Bayesian Prediction Bounds in Accelerated Life Testing: Weibull Models with Two Levels of Acceleration

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Abstract

Bayesian lower prediction bounds for a future observation from a Weibull distribution at the design level of stress using Type II censored data from two levels of accelerated stress is considered. The scale parameter of the Weibull distribution is assumed to have an inverse power relationship with the levels of stress while the shape parameter is assumed to be a constant. OpenBUGS is used to calculate Bayesian estimates of the parameters in a simulation study. A previous simulation study (Jayawardhana and Samaranayake, 2003) provides comparable results using Maximum Likelihood Predictive Density Method. Results of the current simulation study is compared with the results of the study by Jayawardhana and Samaranayake (2003). Proposed method will be illustrated through a well-known data set on breakdown times for insulating fluids (Nelson, 1972).

Key words: Accelerated tests, Weibull distribution, Inverse power law, Bayesian methods

I. Introduction

Most modern products are made with high reliability. Manufacturers spend large amount of resources to increase quality of their products and to be competitive. Reliability studies involve characterizing life distributions of materials or components. Accelerated life testing is a common way to collect failure data from products otherwise will last for a long time without a failure under normal use conditions. Typical stress factors on products are humidity, mechanical load, pressure, temperature, voltage, and vibration. Subjecting products to higher than designed levels of stress, researchers expect to have failures of products in a short period of time. Information collected during the short period of time is used to extrapolate the life distribution at normal use conditions. One practical difficulty of accelerated life testing is to decide the acceleration range so that higher level of acceleration is within physically possible range of stress and lower level of acceleration is as low as possible to get a reasonable number of failures to provide enough information. One theoretical difficulty is to figure out the physical relationship or the mathematical relationship between the parameters of the product life and stress level. Commonly used such relationships are Eyring model, inverse power law model, and Arrhenius model. Engineers use results of accelerated life tests to predict product life at normal use conditions.

Estimation of lower quantile points of the life distribution of a product at the normal use level for purposes such as warranty assurance, evaluation of early failures, making specification limits, making service plans, and cost analysis is a common objective of accelerated life tests. Parametric accelerated life testing models have two components: 1) a parametric distribution for the life of the unit; and 2) a physical relationship between the stress level and parameters of the distribution.

Literature on Bayesian accelerated life testing is not as frequent as that based on frequentist methodology. Nelson (1990) and Meeker and Escobar (1998) are good references for accelerated life testing. With advances in simulation-based computational tools, research on Bayesian accelerated life testing is becoming more common than before. Achcar and Louzado-Neto (1992) presents Bayesian theory for the Weibull life model with the Eyring model, inverse power law model, and Arrhenius model using the Laplace approximations. Barbosa and Louzada-Neto (1994), assuming Type II censoring, estimate the mean lifetime of units under normal working conditions. Their model structure is expressed in terms of generalized linear models. Mattos and Migon (2001), present theory for a full Bayesian analysis and used Gibbs sampler to analyze a data set published by Nelson (1972). They use adaptive rejection sampling method to draw samples. Leon et al. (2007), show how to use Bayesian methods to make inferences from accelerated life data with random effects. Jayawardhana and Samaranayake (2003) report results of a simulation study using Maximum Likelihood Predictive Density method. In their method, they substitute for the scale parameter with the maximum likelihood estimate of it in the likelihood equation and reduce the number of parameters to be estimated by one. They estimate the shape parameter using maximum likelihood estimator and simple estimator and present simulation results for each case.

II. The Proposed Method

We assume that the product life X has a Weibull distribution with a scale parameter θ and a shape parameter β. We also assume that the scale parameter θ is related to the stress V by $\theta = \eta_0\, V^{-\eta_1}$(inverse power law model), where η_0 and η_1 are positive constants and the shape parameter β is a constant for each level of stress. We focus on Type II censoring at each level of accelerated stress. Let the design level of stress, lower level of accelerated stress, higher level of accelerated stress, number of items subject to lower level of accelerated stress, number of items subject to upper level of accelerated stress, Type II censored number of items at lower level of accelerated stress, and Type II censored number of items at higher level of accelerated stress respectively be V_D, V_L, V_H, n_L, n_H, r_L, and r_H. We assume lifetime samples $x_{L1}, x_{L2}, \dots x_{Ln_L}$ and $x_{H1}, x_{H2}, \dots, x_{Hn_H}$ from lower and upper level of acceleration are independent within the samples and between samples. Weibull cumulative distribution function is given by $F(x) = 1 - exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}; x > 0, \beta > 0, \theta > 0$ and the p^{th}percentile of the Weibull distribution at the design level of stress is given by $X_p = \theta_D\left[ln\left(\frac{1}{1-p}\right)\right]^{\frac{1}{\beta}}$ where $\theta_D = \eta_0\, V_D{}^{-\eta_1}$. We propose to estimate parameters β, η_0 and $\eta_1$using Gibbs sampling.

III. Monte Carlo Simulation

We parameterize the stress levels in such a way that V_D is equal to 1. In practice, this can be done by dividing all the stress levels by the design level of stress. The advantage of this parameterization is that we do not have to estimate η_1 unless we want to study the accuracy in a simulation study. We assume reasonable values for the parameters $\beta, \eta_0, and\ \eta_1$and sample sizes presented by Jayawardhana and Samaranayake (2003). Using R programming language, we generate n_LWeibull random numbers from the lower level of stress (with $\theta_L = \eta_0\, V_L^{-\eta_1}$) and n_H from the upper level of stress (with $\theta_H = \eta_0\, V_H^{-\eta_1}$) respectively. For censoring, we order the two data sets in increasing order and select the first $r_L$and r_H from lower and upper level of stress respectively. Using RtoOpenBUGS package, we pass the two data sets to OpenBUGS and estimate the parameters θ and β. Since the equation $\theta = \eta_0\, V^{-\eta_1}$ can be written as $ln(\theta) = ln(\eta_0) - \eta_1 ln(V)$, we relabeled $ln(\eta_0)$ as $Intercept$. Within the OpenBUGS program, we use Gamma priors for the parameters β and η_1 and a normal prior for the parameter $Intercept$. R program and OpenBUGS program for this simulation study are provided in the appendices. Within OpenBUGS, we simulate 5000 iterations with 2500 iterations of burn-in. If there are convergence issues we use Uniform priors with positive values. Estimated values of β, η_1, and $Intercept$ are then passed to the R program to calculate the p^{th}percentile point at the design level of stress which is equal to $\hat{x}_p = \hat{\theta}_D \left[ln\left(\frac{1}{1-p}\right)\right]^{\frac{1}{\hat{\beta}}}$ where $\hat{\theta}_D = exp(Intercept)$. Then we calculate the coverage probability $P(x > \hat{x}_p) = 1 - exp\left\{-\left(\frac{x_p}{\theta_D}\right)^{\beta}\right\}$. This process is repeated 1000 times and the mean and standard deviation of the coverage probabilities are calculated and reported as $E[\mathrm{P}\ (x > \hat{x}_p)]$ and $SD[\mathrm{P}\ (x > \hat{x}_p)]$ respectively.

IV. An Example

Nelson (1972) provides a data set from an experiment consisting of times to breakdown (in minutes) of an insulating fluid subjected to various constant elevated voltages. The data from the experiment is presented in Table 1.

Table 1: Failure Times for Insulating Fluid at Various Voltages

26 Kv	28 Kv	30 Kv	32 Kv	34 Kv	36 Kv	38 Kv
5.79	68.85	7.74	0.27	0.19	0.35	0.09
1579.52	108.29	17.05	0.40	0.78	0.59	0.39
2323.70	110.29	20.46	0.69	0.96	0.96	0.47
	426.07	21.02	0.79	1.31	0.99	0.73
	1067.6	22.66	2.75	2.78	1.69	0.74
		43.40	3.91	3.16	1.97	1.13
		47.30	9.88	4.15	2.07	1.40
		139.07	13.95	4.67	2.58	2.38
		144.12	15.93	4.85	2.71	
		175.88	27.80	6.50	2.90	
		194.90	53.24	7.35	3.67	
			82.85	8.01	3.99	
			89.29	8.27	5.35	
			100.58	12.06	13.77	
			215.10	31.75	25.50	
				32.52		
				33.91		
				36.71		
				72.89		

Using graphical methods and data from all the levels of acceleration, Nelson estimates β to be 0.81 and θ to be 63,000 at the design level of stress (20 Kv). Using our equation for x_p and Nelson's parameter estimates, we would calculate $\hat{x}_{0.01}$ to be

$$\hat{x}_{0.01} = 63{,}000\left[\ln\left(\frac{1}{0.99}\right)\right]^{\frac{1}{0.81}} \approx 215.23 \text{ minutes.}$$

Jayawardhana & Samaranayake also provide a solution to this problem using data from only two different levels of stress ($V_L = 30$ Kv and $V_H = 36$ Kv), estimating β to be 0.8566 and η_1 to be 15.53, and then using the equation for $\hat{x}_p$ to calculate $\hat{x}_{0.01} = 180.17$. They also propose an ad-hoc adjustment to the equation that replaces $r_L + r_H$ with $r_L + r_H + 4$ in the equation for $\hat{x}_p$. With this adjustment, they calculate $\hat{x}_{0.01}$ to be 152.44.

Using OpenBUGS to perform our Bayesian estimation of the parameters, we estimate the parameters using data from only two different levels of stress, $V_L = 30$ Kv and $V_H = 36$ Kv (levels of stress were scaled appropriately so that $V_D = 1.0$). Our estimates are $\hat{\beta} = 0.9475$, $\hat{\eta}_0 = 26680.0$, and $\hat{\eta}_1 = 14.54$. Using these parameter estimates, we calculate

$$\hat{x}_{0.01} = 26680\left[ln\left(\frac{1}{0.99}\right)\right]^{\frac{1}{0.9475}} = 207.81.$$

Notice that, our estimate for β is larger than both the estimate produced by Nelson and the estimate produced by Jayawardhana & Samaranayake (2003).

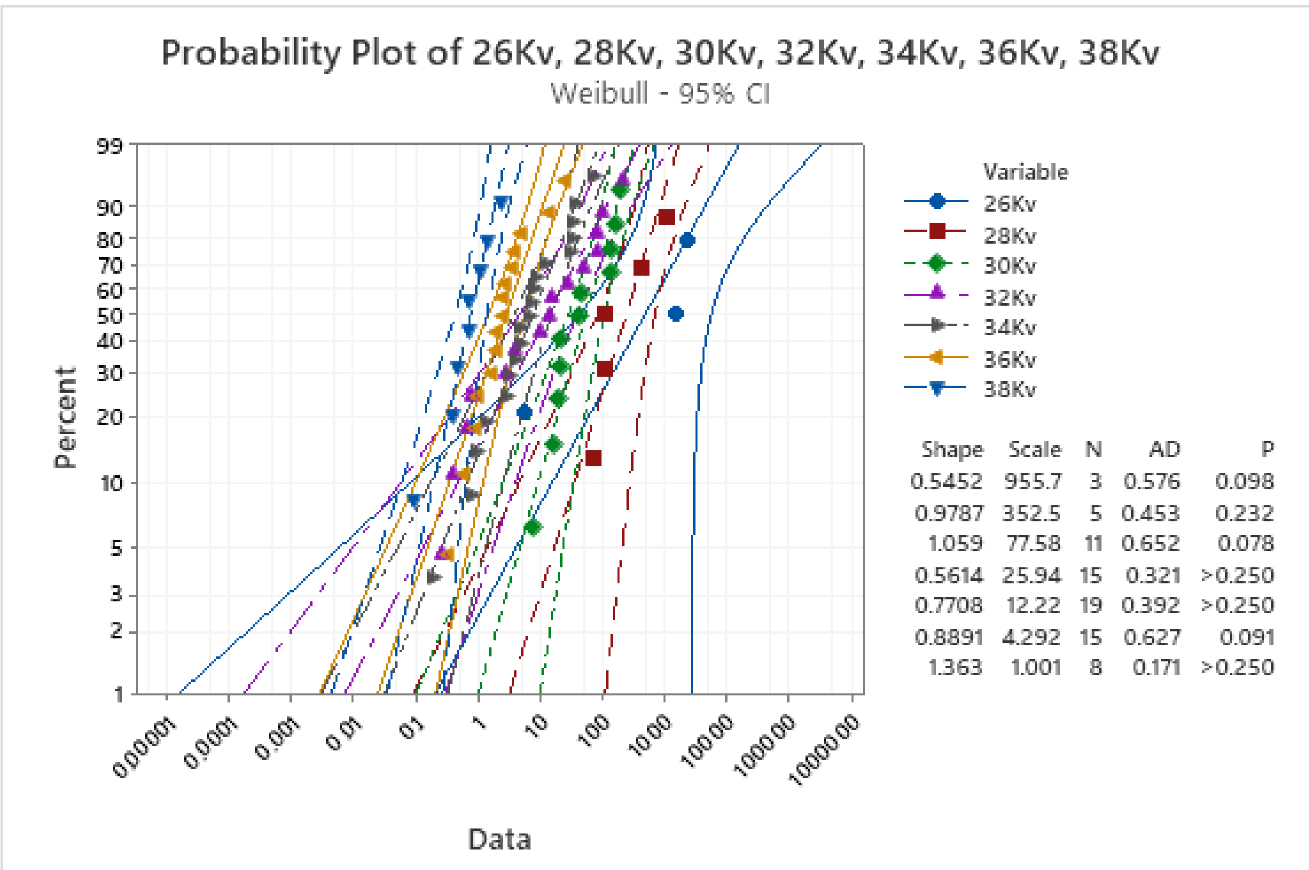

Figure 1: Weibull Probability Plot of Nelson's Data

Table 2: Comparison of Results from Three Studies

Parameter Estimates	Nelson Using All Levels	Jayawardhana and Samaranayake Using 30 & 36 Kv	Jayawardhana and Custer Using 30 & 36 Kv
$\hat{\beta}$	0.81	0.86	0.95
$\widehat{\eta_0}$	63,000	39,954*	26,680
$\widehat{\eta_1}$		15.53	14.54
$\widehat{x_p}$	215.23*	180.17	207.81

*These numbers are not reported by the authors.

Jayawardhana and Samaranayake do not report the value of $\hat{\eta}_0$ but we calculate it to be 39,954.25 without their ad-hoc adjustment.

V. Conclusion

We propose an easy method to calculate a percentile point of a Weibull life distribution of a product using data from two levels of acceleration of stress. A comparison with Jayawardhana and Samaranayake (2003) simulation results reveals that the percentile points calculated using Bayesian methods and Maximum Likelihood Predictive Density Method are both slightly over estimated consistently. Jayawardhana and Samaranayake use unbiasing constants in their simulation study and propose an easy to use ad-hoc adjustment to lower the percentile points but we do not have such a suggestion or way to correct for biased parameters. Other simulation studies we conducted show that Bayesian estimation of Weibull parameters even without acceleration of stress tend to be biased depending on the choice of the prior distributions and sample size.

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Table 3: Estimated Coverage Probabilities using Bayesian Estimates of β

	$p = 0.10$		$p = 0.05$	
	$E[\mathrm{P}\ (x > \hat{x}_{0.10})]$	$SD[\mathrm{P}\ (x > \hat{x}_{0.10})]$	$E[\mathrm{P}\ (x > \hat{x}_{0.05})]$	$SD[\mathrm{P}\ (x > \hat{x}_{0.05})]$
$V_L = 1.5, V_H = 3.5$				
20 20 10 20	0.876	0.036	0.931	0.027
40 40 10 40	0.900	0.022	0.949	0.015
40 40 40 40	0.897	0.019	0.948	0.013
$V_L = 1.5, V_H = 5.5$				
20 20 10 20	0.877	0.036	0.932	0.027
40 40 10 40	0.891	0.024	0.945	0.016
40 40 40 40	0.897	0.019	0.947	0.014
$V_L = 2.0, V_H = 4.0$				
20 20 10 20	0.877	0.036	0.931	0.027
40 40 10 40	0.917	0.019	0.960	0.012
40 40 40 40	0.901	0.018	0.950	0.013
$V_L = 2.0, V_H = 6.0$				
20 20 10 20	0.875	0.038	0.931	0.028
40 40 10 40	0.909	0.020	0.955	0.014
40 40 40 40	0.898	0.019	0.948	0.014
$V_L = 2.5, V_H = 4.5$				
20 20 10 20	0.878	0.038	0.932	0.028
40 40 10 40	0.926	0.017	0.966	0.011
40 40 40 40	0.914	0.152	0.959	0.010
$V_L = 2.5, V_H = 6.5$				
20 20 10 20	0.875	0.037	0.930	0.027
40 40 10 40	0.917	0.019	0.961	0.012
40 40 40 40	0.901	0.018	0.950	0.012

Table 4: Estimated (Unmodified) Coverage Probabilities Using MLEs and Simple Estimates of β

(Reproduced from Jayawardhana and Samaranayake, 2003)

	$p = 0.10$		$p = 0.05$	
	MLE	Simple Estimator	MLE	Simple Estimator
$V_L = 1.5, V_H = 3.5$				
20 20 10 20	0.895	0.898	0.946	0.948
40 40 10 40	0.897	0.896	0.948	0.948
40 40 40 40	0.896	0.897	0.947	0.948
$V_L = 1.5, V_H = 5.5$				
20 20 10 20	0.896	0.899	0.946	0.949
40 40 10 40	0.899	0.898	0.949	0.949
40 40 40 40	0.897	0.899	0.948	0.949
$V_L = 2.0, V_H = 4.0$				
20 20 10 20	0.889	0.890	0.943	0.944
40 40 10 40	0.890	0.889	0.945	0.944
40 40 40 40	0.893	0.895	0.945	0.947
$V_L = 2.0, V_H = 6.0$				
20 20 10 20	0.892	0.896	0.944	0.947
40 40 10 40	0.896	0.895	0.948	0.947
40 40 40 40	0.896	0.897	0.947	0.947
$V_L = 2.5, V_H = 4.5$				
20 20 10 20	0.873	0.877	0.934	0.937
40 40 10 40	0.878	0.873	0.938	0.936
40 40 40 40	0.889	0.891	0.943	0.944
$V_L = 2.5, V_H = 6.5$				
20 20 10 20	0.889	0.892	0.943	0.945
40 40 10 40	0.891	0.890	0.945	0.945
40 40 40 40	0.894	0.894	0.946	0.946

APPENDIX A

R Function to Simulate Type II Censored Weibull Data and Calculate Coverage Probabilities

```
cen<-function(N,VL,VH,rL,rH,p){
model.file.weibull<-#Location of OpenBUGS model on computer#
I<-1000
M<-2
V<-c(VL,VH)
Y<-matrix(NA,nrow=M,ncol=N,byrow=TRUE)
beta<-2
intercept<-0
eta0<-exp(intercept)
eta1<-2
theta<-eta0/(V^eta1)
r<-c(rL,rH)
L<-NULL
weib<-matrix(NA,nrow=I,ncol=3)
betahat<-NULL
intercepthat<-NULL
eta1hat<-NULL
eta0hat<-NULL
zp<-NULL
coverage<-NULL
for(i in 1:I){
for(m in 1:M){
Y[m,]<-c(sort(rweibull(N,beta,theta[m]))[1:r[m]],rep(NA,N-r[m]))
}data<-rbind(Y[1,],Y[2,])
L<-list(data=data,V=V,N=N,M=M)
parameters<-c("beta","eta1","intercept")
inits<-list(beta=1,eta1=1,intercept=1)
weibull.sim<-
bugs(data=L,parameters,inits=inits,model.file=model.file.weibull,n.iter=5000,n.burnin=2500,debug=FAL
SE)
weib[i,]<-weibull.sim$summary[c(1,2,3),1]
betahat[i]<-weib[i,1]
intercepthat[i]<-weib[i,2]
eta1hat[i]<-weib[i,3]
eta0hat[i]<-exp(intercepthat[i])
zp[i]<-eta0hat[i]*((log(1/(1-p)))^(1/betahat[i]))
coverage[i]<-exp(-(zp[i]/eta0)^beta)
print(i)
}print(mean(coverage))
print(sd(coverage))
}
```

APPENDIX B

OpenBUGS Program for R Program in Appendix A

```
model{
for (i in 1:M){
theta[i]<-exp(intercept-eta1*log(V[i]))
lambda[i]<-pow(theta[i],-beta)
for (j in 1:N){
data[i,j]~dweib(beta,lambda[i])
}
}
eta1~dgamma(1,1)
intercept~dnorm(0,0.001)
beta~dgamma(1,0.3)
}
```

APPENDIX C

OpenBUGS Program for Example from Nelson (1972)

```
model{
for (i in 1:N){
  theta[i]<-pow(V[i],eta1)/eta0
   for (j in 1:r[i])
     {
        Y[i,j]~dweib(beta,theta[i])
}
}
eta0~dunif(0,50000)
eta1~dunif(0,100)
beta~dunif(0,10)
}
#data
list(N=2, r=c(11,15), V=c(1.5,1.8),
Y=structure(.Data=c(7.74,17.05,20.46,21.02,22.66,43.40,47.30,139.07,144.12,175.88,194.90,NA,NA,NA
,NA,
0.35,0.59,0.96,0.99,1.69,1.97,2.07,2.58,2.71,2.90,3.67,3.99,5.35,13.77,25.50),.Dim=c(2,15)
)
)
```