A Process Control Model with Misclassifications and Acceptance Based on Clustering

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Abstract
A process control model is studied in which every h^{th} item is selected and subjected to repeated classifications because there is the possibility of misclassification. A final judgment of conforming is based on clustering of conforming classifications. Markov chain methods are used to investigate long term and various short term properties of the process.

Key Words: quality, scan, Markov chains, misclassification

1. Introduction

Various authors including Taguchi, Elsayed, and Hsiang(1989) and Taguchi, Chowdhury, and Wu (2004) have considered models of on-line process control by attributes in which every h^{th} item produced is inspected. In these, the process is initially assumed to be in control and to have a fraction of items conforming to specifications denoted by p_1. When the process goes out of control the fraction conforming shifts to p_2 ($< p_1$). When an inspected item is judged nonconforming, the process is stopped and a search is made to find an assignable cause.

Other authors have investigated modifications of this or models in a similar spirit. For example, in Nayebpour and Woodall (1993) the random time until the shift from p_1 to p_2 is assumed to follow a geometric distribution. Items produced were modeled as independent and identically distributed trials with a constant probability of π for each item to be the first one produced with the shifted (i.e., smaller) fraction conforming. Because only every h^{th} item is inspected, the first item produced under this shifted fraction conforming value might not be inspected and thus there might be items produced before there is a chance of this shift being detected.

Borges, Ho, and Turnes (2001) have argued that the inspection process itself can be subject to diagnostic errors so that a classification can result in a conforming item being mistakenly classified as nonconforming. We use p_{CN} to denote the probability of this misclassification. It is also possible that a nonconforming item can be classified as conforming and we use p_{NC} to represent the probability of this misclassification. We will also define p_{CC} (p_{NN}) to be the probability of correct classification that a conforming (nonconforming) item is classified as conforming (nonconforming). This possibility of misclassification leads to the notion that repeated classifications of each inspected item should be made prior to making the final judgment of whether the item is conforming or nonconforming. When the item has been judged in this final determination to be nonconforming, the process is then judged out of control and is stopped and a search for an assignable cause is done. If the process is actually out of control, we assume that an assignable cause is found and corrected and that the process is put back in control and the process model continues as before. Since there is also the possibility that an item is judged to be nonconforming and the process is judged out of control, even though it actually is not. In this case, it is still stopped for a search for an assignable cause, but finding none, the process is then restarted and it is assumed that the process has not somehow been put out of control by the stopping and searching for a cause. On the other hand, it is also possible that the process goes out of control, but this is not detected when an item is subjected to inspection through repeated classifications in which case it remains out of control until this is detected with a later item.

In Trindade, Ho, and Quinino (2007), the final judgment of whether the inspected item is conforming, and thus whether the process is in control, was based on majority rule in a pre-specified number of repeated classifications. In Quinino, Colin, and Ho (2010), an item was judged to be conforming and the process to be in control if and only if there were k classifications as conforming before f classifications as nonconforming, where k and f are some pre-specified positive integers. We will use the acronym TCTN because the decision is based on the total number of classifications as conforming and nonconforming. Griffith and Smith (2017, 2018) further studied this rule and another rule called CCTN in which the final judgment is conforming if there are k consecutive conforming classifications prior to a total of f nonconforming classifications.

In this paper, we study a compromise between the TCTN and CCTN protocols called the ScanCTN rule. Griffith and Smith (2016) considered some aspects of this and the present paper extends those results to a short term and long-term analysis using Markov chains. For the ScanCTN protocol the item is judged conforming if k conforming classifications are achieved within w consecutive classifications prior to observing f total nonconforming classifications among all classifications. Likewise, the item is judged nonconforming if f total nonconforming classifications among all classifications occur prior to k conforming classifications within w consecutive classifications. This rule eases the restriction on consecutive classifications from the CCTN rule in the sense that the count for classifications does not necessarily return to zero when a nonconforming classification occurs. This rule requires more consistency in the way conforming classifications are obtained than the TCTN rule.

2. State Space and Transition Probabilities

In order to assist with the readability of this section we will define the following notation for the ScanCTN rule.

- We assume that the production process of items is modeled by independent and identically distributed Bernoulli trials having a constant probability π for each item to be the first item produced after the shift of the fraction conforming. Every h^{th} item is inspected.
- Let $\theta = 1 - (1 - \pi)^h$. So, $1 - \theta = (1 - \pi)^h$ is the probability that the process has remained in control while those h items have been produced.
- f = total number nonconforming classifications for final judgment of nonconforming
- w = number of classifications in the window or scan
- k = number of conforming classifications within the window, w, for final judgment of conforming

- p = probability that the classification of an item is judged "conforming"
- $q = 1\text{-}p$
- $\{X_n\}$ = Markov-Chain where $X_n = (x_1, x_2, x_3, x_4, \ldots, x_w, s, h)$
- x_i = classification result (1 = conforming, 0 = nonconforming) for i = 1 to w-1
- x_w = the outcome of the w^{th} classification within the window of w
- s = total number of conforming classifications in the entries x_1 to x_w
- h = total number of nonconforming classifications

In the case of the ScanCTN, the item is judged conforming if k conforming classifications are achieved within w consecutive classifications prior to observing f total nonconforming classifications among all classifications. Likewise, the item is judged nonconforming if f total nonconforming classifications among all classifications occur prior to k total conforming classifications within w consecutive classifications. Consider the Markov Chain $\{X_n\}$ where $X_n = (x_1, x_2, x_3, x_4, \ldots, x_w, s, h)$ means that after the n^{th} classification the first w-1 entries, x_1 to x_{w-1}, contain the classification results (1 = conforming classification, 0 = nonconforming classification) of the previous w-1 classifications, the w^{th} entry, x_w, is the classification result of the w^{th} trial, s counts the total number of conforming classifications in the entries x_1 to x_w, and h is the total number of nonconforming classifications among all the classifications. When $n < w$, the n^{th} entry will contain the classification of the n^{th} trial, we will let NA be a placeholder in the entries x_{n+1} to x_w. Obviously, the index s is not necessarily needed but it does aid in the computation. The probability of conforming classification is given by p and probability of nonconforming classification is given by q = 1- p.

It is a rather difficult task to write out the state space in set notation and the transition probabilities and state space in a simple diagram even for small values of k, w, and f. Therefore, to aid in understanding the states involved in this Markov Chain, we have listed the absorbing and transient states in Table 1 for a ScanCTN rule where $k = 3$, $w = 4$ and $f = 3$.

For $n = 1$: $P(X_1 = (1,\text{NA},\text{NA},\ldots\text{NA},1,0) \mid X_0 = (\text{NA},\text{NA},\text{NA},\ldots\text{NA},0,0)) = p$
$P(X_1 = (0,\text{NA},\text{NA},\ldots\text{NA},0,1) \mid X_0 = (\text{NA},\text{NA},\text{NA},\ldots\text{NA},0,0)) = q$

$n = 2$: $P(X_2 = (1,1,\text{NA},\ldots\text{NA},2,0) \mid X_1 = (1,\text{NA},\text{NA},\ldots\text{NA},1,0)) = p$
$P(X_2 = (1,0,\text{NA},\ldots\text{NA},1,1) \mid X_1 = (1,\text{NA},\text{NA},\ldots\text{NA},1,0)) = q$
$P(X_2 = (0,1,\text{NA},\ldots\text{NA},1,1) \mid X_1 = (0,\text{NA},\text{NA},\ldots\text{NA},0,1)) = p$
$P(X_2 = (0,0,\text{NA},\ldots\text{NA},0,2) \mid X_1 = (0,\text{NA},\text{NA},\ldots\text{NA},0,1)) = q$

Etc. for $n \le w$

For $n > w$, $P(X_n = (b,c,d,e,\ldots,1,s+1,h) \mid X_{n-1} = (a,b,c,d,\ldots,g,s,h)) = p$ if $a = 0$
$P(X_n = (b,c,d,e,\ldots,1,s,h) \mid X_{n-1} = (a,b,c,d,\ldots,g,s,h)) = p$ if $a = 1$
$P(X_n = (b,c,d,e,\ldots,0,s,h+1) \mid X_{n-1} = (a,b,c,d,\ldots,g,s,h)) = q$ if $a = 0$
$P(X_n = (b,c,d,e,\ldots,0,s-1,h+1) \mid X_{n-1} = (a,b,c,d,\ldots,g,s,h)) = q$ if $a = 1$

For the Markov chain there are absorbing (recurrent) states, which correspond to the termination of the rule. Let A denote the set of absorbing states and a denote the number of absorbing states. In fact, the singleton sets consisting of each of these absorbing states are recurrent classes. The remaining states are transient which we will denote by T and likewise the number of transient states by t. Written in canonical form, the one-step transition probability matrix $\mathbf{P}$ for the Markov chain is $\begin{bmatrix} \mathbf{P_1} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}$, where $\mathbf{P_1}$ is the $a \times a$ identity matrix for the absorbing states, **R** is a $t \times a$ matrix containing the one-step probabilities of the transient states to the recurrent (absorbing) states, **Q** is a $t \times t$ matrix containing the one-step probabilities among the transient states, and **0** is the $a \times t$ zero matrix. The one-step probabilities of **R** and **Q** are determined by the transition probabilities given for each test. The first row of **Q** contains the one step transition probabilities from state (NA,NA,NA,…NA,0,0).

To compute the moments of the rule length, we will define the following notation. Since elements of T appear as subscripts, we will use *i* and *j* as typical elements of T. However, it should be noted that when we do so, each of *i* and *j* refer to an ordered $(w + 2)$ tuple. Let,

- $\mathbf{I}_{t \times t}$ = identity matrix of dimension $t \times t$
- $\mathbf{M}_{t \times t} = (\mathbf{I}_{t \times t} - \mathbf{Q}_{t \times t})^{-1}$- the fundamental matrix of dimension $t \times t$
- $\mathbf{e}_m$ = column vector of length t where the m^{th} element is one and the remaining elements are zero.
- $\mathbf{e}_m'$ is defined to be the transpose of $\mathbf{e}_m$
- $\mathbf{u}_{\{NS\}}$ = column vector where all the elements corresponding to the final judgment of nonconforming states are one, and the remainder of the elements are zero.
- $\mathbf{1}_z$ = column vector of ones of length z
- N_{ij} = random variable that represents the number of times the process visits state j before it eventually enters a recurrent state, having initially started from state i $(i,j \in$ T).
- $\mu_{ij} = E(N_{ij})$ for $i,j \in$ T.
- $\mathbf{M}_\rho = \left\lfloor \sum_{j \in T} \mu_{ij} \right\rfloor = \mathbf{M}\,\mathbf{1}_t$ = column vector such that the m^{th} element is the sum of the m^{th} row of **M**
- $\mathbf{M}_{\rho^2} = \left[\left(\sum_{j \in T} \mu_{ij} \right)^2 \right] = \text{diag}\,(\mathbf{M}_\rho)\,\mathbf{M}_\rho$ - column vector such that the m^{th} element is the square of the sum of the m^{th} row of **M.** Note: diag $(\mathbf{M}_\rho)$ is a diagonal matrix whose entries are the corresponding entries of $\mathbf{M}_\rho$.

Using the notation of the preceding section, the geometric distribution as a waiting time distribution, and basic probability results such as the law of total probability, we can obtain a number of results. These results are based on formulas in Bhat[18]. Propositions 1,2, and 3 are from Griffith and Smith (2016) and are repeated here without proof for completeness.

Proposition 1: If the item being inspected is conforming (nonconforming), the probability that it is judged to be conforming is

$P(judged\ conforming|\ actually\ conforming) =$
$ScanCTN(p_{CC}) = 1 - \boldsymbol{e}1'\boldsymbol{MR}\,\boldsymbol{u}\{NS\}$ where $p = p_{CC}$.

$P(judged\ conforming|\ actually\ nonconforming) =$
$ScanCTN(p_{NC}) = 1 - \boldsymbol{e}1'\boldsymbol{MR}\,\boldsymbol{u}\{NS\}$ where $p = p_{NC}$

Proposition 2: If the process is in control, the probability that it is judged to be in control is

$$P_{II} = P(judged\ in\ control|\ actually\ control) = p_1 ScanCTN(p_{CC}) + (1 - p_1) ScanCTN(p_{NC})$$

Proposition 3: If the process is out of control, the probability that is judged to be in control is

$$P_{OI} = P(judged\ in\ control|\ out\ of\ control) = p_2 ScanCTN(p_{CC}) + (1 - p_2) ScanCTN(p_{NC})$$

Proposition 4: When the process is out of control, the average run length is $\frac{1}{1-P_{OI}}$.

Proof: This is geometric distribution with parameter $1 - P_{OI}$.

Proposition 5: When the process is in control, the average run length is $\frac{1}{1-P_{II}}$.

Proof: This is geometric distribution with parameter $1 - P_{II}$.

3. Short Term Analysis Using Markov Chains

We now turn our attention to the short term analysis of this online process control with the ScanCTN protocol and will how to use Markov Chains to investigate the probability of judging the process to be out of control when it is in control as well as judging it to be out of control when it is out of control. Also we will explain a Markov chain approach to studying the distribution of the time until the process is declared out of control using first passage probabilities. For this purpose, we create a Markov Chain whose state space consists of four ordered-pairs whose elements are one or zeros. We use a 1 to stand for in control and a 0 to stand for out of control. The first coordinate is the actual state of the process and second coordinate is the judgment. As an example, (1,1) means that at a decision point the process is in control and judged to be in control. Whereas, (0,1) means that the process is actually out of control but judged to be in control. If we let $\theta = 1 - (1-\pi)^h$ then $1 - \theta = (1-\pi)^h$ is the probability that the process has remained in control while those h items have been produced. The one-step probability matrix for the transitions of this Markov chain is given in the following transition matrix.

$$\begin{array}{c} \\ (1,1) \\ (0,1) \\ (1,0) \\ (0,0) \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (0,1) & (1,0) & (0,0) \end{array} \\ \begin{pmatrix} (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ 0 & P_{OI} & 0 & 1-P_{OI} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

We can use first-passage probabilities to find the probability distribution of the time until the process is declared out of control by finding the probability of first reaching each absorbing state in n steps and then adding these probabilities to obtain the probability that it takes n steps (cycles of item inspections) to declare the process out of control. One can also use first-step analysis to find the probability of absorption into (1,0) and into (0,0). Note: P_{IO} = 1 - P_{II} and P_{OO} = 1 - P_{OI}.

4. Long Term Analysis Using Markov Chains

Markov chains can also be used to study the long-term behavior of this process control. The process is judged out of control whenever we reach either state (1,0) or state (0,0). When the cause is found and corrected or when it is determined that the process is actually in control and there is no cause, the process is then put back online and the one step transition probabilities are like those from state (1,1). Hence, to analyze the long term behavior of the process, we can use a one-step transition probability matrix in which the rows in the matrix that correspond to transitions out of (1,0) and (0,0) are identical to those out of state (1,1). The one-step transition probability matrix useful for long term analysis is given below.

$$\begin{array}{c} \\ (1,1) \\ (0,1) \\ (1,0) \\ (0,0) \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (0,1) & (1,0) & (0,0) \end{array} \\ \begin{pmatrix} (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ 0 & P_{OI} & 0 & 1-P_{OI} \\ (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \end{pmatrix} \end{array}$$

Note that this one-step transition probability matrix is that of an irreducible, aperiodic, positive recurrent Markov chain and therefore the limiting probabilities exist and are independent of the starting state. These limiting probabilities also have the interpretation of being the long-term proportion of time spent in each state. These can be found by solving a system of linear equations.

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