Some Results on the Additivity and Multiplication Order Preserving Properties of Stochastic Orders

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Abstract
The usual order ($\leq$) on the real numbers is both additive and multiplicative. These properties are preserved when comparing these constant random variables in the usual stochastic order. However, the statements for the non-constant real-valued random variables have still been missing in the literature. To examine the existential conditions of order preserving of additivity and multiplication of stochastic orders on the real-valued random variables a set of six stochastic orders including the usual, hazard rate, moment, Laplace transform, convolution and increasing convex order were considered; and, under independence assumption their order preserving property statuses were discussed. The results indicated that while the usual, the moment and the Laplace transform order are both additive and multiplicative, the hazard rate and the increasing convex order preserve these properties partially and the convolution order is only additive. As a conclusion, additivity and multiplication order preserving status of stochastic orders over real-valued random variables vary by the type of the order.

Key Words: Inequalities, Stochastic orders

1. Introduction

In their 1934 book on inequalities, Hardy, Littlewood and Polya introduced the concept of majorization as one of the fundamental building blocks of stochastic orders. Later in 1955, Lehmann introduced the concept of stochastic orders on real valued random variables, (Mosler and Scarsini, 1993). Since then, inspired by their application in many fields there has been a growing literature on these orders specially from 1994. They have been playing a key role in comparison of different probability models in wide range of research areas such as survival analysis, reliability, queuing theory, biology and actuarial science. They serve as an informative comparison criteria between different distributions much more effective than low informative distributional point comparisons of means, medians, variance and IQRs. In almost all cases, they have the same fundamental properties of usual order $\leq$ in real numbers including reflexivity, anti-symmetry, transitivity.

This paper deals with exploring another aspect of stochastic orders including order preserving additive and multiplicative properties. Our investigation originates from natural existence of earlier fundamental properties in both the set of real numbers equipped with usual order and the set of real-valued random variables equipped with one of stochastic orders. Given existence of the mentioned order preserving properties on the set of real numbers equipped with the usual order, there was a missing investigation on their parallel existential conditions for the case of the set of real-valued random variables equipped with a stochastic order.

This paper is divided into three sections: preliminaries, order preserving properties of usual stochastic order; and, order preserving properties of other stochastic orders. In the first

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section, we provide the required definitions and established results for the next two sections. Then in the second section, we establish order preserving additive and multiplicative properties for the usual stochastic order. Finally, in the last section we discuss these properties for five other stochastic orders including the hazard rate order, the moment order, the Laplace transformation order, the convolution order and the increasing convex order.

2. Preliminaries

The reader who has studied the concepts of order and stochastic order is well acquainted with the following definitions and results. For an essential account of the mentioned concepts, see (Davey and Priestly, 2002; Belzunce and Martinez-Riquelme, 2016; and Shaked and Shantikumar, 2010). We begin with some definitions:

Definition 2.1. *Let P be a set. An order on P is a binary $\leq$ on P such that for all $x, y, z \in P$:*
(i) $x \leq x$ (reflexivity),
(ii) $x \leq y$ and $y \leq x$ imply $x = y$ (anti-symmetry),
(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).

Definition 2.2. *Let P and Q be ordered sets. A map $\phi : P \to Q$ is said to be order preserving whenever $x \leq y$ in P implies $\phi(x) \leq \phi(y)$, in Q.*

When $P = Q$ and for fixed $z \in P$ the maps $\phi_z^{add}(x) = x+z$ and $\phi_z^{mul}(x) = xz(0 < z)$ are order preserving, it is said that the order $\leq$ has additivity and multiplication properties.

It is trivial that for the case $P = \mathbb{R}$ equipped with its usual order $\leq$, the order has all of reflexivity, anti-symmetry, transitivity, additivity and multiplication properties. From now onward, throughout this paper it is assumed that P is the set of all real valued random variables. For the orders related to this P we begin with one of the most well-known ones:

Definition 2.3. *Let X and Y be two real-valued random variables with finite means and associated Cumulative Distribution Functions F_X, and F_Y, respectively. X is said to be less or equal than Y in the usual stochastic order denoted by $X \leq_{st} Y$, if:*

$$F_X(t) \geq F_Y(t) \quad (-\infty < t < \infty).$$

It has been mentioned in the literature that usual stochastic order is reflexive, anti-symmetric and transitive (Belzunce and Martinez-Riquelme, 2016). Regarding order preserving maps on P we have (Shaked and Shantikumar, 2010):

Theorem 2.4. *If $X \leq_{st} Y$ and $\phi : \mathbb{R} \to \mathbb{R}$ is any increasing function, then $\phi(X) \leq_{st} \phi(Y)$.*

In particular, for constant random variable Z_0, $\phi_{Z_0}^{add}$ and $\phi_{Z_0}^{mul}$ are order preserving and hence the usual stochastic order is additive and multiplicative in this special case. However, the case in general is still unknown. The other orders of interest in this work are introduced in the following definition:

Definition 2.5. *Let X and Y be two real-valued random variables with finite means and associated Cumulative Distribution Functions F_X and F_Y, respectively. X is said to be less or equal than Y in the:*

(i) hazard rate order denoted by $X \leq_{hr} Y$*, if for the hazard function* $r(t) = \frac{\frac{dF(t)}{dt}}{1-F(t)}$ *we have:*

$$r_X(t) \geq r_Y(t) \text{ for all } -\infty < t < \infty.$$

(ii) moment order denoted by $X \leq_m Y$*, if with assumption* $0 \leq X, Y$ *we have:*

$$E(X^m) \leq E(Y^m) \text{ for all } (m \in \mathbb{N}).$$

(iii) Laplace transformation order denoted by $X \leq_{Lt} Y$*, if with assumption* $0 \leq X, Y$ *we have:*

$$L_X(s) \geq L_Y(s) \text{ for all } 0 < s < \infty.$$

(iv) convolution order denoted by $X \leq_{conv} Y$*,if for some non-negative independent random variable* U *of* X*:*

$$Y =_{st} X + U.$$

(v) increasing convex order denoted by $X \leq_{icx} Y$*,if :*

$$E(g(X)) \leq E(g(Y)) : \text{ for all increasing convex functions } g.$$

Remark 2.6. *This work does not cover the moment generating function order* $\leq_{mgf}$ *as it is equivalently associated with the Laplace transformation order* $\leq_{Lt}$ *(i.e.* $X \leq_{mgf} Y$ *if an only if* $-X \geq_{Lt} -Y$*). All results valid for the later are valid for the former too.*

An straightforward verification shows that considering the subset P_0 of all constant real-valued random variables of P equipped with one of above six stochastic orders, the order has both of mentioned order preserving properties. This is a parallel result to the case of real numbers $\mathbb{R}$ equipped with the usual order $\leq$. The following lemmas will be useful in the proof of the next sections theorems, (Belzunce and Martinez-Riquelme, 2016; Shaked and Shantikumar, 2010).

Lemma 2.7. *Let* X, Y *be two real valued random variables with associated Laplace transforms* L_X, L_Y*, respectively. Then,* $X \leq_{conv} Y$ *if and only if for* $\phi_{X,Y} = \frac{L_Y}{L_X}$ *we have:*

$$(-1)^n.\phi^{(n)}_{X,Y}(s) \geq 0 \text{ for all } (0 < s < \infty,\ n \in \mathbb{N})$$

Lemma 2.8. *Let* X, Y *be two real valued random variables with associative CDFs* F_X, F_Y*, respectively. Then,* $X \leq_{icx} Y$ *if and only if:*

$$\int_x^\infty (1 - F_X(t))dt \leq \int_x^\infty (1 - F_Y(t))dt, \text{ for all } (-\infty < x < \infty).$$

We conclude this section with a remark on the independency of involved random variables:

Remark 2.9. *Considering any pair of real numbers* x_0, y_0 *as fixed random variables* $X = x_0$ *and* $Y = y_0$*, it is trivial that any fixed random variable* $Z = z_0$ *is independent from each of them. Hence, in the next two sections we may assume the given random variables* Z *is independent from* X *and* Y*.*

3. Order Preserving Properties of Usual Stochastic Order

This section deals with general properties of usual stochastic order. The first three properties has been established as mentioned above. We discuss the later two properties and present an evidence of the necessity of the independence condition mentioned in Remark2.9.

Theorem 3.1. *The usual stochastic order is (i) additive; and, (ii) multiplicative .*

Proof. To prove (i), first let $X \leq_{st} Y$, and Z be a independent random variable from X, Y. Consequently:

$$F_{Y+Z}(t) = \int_{z=-\infty}^{\infty} F_Y(t-z)dF_Z(z) \leq \int_{z=-\infty}^{\infty} F_X(t-z)dF_Z(z) = F_{X+Z}(t) \quad (-\infty < t < \infty)$$

Hence, $X + Z \leq_{st} Y + Z$.

To prove (ii), we assume $X \leq_{st} Y$, and $0 \leq_{st} Z$. Then, by definition, $F_X(t) \geq F_Y(t)$ $(-\infty < t < \infty)$ and $P(Z < 0) = 0$. Consequently,

$$\begin{aligned} F_{XZ}(t) &= P(XZ \leq t) \\ &= P(XZ \leq t, Z \geq 0) + P(XZ \leq t, Z < 0) \\ &= P(X \leq \frac{t}{Z}, Z \geq 0) \\ &= \int_{z=0}^{+\infty} F_X(\frac{t}{z})dF_Z(z) \\ &\geq \int_{z=0}^{+\infty} F_Y(\frac{t}{z})dF_Z(z) \\ &= F_{YZ}(t), \quad (-\infty < t < \infty). \end{aligned}$$

Thus, $XZ \leq_{st} YZ$.
□

Remark 3.2. *The assumption of independence is necessary for the usual stochastic order additivity. As the counterexample, let $X \sim N(0, \frac{1}{2}), Y \sim N(1, \frac{1}{2})$ be two independent normal random variables. Then, for the CDF of the standard normal distribution Φ, we have:*

$$F_X(t) = \Phi(\frac{t}{\sqrt{\frac{1}{2}}})) \geq \Phi(\frac{t-1}{\sqrt{\frac{1}{2}}})) = F_Y(t), \quad (-\infty < t < \infty),$$

implying: $X \leq_{st} Y$. On the other hand, take $Z = -X$, consider $Y - X \sim N(1, 1)$, and consequently:

$$F_0(t) = 1_{(-\infty,0)}(t) = 0 < \Phi(t-1) = F_{Y-X}(t), \quad (-\infty < t < 0),$$
$$F_0(t) = 1_{[0,\infty)}(t) = 1 > \Phi(t-1) = F_{Y-X}(t), \quad (0 \leq t < \infty).$$

yielding: $0 \nleq_{st} Y - X$. Figure 1. presents the CDFs of the involved random variables.

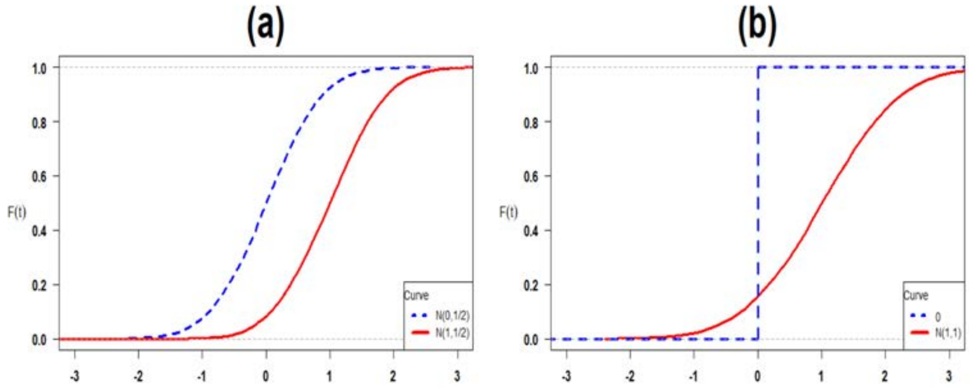

Figure 1: Cumulative Function Distributions of random variables in : (a) Original random variables; (b) added random variables.

4. Order Preserving Properties of Other Stochastic Orders

As we mentioned, there other types of stochastic orders in the literature of particular interest. In this section we deal with the similar problem of determining their main five general properties as the case of usual stochastic order for the hazard rate order ($\leq_{hr}$), the moment order ($\leq_m$), the Laplace transformation order ($\leq_{Lt}$), the convolution order ($\leq_{conv}$) and the increasing convex order ($\leq_{icx}$). First of all, for the case of reflexivity, anti-symmetry and transitivity we have:

Theorem 4.1. *The hazard rate order, moment order, Laplace transformation order, convolution order and increasing convex order are all reflexive, anti-symmetric and transitive.*

Proof. The proof of reflexivity and anti-symmetry properties are trivial. The proof of transitivity is straightforward for all the mentioned types of stochastic orders except the convolution order. For the case of convolution order, let $X \leq_{conv} Y$ and $Y \leq_{conv} Z$. Then, by two applications of Lemma2.7 and an application of Leibniz derivative rule for multiplication we have:

$$\begin{aligned}
(-1)^n.\phi_{X,Z}^{(n)}(s) &= (-1)^n.(\phi_{X,Y} * \phi_{Y,Z})^{(n)}(s) \\
&= (-1)^n.\sum_{k=0}^{n}(\binom{n}{k}(\phi_{X,Y})^{(k)} * (\phi_{Y,Z})^{(n-k)})(s)) \\
&= \sum_{k=0}^{n}(\binom{n}{k}(-1)^k.(\phi_{X,Y})^{(k)} * (-1)^{(n-k)}.(\phi_{Y,Z})^{(n-k)}(s)) \\
&\geq 0, \ \ (0 < s < \infty, \ n \in \mathbb{N}).
\end{aligned}$$

Hence, by another applications of Lemma2.7, it follows that $X \leq_{conv} Z$. □

Next, we deal with the additivity. For the hazard rate order $\leq_{hr}$, an special case of additive property has been proved whenever Z is IFR (i.e. r_Z is increasing), (Shaked and Shantikumar, 2010) (Lemma 1.B.3). But the general case is still unsolved. For the other four orders we have:

Theorem 4.2. *The moment order, Laplace transformation order, convolution order and the increasing convex order are all additive.*

Proof. First, for the moment order, the proof is straightforward application of binomial identity, and the independence condition. Indeed, let $X \leq_m Y$ and $X, Y > 0$. Then, for any independent $0 \leq_m Z$ we have:

$$\begin{aligned} E((X+Z)^m) &= E(\sum_{k=0}^{m} C_k^m X^k Z^{m-k}) = \sum_{k=0}^{m} C_k^m E(X^k) E(Z^{m-k}) \\ &\leq \sum_{k=0}^{m} C_k^m E(Y^k) E(Z^{m-k}) = E(\sum_{k=0}^{m} C_k^m Y^k Z^{m-k}) \\ &= E((Y+Z)^m) \ \ \forall m \in \mathbb{N}. \end{aligned}$$

Thus, $X + Z \leq_m Y + Z$.

Second, for the Laplace transformation, the proof is straightforward from of the Laplace transformation multiplication of sum of independent random variables. Indeed, let $X \leq_{Lt} Y$ and $0 <_{Lt} Z$ be independent. Then we have:

$$\begin{aligned} L_{X+Z}(t) &= E(e^{-t(X+Z)}) = E(e^{-tX} e^{-tZ}) = E(e^{-tX}) E(e^{-tZ}) \\ &\leq E(e^{-tY}) E(e^{-tZ}) = E(e^{-tY} e^{-tZ}) = E(e^{-t(Y+Z)}) \\ &= L_{Y+Z}(t), \forall t \in \mathbb{R}, \end{aligned}$$

implying $X + Z \leq_{Lt} Y + Z$.

Third, we consider the convolution order. Let $X \leq_{conv} Y$ and Z be independent from X, Y. Then, by the later condition it follows:

$$\phi_{X+Z,Y+Z}(s) = \frac{L_{Y+Z}(s)}{L_{X+Z}(s)} = \frac{L_Y(s) L_Z(s)}{L_X(s) L_Z(s)} = \phi_{X,Y}(s), \ \ (0 < s < \infty).$$

Now, two applications of Lemma2.7 and former condition implies $X + Z \leq_{conv} Y + Z$.

Finally, we consider the increasing convex order. Let $X \leq_{icx} Y$ and Z be independent from X, Y. Then by Lemma2.8 and two times application of Fubini's theorem it follows that:

$$\begin{aligned} \int_x^\infty (1 - F_{X+Z}(t))dt &= \int_x^\infty (1 - \int_{z=-\infty}^{\infty} F_X(t-z) dF_Z(z))dt \\ &= \int_x^\infty \int_{z=-\infty}^{\infty} (1 - F_X(t-z)) dF_Z(z) dt \\ &= \int_{z=-\infty}^{\infty} \int_x^\infty (1 - F_X(t-z)) dt dF_Z(z) \\ &\leq \int_{z=-\infty}^{\infty} \int_x^\infty (1 - F_Y(t-z)) dt dF_Z(z) \\ &= \ldots \\ &= \int_x^\infty (1 - F_{Y+Z}(t))dt \ \text{ for all } \ (-\infty < x < \infty). \end{aligned}$$

Now, by another application of Lemma2.8 it follows that $X + Z \leq_{icx} Y + Z$.

□

Finally, we discuss multiplication. For the case of hazard rate order $\leq_{hr}$, the special case for the IFR random variable Z can be proved similarly to the case of additivity. We have the following key theorem:

Theorem 4.3. *The moment order, the Laplace transformation order, and the increasing convex order(for special case) are all multiplicative.*

Proof. First of all, for the moment order the proof is trivial from independence condition.

Second, for the Laplace transformation order, let $X \leq_{Lt} Y$ and $0 <_{Lt} Z$ be independent from X, Y. Then, we have:

$$L_{XZ}(s) = \int_{z=0}^{\infty} L_X(sz)dF_Z(z) \geq \int_{z=0}^{\infty} L_Y(sz)dF_Z(z) = L_{YZ}(s) \quad (0 < s < \infty).$$

Hence, $XZ \leq_{Lt} YZ$.

Finally, for the case of increasing convex order, let $X \leq_{icx} Y$ and $0 \leq_{st} Z$(and so $0 \leq_{icx} Z$) be independent from X, Y. Then, using Fubini's theorem we have:

$$\begin{aligned}
\int_x^{\infty}(1 - F_{XZ}(t))dt &= \int_x^{\infty}(1 - \int_0^{\infty} F_X(\frac{t}{z})dF_Z(z))dt \\
&= \int_x^{\infty}\int_0^{\infty}(1 - F_X(\frac{t}{z})dF_Z(z)dt \\
&= \int_0^{\infty}\int_x^{\infty}(1 - F_X(\frac{t}{z})dtdF_Z(z) \\
&\leq \int_0^{\infty}\int_x^{\infty}(1 - F_Y(\frac{t}{z})dtdF_Z(z) \\
&= \dots \\
&= \int_x^{\infty}(1 - F_{XZ}(t))dt, \ (-\infty < x < \infty).
\end{aligned}$$

Consequently, by another application of Lemma2.8, it follows that $XZ \leq_{icx} YZ$. □

Remark 4.4. *In the above theorem, the proof or counterexample for multiplicity of the increasing convex order remains an open question.*

However, for the case of multiplicativity for the convolution order the situation is different.

Remark 4.5. *The convolution order is not multiplicative. As the counterexample, let* $X \sim exp(1), Y \sim exp(\frac{1}{2})$, *and* $Z \sim Bernoulli(\frac{1}{2})$, *be independent from* X, Y. *Since* $\phi_{X,Y}(s) = \frac{s+1}{2s+1}$, *for all* $0 < s < \infty$ *and* $(-1)^n\phi_{X,Y}^{(n)}(s) = \frac{n!.2^{n-1}}{(2s+1)^{n+1}} > 0$, *for all* $0 < s < \infty$; *by an application of Lemma2.7,* $X \leq_{conv} Y$. *However, a simple conditioning on* Z *yields* $\phi_{XZ,YZ}(s) = \frac{s^2+2s+1}{s^2+2.5s+1}$, *for all* $0 < s < \infty$. *But,* $\phi'_{XZ,YZ}(s) = \frac{0.5(s^2-1)}{(s^2+2.5s+1)^2}$, *changes signs from* $0 < s < 1$ *to* $1 \leq s$. *Consequently, by another application of Lemma2.7, it follows that* $XZ \nleq_{conv} YZ$.

Table.1 summarizes the main results in both sections:

Table 1: A Summary of order preserving properties of six stochastic orders

	Order Preserving Property	
Order Type	Addition	Multiplication
Usual $\leq_{st}$	Yes	Yes
Hazard Rate $\leq_{hr}$	*	*
Moment $\leq_{m}$	Yes	Yes
Laplace transformation $\leq_{Lt}$ [†]	Yes	Yes
Convolution $\leq_{conv}$	Yes	No
Increasing Convex $\leq_{icx}$	Yes	*

[*] Special cases were proved to be "Yes". [†] The results hold for the moment generating function order $\leq_{mgf}$ as well.

Future Work. There are other types of stochastic orders that this study did not cover their general properties.Some of them included mean residual life order $\leq_{mrl}$, harmonic mean residual life order $\leq_{hmrl}$, Lorenz order $\leq_{Lorenz}$, dilation order $\leq_{dil}$, dispersive order $\leq_{disp}$, execessive wealth order $\leq_{ew}$, peakedness order $\leq_{peak}$, starshaped order $\leq_{ss}$, pth order $\leq_{p}$, star order $\leq_{*}$, superadditive order $\leq_{su}$, factorial moment order $\leq_{fm}$, and total time on test order $\leq_{ttt}$, (Belzunce and Martinez-Riquelme, 2016; Shaked and Shantikumar, 2010). It may be of particular interest for the researchers to investigate their existential conditions for non-trivial order preserving additive and multiplicative properties.

Author Contribution: Author contributed to the study in the following manner: Conceptualization, M.S.; methodology, M.S.; formal analysis, M.S.; validation, M.S.; investigation, M.S.; writing—original draft preparation, M.S.; writing—review and editing, M.S.

Funding: This work has no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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