Unfolding the Instantaneous Effect of Each Probability Density Function/Process in A Mixture Probability Distribution/Process

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Abstract
The role of each of the probability density function/process in a mixture probability distribution/process has been unfolded to demonstrate to what extent it does contribute to each partition of the total probability and the results of which factors of each probability density function are participating in that mixture probability distribution/process. It has been observed that it is a result of the joint effect of how steep each distribution/process is compared to the other ones and what are the effects of each of the densities over several partitions.

Key Words: Differentiation, Integration, Leibnitz Theorem, Multinomial Distribution

1. Introduction

Mixture distribution was first coined in 1894. Several authors defined mixtures of distributions and studied various mixture distributions which they called several finite and infinite mixture distributions.

Most of them investigated additive mixture rather than multiplicative mixture. But in real life multiplicative mixture is more representative than the additive mixtures, since in multiplicative mixture distribution, appropriate randomness in both mixing and mixtured distribution is considered. None of the authors (1992, 2004) of mixture distributions demonstrated what is the clear automated role and contribution of each of the probability density functions in a Mixture probability distribution and to what extent each probability density function contributes to several partitions of total mixture probability distribution and what are the influences of different factors of each mixing probability density function participating in that mixture probability distribution.

As such, the aim of this paper is to unmask the explicit role(s) of each density function played in a finite mixture distribution.

2. Construction of the Existing Mixture Distributions

A mixture distribution is a weighted average of probability distribution of positive weights that sum to one. The weights themselves comprise a probability distribution called the mixing distribution. Due to the property of weights, a mixture is a probability distribution. The parameter θ of a family of distributions, given by the density by the density function

$f(x;\theta)$, is itself a subject to the change variation. The mixing distribution $g(x;\theta)$ is then a probability distribution on the parameter of the distributions. The general formula for the finite mixture is $\sum_{i=1}^{k} f(x;\theta_i)g(\theta_i)$ and the infinite counterpart is $\int f(x;\theta)g(\theta)d\theta$ where g is the density function.

3. Finite Mixture Distribution along with its Instantaneous Effects

The Mixture Probability Mass Function of the mixture of two continuous probability density functions f_1 and f_2 of the mixture random variable X is given by

$$\mathrm{P}\,(X=r)=\frac{\binom{n}{x}{f_1}^{(n-r)}{f_2}^{(r)}}{(f_1f_2)^{(n)}} \qquad \forall\, r=0,\ 1,\ 2,\ldots,\ n, \tag{1}$$

where, each of f_1 and f_2 is a n times differentiable density function of x. Here, ${f_1}^{(n-r)}$ is the $(n-r)$ th derivative and ${f_2}^{(r)}$ is the r th derivative of the functions f_1 and f_2 respectively. $(f_1f_2)^{(n)}$ is the n^{th} derivative of the multiple of functions f_1 and f_2. Each of f_1 and f_2 is a n degree polynomial density function of x. Each of f_1 and f_2 is n times differentiable with respect to x. The term $\frac{{f_1}^{(n-r)}{f_2}^{(r)}}{(f_1f_2)^{(n)}}$ is the contribution of the r^{th} term to the coefficient of the Binomial expansion responsible for how steep is the polynomial function f_2 and its successive contribution in the joint slope of the term after differentiation(s). Here, $f_1$and f_2 are not complimentary probability functions. f_1+f_2 is not necessarily 1, since both are continuous functions of x.

$$E(x)=\mathrm{n}\frac{\left[\left\{\frac{d}{dx}(f_2(x))\right\}\{\int f_1(x)\,\mathrm{d}x\}\right]^{(n-1)}}{[(f_2(x))(f_1(x))]^{(n)}}, \tag{2}$$

$$E(x(x-1))=\mathrm{n(n-1)}\frac{\left[\left\{\frac{d^2}{dx^2}(f_2(x))\right\}\{\int F_1(x)\,\mathrm{d}x\}\right]^{(n-2)}}{[(f_2(x))(f_1(x))]^{(n)}},$$

$$V(x)=\mathrm{n(n-1)}\frac{\left[\left\{\frac{d^2}{dx^2}(f_2(x))\right\}\{\int F_1(x)\,\mathrm{d}x\}\right]^{(n-2)}}{[(f_2(x))(f_1(x))]^{(n)}}+\mathrm{n}\frac{\left[\left\{\frac{d}{dx}(f_2(x))\right\}\{\int f_1(x)\,\mathrm{d}x\}\right]^{(n-1)}}{[(f_2(x))(f_1(x))]^{(n)}}-\left(\mathrm{n}\frac{\left[\left\{\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{f}_2(\mathrm{x}))\right\}\{\int \mathrm{f}_1(\mathrm{x})\,\mathrm{dx}\}\right]^{(\mathrm{n}-1)}}{[(\mathrm{f}_2(\mathrm{x}))(\mathrm{f}_1(\mathrm{x}))]^{(\mathrm{n})}}\right)^2, \tag{3}$$

Similarly, the Mixture Probability Mass Function of the mixture of k continuous density functions f_1,f_2, …, f_k of the mixture random variable X is given P(r_1 number of successes according to the density f_1, r_2 number of successes according to f_2,…, r_k number of successes according to f_k in n trials),

$$P\,(X=r_i)=\frac{\binom{n}{r_1,r_2,\ldots,r_k}{f_1}^{(r_1)}{f_2}^{(r_2)}\ldots\ldots{f_k}^{(r_k)}}{(f_1f_2\ldots\ldots f_k)^{(n)}} \qquad \forall r_i=0,\ 1,\ 2,\ldots,\ n. \tag{4}$$

Each of f_1, f_2, …, f_k is a n degree polynomial function of x. Each of f_1, f_2, …, f_k is n times differentiable with respect to x. The term $\frac{{f_1}^{(r_1)}{f_2}^{(r_2)}\ldots\ldots{f_k}^{(r_k)}}{(f_1f_2\ldots\ldots f_k)^{(n)}}$ is the contribution of the ${r_i}^{\text{th}}$ term to the coefficient of the Multinomial expansion responsible for how steep is the f_i polynomial function and its successive contribution in the joint slope of the term after differentiation(s). Here, for f_1, f_2,

…, f_k ; $f_1 + f_2 + \cdots + f_k$ is not necessarily 1, since each of them is a continuous function of x. The moments of the finite k-mixture distribution is

$$E(X_i) = \mathrm{n}\frac{\left[\left\{\frac{d}{dx}(f_i(x))\right\}\{\int f_1(x)\,\mathrm{d}x\}\ldots\ldots\{\int f_k(x)\,\mathrm{d}x\}\right]^{(n-1)}}{[(f_k(x))\ldots(f_1(x))]^{(n)}}, \tag{5}$$

4. Relation to Traditional Binomial and Multinomial Distributions

Since each of the functions f_1 , f_2 is a function of x and at least n times differentiable with respect to x, each term of Binomial expansion demonstrates the joint slope of their product. Here each term of the Binomial expansion expresses how much of the total probability is being distributed to different binomial terms according to $\frac{\binom{n}{r} f_1{}^{(n-r)} f_2{}^{(r)}}{(f_1 f_2)^{(n)}}$ 100 %.

Since each of the K density functions f_1, f_2, …, f_k is a function of x and at least n times differentiable, each term of the expansion demonstrates the joint slope of their product term. Here each term of the Multinomial expansion expresses how the total probability is being distributed to different terms according to $\frac{\binom{n}{r_1, r_2, \ldots, r_k} f_1{}^{(r_1)}(x) f_2{}^{(r_2)}(x) \ldots\ldots f_k{}^{(r_k)}(x)}{(f_1 f_2 \ldots\ldots f_k)^{(n)}(x)}$ 100 %.

Unlike two complementary related fixed probability of success and that of failure, f_1 and f_2 are two probability density success functions. In traditional Binomial distribution, probability of obtaining a fixed number of successes depends on how many number of successes one is interested to find and what is the extent of the probability of getting a success. But in the proposed Finite Mixture Distribution, the probability of obtaining a fixed number of successes according to a success function depends on how many number of successes one is interested to find and what is the product of the rates of the forces of that success function and the other success function.

5. Connection to the Generalized Leibnitz Theorem

The probability mass function in equation (1) must satisfy the fundamental rule of a probability distribution which is $\sum_{x=1}^{n} \frac{\binom{n}{x} f_1{}^{(n-x)} f_2{}^{(x)}}{(f_1 f_2)^{(n)}} = 1$. So, it immediately gives the following equation after cross multiplication as below

$$(f_1 f_2)^{(n)} = \sum_{x=1}^{n} \binom{n}{x} f_1{}^{(n-x)} f_2{}^{(x)} \tag{6}$$

The left-hand side of the equation (6) is the numerator of the probability mass function in equation (1). This equation is also known as the Leibniz theorem in Calculus due to Gottifried Leibnitz (Stewart, J. 2020) stating how to find the n^{th} derivative of the product of two n-differentiable functions f_1 and f_2 each of which is a function of x .The generalized form of Leibnitz theorem also can also be obtained from the generalized Finite Mixture Distribution via the following equation

$$(f_1 f_2 \dots \dots f_k)^{(n)} = \sum_{x_i} \binom{n}{x_1, \dots . x_i, \dots, x_k} f_1^{(x_1)} \dots \dots . . f_i^{(x_i)} \dots . \dots f_k^{(x_k)} \quad (7)$$

6. Some Examples of Finite Mixture Distributions for Two Distributions

Suppose that we have two exponential density functions $f_1(x) = e^{-x}$ and $f_2(x) = e^{-2x}$. We want to observe the probability distribution of x number of successes according to function $f_2(x)$ if there are n total number of trials. For, $n = 1$, the probability of r success is $\frac{1}{3}, \frac{2}{3}$ where $r = 0,\ 1$ respectively. Similarly, for 2 total number of trials, the probability distribution for 0, 1, 2 number of success(es) are $\frac{1}{9}, \frac{4}{9}, \frac{4}{9}$. For 3 trials the probability distribution for or 0, 1, 2, 3 number of success(es) are $\frac{1}{27}, \frac{6}{27}, \frac{12}{27}, \frac{8}{27}$ respectively. Interestingly, the location parameter is every time $\frac{2}{3}$. It is also evident for the weighted average for the mixture Bernoulli variate for $n = 1$.

Let we have two other density functions $E(1)$ and $N(0,1)$ from the exponential and the standard normal distributions respectively. If there are n trials, the probability of x number of successes according to function $N(0,1)$ can be obtained. For, $n = 1$, the probabilities of 0 and 1 success are $\frac{1}{1+x}$, $\frac{x}{1+x}$ respectively. So, the location parameter is $\frac{x}{1+x}$. Similarly, for 2 trials the probability distribution for 0, 1, 2 success(es) are $\frac{e^{-\frac{1}{2}x^2-x}}{\sqrt{2\pi}}$, $\frac{2xe^{-\frac{1}{2}x^2-x}}{\sqrt{2\pi}}$, $\frac{(x^2-1)e^{-\frac{1}{2}x^2-x}}{\sqrt{2\pi}}$. The mean according to the formula (2) is $2\frac{x(x+1)e^{-\frac{1}{2}x^2-x} - e^{-\frac{1}{2}x^2-x}}{e^{-\frac{1}{2}x^2-x}+2xe^{-\frac{1}{2}x^2-x}+(x^2-1)e^{-\frac{1}{2}x^2-x}}$. This mean can also be obtained from the weighted average of 0, 1, 2 with weights $\frac{e^{-\frac{1}{2}x^2-x}}{\sqrt{2\pi}}$, $\frac{2xe^{-\frac{1}{2}x^2-x}}{\sqrt{2\pi}}$, $\frac{(x^2-1)e^{-\frac{1}{2}x^2-x}}{\sqrt{2\pi}}$ respectively.

For the Beta density function $Beta(4,1)$ and the Gamma density function $Gamma(1,1)$, for n trials, the probability of x number of successes according to function $Gamma(1,1)$ can be obtained. For, $n = 2$, the mean number of successes is $2\frac{60x^2e^{-x}-20x^3e^{-x}}{60x^2e^{-x}-40x^3e^{-x}+5x^4e^{-x}}$.

7. Further Application of the Proposed Finite Mixture Distribution in Generalized Linear Model and Mixture Stochastic Processes

The Generalized Linear Model for the proposed Finite Mixture Distribution will be

$$h(g_2(f_2), g_1(f_1)) = \sum_{j=1}^{p} x_{ij}\,\beta_j \qquad \forall\ i = 1,2,\dots,n$$

where g_2, g_1 are some functions of f_2, f_1 respectively and h is a non-linear function of g_2, g_1. There are p predictors whereas x_j is the j^th predictor along-with the j^th slope parameter β_j.

The Mixture Probability Mass Function of the mixture of two continuous processes $f_1(t)$ and $f_2(t)$ of the mixture random family of random variable X(t) is given by

P(r number of successes according to the density function f_2 in n trials)

$$= \mathrm{P}\ (X(t) = \mathrm{r}) = \frac{\binom{n}{r} {f_1}^{(n-r)}(t) {f_2}^{(r)}(t)}{(f_1 f_2)^{(n)}(t)}\ , \qquad \forall\, r = 0,\ 1,\ 2, \ldots,\ n$$

Here, ${f_1}^{(n-r)}(t), {f_2}^{(r)}(t)$ are the Stochastic Differentials of the densities $f_1\ \ (t), f_2\ \ (t)$ of the Stochastic Processes.

Conclusion

The proposed way of generating finite mixture distribution(s) can be used for mixing two (or more than two) continuous probability distribution where each of binomial expansion represents how the contribution of the r th term to the coefficient of the Binomial expansion is responsible for the steepness of the density functions and their successive contributions to the joint slope(s) after differentiation(s).

The distribution of the product of two continuous probability density functions is splitted to several partitions showing the momentum of the distribution at several segments for a specific number of success(es) due to how steep a density function is and how the other density function affects the slope of the first function on the overall joint slope for individual binomial term.

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