARR**	0.3270	0.4086	-0.26731 (0.3946)
	GFACARR	**0.3255**	**0.4077**	
NIKKEI225	**FACARR**	0.3149	0.3980	-2.8334 (0.0023)
	GFACARR	**0.3095**	**0.3913**	

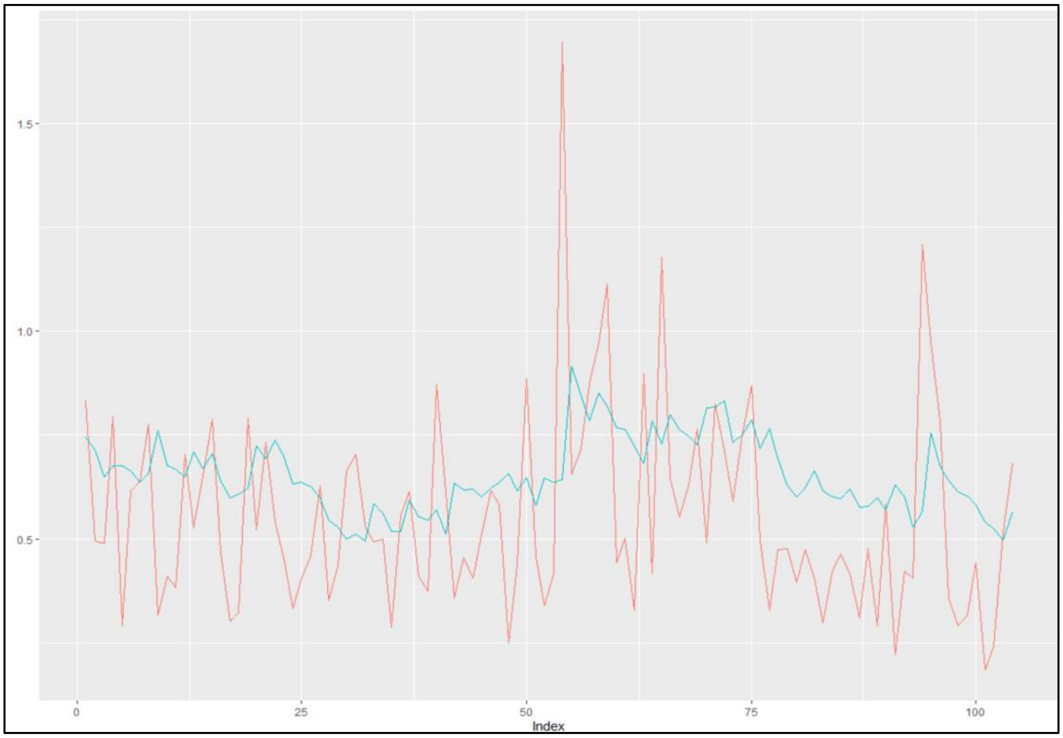

Figure 7: Out-of-sample forecasted values by GFACARR (green) for the S&P500 (red) index.

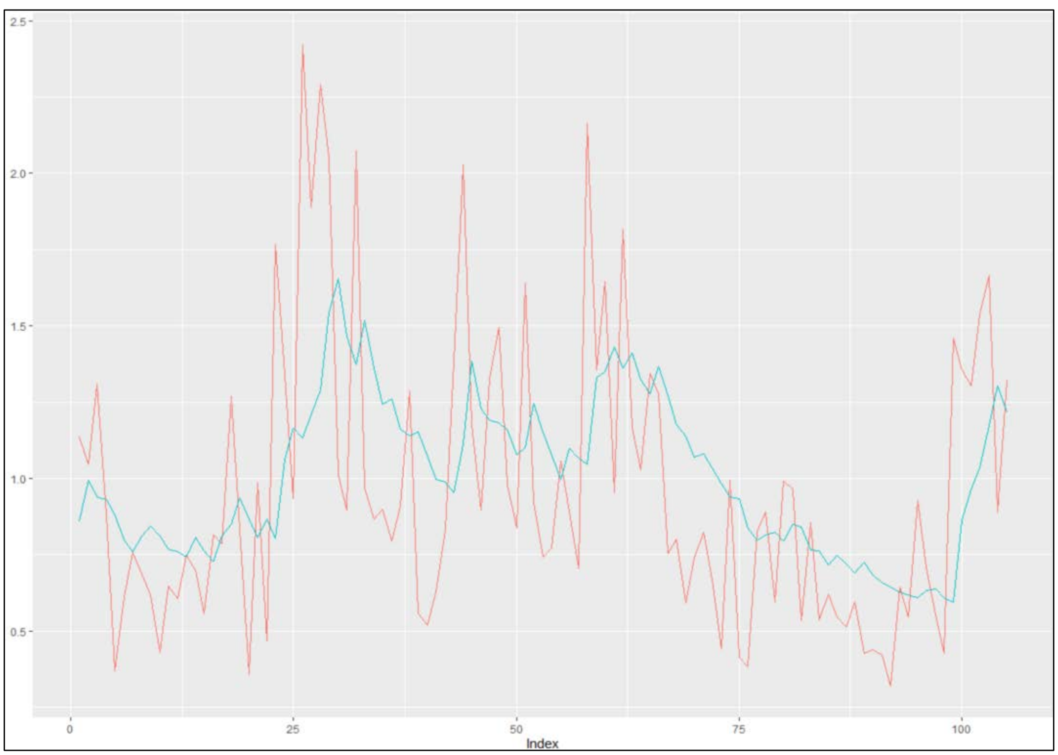

Figure 8: Out-of-sample forecasted values by GFACARR (green) for the CAC40 (red) index.

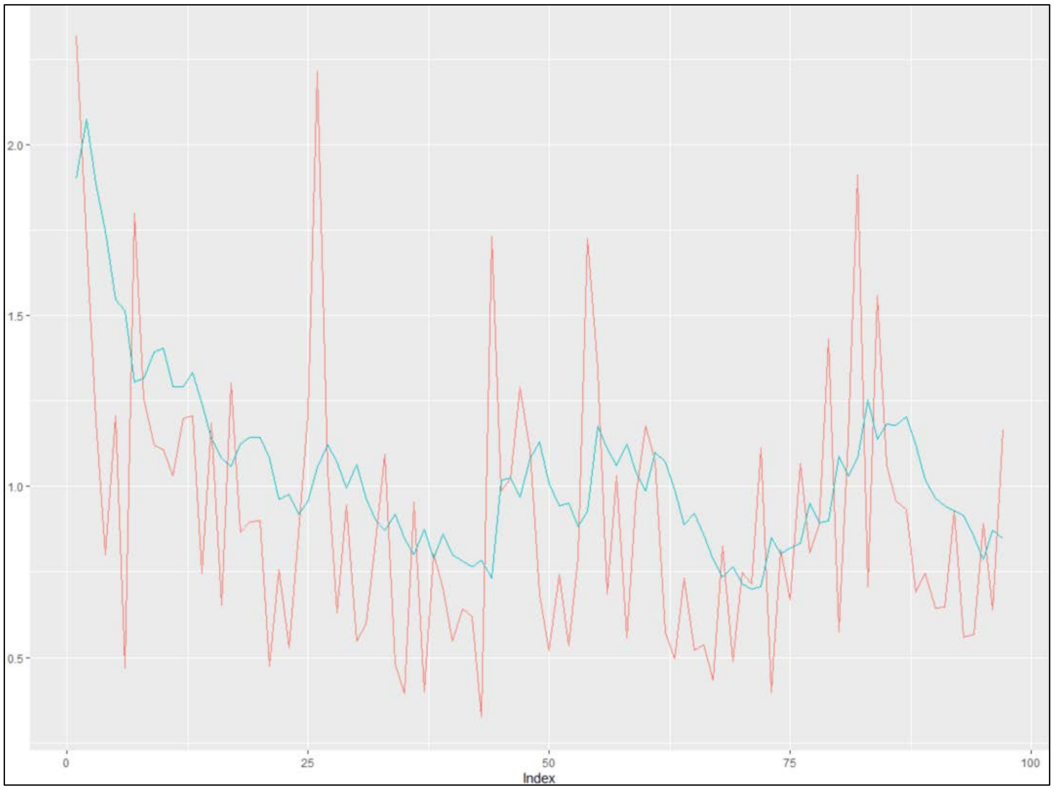

Figure 9: Out-of-sample forecasted values by GFACARR (green) for the NIKKEI225 (red) index.

7. Conclusions

In this paper we proposed the GFACARR model, which is a bivariate CARR type model, to accommodate asymmetric propagation of upward and downward ranges while accommodating a complete dynamic feedback mechanism between these two components and their conditional means. The GFACARR process use both previous downward (upward) price range and conditional mean downward (upward) range values to model the current conditional mean upward (downward) range. Furthermore, GFACARR model is capable of modeling the negative relationship between upward and downward range data which cannot be achieved using the FACARR model. The performance of the proposed model is gauged through an empirical study by using three stocks indices, namely S&P500, CAC40 and NIKKEI225. Based on the AIC value, the GFACARR model seems to provide a better predictions of the overall price range compared to its counterpart, the FACARR model. According to the performance evaluation indicators we employed, for GFACARR model has relatively lower predicting errors when predicting the overall range and it performs better at predicting both upward and downward ranges during recession periods. However, in some non-recession situations, FACARR has slightly better performance in in-sample predictions than the GFACARR model with respect to predicting upward or downward ranges. Smaller forecasting error values are obtained for the out of sample price range data by GFACARR model for two indices studies and the performance of the two models are not statistically significant for the third index. Overall, the results indicate that GFACARR model beats FACARR model for out of sample forecasting.

8. References

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