On the Existence of the Optimal Step-stress ALT under Progressive Type-I Censoring

David Han*

Abstract
In reliability engineering, the accelerated life test is not only getting increasingly popular but also necessary as it quickly yields information on the lifetime distribution of a highly reliable product in a short period of time by conducting the life test at more extreme stress levels than normal operating conditions. Through extrapolation, the lifetime distribution at the usage stress is then estimated with an appropriate regression model. In this work, we revisit the problem of the design optimization for a general k-level step-stress accelerated life test under progressive Type-I censoring with an equi-spaced step duration Δ for the design simplicity. Allowing the intermediate censoring to take place at each stress change time point ($viz.$, $i\Delta$, $i = 1, 2, \ldots, k$), the existence of the optimal stress duration is demonstrated under various design criteria including A-optimality and E-optimality in addition to D-optimality, T-optimality, and C-optimality. The existence of these optimal designs is investigated in detail for exponential lifetimes with a single stress variable.

Key Words: accelerated life tests, design of experiment, Fisher information, order statistics, progressive Type-I censoring, step-stress loading

1. Introduction

Thanks to the ever improving manufacturing process and technology, products and devices are becoming highly reliable with substantially long life-spans these days, which makes the standard life tests at normal operating conditions practically unfeasible. For gaining sufficient information about the lifetime distribution of a product or even a prototype, such tests are too time-consuming and costly to the industrial markets. For these reasons, the accelerated life test (ALT) is not only getting increasingly popular but also necessary as it quickly yields information on the lifetime distribution of a highly reliable product in a short period of time; see, for example, Chernoff [1], Nelson and Meeker [2], Nelson [3], Meeker and LuValle [4], Meeker and Escobar [5], Bagdonavicius and Nikulin [6]. By conducting the life test at more extreme stress levels than normal operating conditions, more failures can be collected rapidly. The lifetime distribution at the usage stress is then estimated with an appropriate stress-response regression model.

As a particular class of ALT, the (step-up) step-stress test implements a special stress loading scheme where the stress levels are sequentially increased at some prefixed time points until the termination time of the test. During the past decades, the inference and design optimization for the step-stress ALT have attracted great attention in the statistical reliability and engineering literature. Miller and Nelson [7] initiated a formal research in this direction and studied the optimal planning of a simple step-stress ALT while Bai *et al.* [8] investigated the optimization of a simple step-stress ALT with censoring. Later, Khamis and Higgins [9, 10] extended their results to a three-step step-stress ALT. In the meantime, Meeker and Hahn [11], Meeker [12] compared various ALT plans to estimate the reliability function at a design stress with Type-I censored failure data from different lifetime distributions including Weibull and lognormal. The optimal ALT with a non-constant scale parameter was studied by Meeter and Meeker [13], and then, Escobar and Meeker [14] explored planning ALT with two or more experimental factors while Yeo and Tang [15]

*Department of Management Science and Statistics, University of Texas at San Antonio, TX 78249

researched designing a step-stress ALT with a target acceleration-factor. On the inferential side, Balakrishnan and Han [16, 17], Han and Balakrishnan [18] studied the exact point and interval estimations for a simple step-stress ALT with competing risks under Type-I and Type-II censorings; see Liu and Qiu [19], Han and Kundu [20] as well. Lately, Zhang and Meeker [21] developed the Bayesian methods for planning ALT while Meeker *et al.* [22] used the ALT results to predict product field reliability. Lee *et al.* [23] also assessed the lifetime performance index of exponential products with a step-stress ALT as an application. More recently, Han and Ng [24], Han [25, 26] formulated the optimal designs and compared the efficiency of a step-stress ALT against a constant-stress ALT under conventional time and cost constraints.

As indicated by the reliability literature, time and budgetary constraints are practical aspects of any life testing, and for these reasons, censored sampling is usually demanded in practice. A generalized censoring scheme known as progressive Type-I censoring allows functional test units to be successively removed from a life test at some prefixed non-terminal time points. Those withdrawn, unfailed units can then be utilized in other tests in the same or at a different facility; see, for example, Gouno *et al.* [27], Han *et al.* [28], and Balakrishnan *et al.* [29]. Surprisingly, progressively censored sampling has not gained much popularity in ALT despite its flexibility and efficient utilization of the available resources compared to the traditional censoring methods, partly due to its distributional complexity rendering its statistical analysis rather difficult; see Cohen [30], Lawless [31].

In this work, we revisit the problem of the design optimization for a general k-level step-stress ALT under progressive Type-I censoring. For the design simplicity, an equi-spaced step duration Δ is considered along with the popular log-linear relationship between the mean lifetime parameter and stress level as well as the accelerated failure time (AFT) model for the effect of changing stress. For deriving the analytical tractable results here, the lifetimes of units are assumed to follow an exponential distribution at each stress level. Although simple, the exponential distribution is a good approximate model for numerous practical applications, including the decay time of a radioactive particle, the waiting time for service calls, the default time in credit risk modeling, and the distance between mutations on a DNA strand. In electrical and mechanical engineering, it has been successfully used to model the lifetime of an electric circuit and a semiconductor. Reliability theory and reliability engineering also make extensive use of the exponential distribution since its memoryless property renders it well-suited for modeling the constant hazard rate portion of the bathtub curve. More importantly, its statistical property serves as a theoretical proof of concept for other popular lifetime distributions such as gamma and Weibull, which is also the case based on the research outcomes of this study.

Allowing the intermediate censoring to take place at each stress change time point ($viz.$, $i\Delta$, $i = 1, 2, \ldots, k$), the existence of the optimal stress duration is demonstrated under five different design criteria including A-optimality and E-optimality in addition to D-optimality, T-optimality, and C-optimality. The existence of these optimal designs is investigated in detail with a single stress variable. Studying the existence of these optimal designs is not only theoretically but also practically important in order to guarantee the design feasibility under general settings as well as to develop and implement an efficient computational search algorithm for the optimal designs. The rest of the paper is organized as follows. Section 2 presents the model description for a general k-level step-stress ALT under progressive Type-I censoring. From the (expected) Fisher information matrix of the regression parameters, the behavior of the component of each entry is studied as a function of the step duration Δ in Section 3. In Section 4, various optimality criteria are defined based on the Fisher information stated in Section 3. Understanding the functional behavior of the Fisher information, the existence of the optimal step duration is also discussed under each optimality criterion.

2. Test Procedure & Joint Distribution

In a step-stress ALT, the stress levels are sequentially increased at some prefixed time points during the experiment. To lay out the procedure of a general k-level step-stress ALT under progressive Type-I censoring, let us first denote $s(t)$ to be the given (transformed) stress loading for ALT, which is a deterministic function of time. Also, let s_H be an upper bound of stress level and s_U be the normal usage stress level. The standardized stress loading is then defined as $x(t) = (s(t) - s_U)/(s_H - s_U)$ for $t \geq 0$ so that the range of $x(t)$ is $[0, 1]$. Now, let us define $0 \equiv x_0 \leq x_1 < x_2 < \cdots < x_k \leq 1$ to be the ordered k standardized stress levels to be used in the ALT. It is further assumed that at any stress level x_i, the lifetime of a test unit is exponentially distributed with the probability density function (PDF) and the cumulative distribution function (CDF) given by

$$f_i(t) = \frac{1}{\theta_i} e^{-t/\theta_i}, \qquad 0 \leq t < \infty, \tag{1}$$

$$F_i(t) = 1 - S_i(t) = 1 - e^{-t/\theta_i}, \quad 0 \leq t < \infty, \tag{2}$$

respectively. At any stress level x_i, it is also assumed that the mean time to failure (MTTF) of a test unit, θ_i, has a log-linear relationship with the corresponding stress level x_i. That is,

$$\log \theta_i = \alpha + \beta x_i, \tag{3}$$

where the regression parameters α and β are to be estimated. This log-linear link is a commonly used and well-studied model for the accelerated exponential distribution model. As Miller and Nelson [7] noted, the log-linear response is simple to understand but it also represents several physics-based life-stress relationships; for instance, Arrhenius, inverse power law, Eyring, temperature-humidity, and temperature-non-thermal.

Now, for $i = 1, 2, \ldots, k$, let n_i denote the number of units failed at stress level x_i in time interval $[(i-1)\Delta, i\Delta)$ while $y_{i,l}$ denotes the l-th ordered failure time of n_i failed units for $l = 1, 2, \ldots, n_i$. Also, let c_i denote the number of units censored at time $i\Delta$ with N_i denoting the number of units operating and remaining on test at the start of stress level x_i. That is, $N_i = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} c_j$. Under this setup, a step-stress ALT under progressive Type-I censoring with a uniform step duration Δ proceeds as follows. A total of $N_1 \equiv n$ test units is initially placed at stress level x_1 and tested until time Δ at which point c_1 surviving items are arbitrarily removed from the test and the stress is changed to x_2. The test is continued on $N_2 = n - n_1 - c_1$ units until time 2Δ, when c_2 items are removed from the test and the stress is changed to x_3, and so on. Finally, at time $k\Delta$, all the surviving items are removed, thereby terminating the life test. Since $n \equiv \sum_{i=1}^{k}(n_i + c_i)$, the number of withdrawn items at time $k\Delta$ is $c_k = n - \sum_{i=1}^{k} n_i - \sum_{i=1}^{k-1} c_i = N_k - n_k$. As a special case, when there is no intermediate censoring ($viz.$, $c_1 = c_2 = \cdots = c_{k-1} = 0$), this situation corresponds to the k-level step-stress ALT under traditional Type-I right censoring. When there is no right censoring ($viz.$, $n_k = N_k$), this situation corresponds to the k-level step-stress ALT under complete sampling. Since the stress-loading is non-constant for the step-stress ALT, an additional model to explain the effect of changing stress is required. In reliability engineering for the exponential distribution, often a suitable choice is the AFT model, also known as the additive accumulative damage model. It generalizes a number of well-known models such as the basic (linear) cumulative exposure model and the PH model.

Based on the AFT model along with (1) and (2), the PDF and CDF of a test unit for the step-stress ALT are

$$\begin{aligned} f(t) &= \left[\prod_{j=1}^{i-1} S_j(\Delta)\right] f_i\big(t-(i-1)\Delta\big) \\ & \quad \text{if} \begin{cases} (i-1)\Delta \le t < i\Delta & \text{for } i = 1,2,\ldots,k-1 \\ (k-1)\Delta \le t < \infty & \text{for } i = k \end{cases} \end{aligned} \tag{4}$$

$$\begin{aligned} F(t) &= 1-\left[\prod_{j=1}^{i-1} S_j(\Delta)\right] S_i\big(t-(i-1)\Delta\big) \\ & \quad \text{if} \begin{cases} (i-1)\Delta \le t < i\Delta & \text{for } i = 1,2,\ldots,k-1 \\ (k-1)\Delta \le t < \infty & \text{for } i = k \end{cases}, \end{aligned} \tag{5}$$

respectively. Upon using (4) and (5), the joint distribution function of the failure counts $\boldsymbol{n} = (n_1, n_2, \ldots, n_k)$ and the set of failure times $\boldsymbol{y} = (\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_k})$ with $\boldsymbol{y_i} = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ is derived as

$$f_J(\boldsymbol{y}, \boldsymbol{n}) = \left[\prod_{i=1}^{k} \frac{N_i!}{(N_i-n_i)!}\right]\left[\prod_{i=1}^{k} \theta_i^{-n_i}\right] \exp\left(-\sum_{i=1}^{k} \frac{U_i}{\theta_i}\right), \tag{6}$$

where

$$U_i = \sum_{l=1}^{n_i} \big(y_{i,l} - (i-1)\Delta\big) + (N_i - n_i)\Delta$$

is the *Total Time on Test* statistic at stress level x_i for $i = 1, 2, \ldots, k$. Applying the log-linear link in (3) to (6), the log-likelihood of (α, β) is obtained as

$$l(\alpha, \beta) = -\alpha \sum_{i=1}^{k} n_i - \beta \sum_{i=1}^{k} n_i x_i - \sum_{i=1}^{k} U_i \exp\big[-(\alpha + \beta x_i)\big], \tag{7}$$

and by differentiating (7) with respect to α and β, the maximum likelihood estimates (MLE) $\hat{\alpha}$ and $\hat{\beta}$ can be obtained as simultaneous solutions to the likelihood equations. Since the MLE $\hat{\alpha}$ and $\hat{\beta}$ do not exhibit explicit formulae, a numerical procedure such as the fixed point iteration, the Newton-Raphson method, or the expectation-maximization (EM) algorithm is required for estimation. More importantly, due to the non-linear nature of the MLE, statistical inferences with these MLE are based on the asymptotic result that $(\hat{\alpha}, \hat{\beta})$ is approximately distributed as a bivariate normal with mean (α, β) and dispersion matrix $\mathbf{I}_n^{-1}(\alpha, \beta)$, where $\mathbf{I}_n(\alpha, \beta)$ is the expected Fisher information matrix of (α, β).

3. Censoring Scheme & Fisher Information

Unlike progressive Type-II censoring scheme, there is an inherent mathematical lapse by prefixing the progressive Type-I censoring scheme $\boldsymbol{c} = (c_1, c_2, \ldots, c_{k-1})$ as pointed out by Balakrishnan and Han [17], Balakrishnan *et al.* [29]. It is due to the fact that there is a positive probability that all the units could fail before reaching the last stress level x_k, resulting in an early termination of the ALT as well as failing to fully implement the censoring scheme. To get around this issue, Gouno *et al.* [27] assumed a large sample size, small global censoring proportions, and a small number of stress levels so that the prefixed number of surviving units could be withdrawn at the end of each stress level. As a consequence, however, they had to restrict the search region for the optimal step duration to

$\{\Delta : A_i(\Delta) > 0,\ i = 2, 3, \ldots, k\}$, where $A_i(\Delta) = \big[1 - \sum_{j=1}^{i-1} \pi_j / G_j(\Delta)\big] G_{i-1}(\Delta) F_i(\Delta)$ with $G_j(\Delta) = \prod_{i=1}^{j} S_i(\Delta)$ and $\pi_i = c_i/n$ is the overall censoring proportion at x_i. A careful analysis of this search region reveals that it guarantees the availability of a sufficient number of live units to be censored at the end of each stress level only on average but not for each sample. Reliability testing, on the other hand, usually runs on a small sample size and may require severe censoring due to cost constraints and facility requirements, which violates the assumptions of Gouno *et al.* [27]. This calls for a practical modification of the progressive censoring scheme in order to ensure its feasibility. A simple suggestion is to decide on fixed proportions of remaining items to be censored at the end of each stress level x_i, say $\boldsymbol{\pi^*} = (\pi_1^*, \pi_2^*, \ldots, \pi_{k-1}^*)$ with $0 \le \pi_i^* < 1$. One could also define $\pi_k^* = 1$ since all the remaining units are removed from the ALT at time $k\Delta$. The actual number of units censored at the end of x_i is then determined by $c_i = \Upsilon((N_i - n_i)\pi_i^*)$ with a discretizing function of choice $\Upsilon(\cdot)$. This allows the ALT to terminate before reaching the last stress level x_k without any mathematical inconsistency. Also, under the proposed censoring mode, the actual censoring scheme $\boldsymbol{c}$ is random since the number of live units at the end of each stage before censoring takes place is random. As a special case, when $\boldsymbol{\pi^*} = (0, 0, \ldots, 0) = \mathbf{0_{k-1}}$, we have $\boldsymbol{c} = \mathbf{0_{k-1}}$, which corresponds to the general k-level step-stress ALT under traditional Type-I censoring.

For mathematical derivations, c_i defined above nevertheless makes it difficult to discover the distributional characteristics of the associated quantities, and thus, for $i = 1, 2, \ldots, k-1$, $c_i = (N_i - n_i)\pi_i^*$ is assumed for simplicity as $\Upsilon((N_i - n_i)\pi_i^*) \approx (N_i - n_i)\pi_i^*$. Based on the log-likelihood obtained in the preceding section, Balakrishnan and Han [17] then derived the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$ as

$$\mathbf{I}_n(\alpha, \beta) = n \begin{pmatrix} I_\alpha(\Delta) & I_{\alpha\beta}(\Delta) \\ I_{\alpha\beta}(\Delta) & I_\beta(\Delta) \end{pmatrix} = n \begin{pmatrix} \sum_{i=1}^{k} A_i(\Delta) & \sum_{i=1}^{k} A_i(\Delta) x_i \\ \sum_{i=1}^{k} A_i(\Delta) x_i & \sum_{i=1}^{k} A_i(\Delta) x_i^2 \end{pmatrix}, \quad (8)$$

where

$$A_i(\Delta) = F_i(\Delta) \prod_{j=1}^{i-1} S_j(\Delta)(1 - \pi_j^*), \quad (9)$$

by utilizing the distributional property that n_i given N_i follows a binomial distribution with parameters N_i and $\dfrac{F(i\Delta) - F\big((i-1)\Delta\big)}{1 - F\big((i-1)\Delta\big)} = F_i(\Delta)$. Analyzing the function $A_i(\Delta)$ in (9) reveals that with $\Delta > 0$, $A_i(\Delta) > 0$ for $i = 1, 2, \ldots, k$. Differentiating it with respect to Δ, we also observe that

$$A_i'(\Delta) = A_i(\Delta)\left[\frac{1}{\theta_i}\frac{S_i(\Delta)}{F_i(\Delta)} - \frac{1}{\delta_{i-1}}\right] = A_i(\Delta)\left[\frac{1}{\theta_i F_i(\Delta)} - \frac{1}{\delta_i}\right],$$

where $\delta_i = \big(\sum_{j=1}^{i} 1/\theta_j\big)^{-1} > 0$ for $i = 1, 2, \ldots, k$. By recursion, one can express $\delta_i = \big(1/\theta_i + 1/\delta_{i-1}\big)^{-1}$. Since $\lim_{\Delta \to 0^+} \dfrac{F_i(\Delta)}{\Delta} = \dfrac{1}{\theta_i}$ using L'Hôpital's rule, we have

$$\lim_{\Delta \to 0^+} A_i(\Delta) = 0 \quad \text{and} \quad \lim_{\Delta \to 0^+} \frac{A_i(\Delta)}{\Delta} = \frac{1}{\theta_i} \prod_{j=1}^{i-1} (1 - \pi_j^*) > 0.$$

Therefore, we see that

$$\lim_{\Delta\to 0^+} A_i'(\Delta) = \lim_{\Delta\to 0^+} \frac{A_i(\Delta)}{\Delta}\left[\frac{1}{\theta_i}\frac{\Delta}{F_i(\Delta)} - \frac{\Delta}{\delta_i}\right] = \frac{1}{\theta_i}\prod_{j=1}^{i-1}(1-\pi_j^*) > 0,$$

implying that $A_i(\Delta)$ initially increases from 0 as Δ increases from 0. With higher censoring proportions $\boldsymbol{\pi}^*$, it would grow more slowly. Since $A_1(\Delta) = F_1(\Delta)$, we also see that

$$\lim_{\Delta\to\infty} A_1(\Delta) = 1 \qquad \text{and} \qquad \lim_{\Delta\to\infty} A_i(\Delta) = 0$$

for $i = 2, 3, \ldots, k$ as well as

$$\lim_{\Delta\to\infty} \frac{A_1'(\Delta)}{A_1(\Delta)} = 0 \qquad \text{and} \qquad \lim_{\Delta\to\infty} \frac{A_i'(\Delta)}{A_i(\Delta)} = -\delta_{i-1}^{-1} < 0$$

for $i = 2, 3, \ldots, k$. Finally, we have $\lim_{\Delta\to\infty} A_i'(\Delta) = 0$, meaning that except for $A_1(\Delta)$, $A_i(\Delta)$ eventually decreases and asymptotically approaches 0 as Δ gets substantially large. On the other hand, $A_1(\Delta)$ keeps increasing and asymptotically approaches 1 as Δ increases.

This makes intuitive sense since each entry of the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$ in (8) is a linear combination of $A_i(\Delta)$'s in (9). Hence, the larger $A_i(\Delta)$ is, the larger the information content is for (α, β). As Δ increases, the test duration gets longer, which gives a higher chance to collect more failures and hence, more information about the lifetime distribution of a test unit. This translates to increasing $A_i(\Delta)$ in the beginning. However, when Δ becomes substantially large, the test duration in the first stress level x_1 becomes too long, and all the test units could fail there. This means that given the sample size n, only θ_1 can be estimated with the highest precision but (α, β) cannot be estimated jointly since no failures could be observed at the stress levels higher than x_1. As noted in Han and Bai [32], at least one failure needs to be observed from at least two different stress levels to guarantee the existence of $\left(\hat{\alpha}, \hat{\beta}\right)$. Otherwise, the parameters are not estimable. Thus, if Δ becomes too large, the joint information for (α, β) starts decreasing and so does $A_i(\Delta)$ back to 0 except for $A_1(\Delta)$. The behavior of $A_1(\Delta)$ is expected to be different as it represents the information from the first stress level x_1. Increasing Δ would raise the chance to collect more failures at x_1, hence more information from x_1. Hence, increasing Δ would only increase $A_1(\Delta)$ to its maximum 1. Furthermore, we have

$$A_i''(\Delta) = -A_i(\Delta)\left[\frac{1}{\theta_i}\frac{S_i(\Delta)}{F_i(\Delta)}\left(\frac{1}{\theta_i} + \frac{2}{\delta_{i-1}}\right) + \frac{1}{\delta_{i-1}^2}\right].$$

For $i = 2, 3, \ldots, k$, when $A_i'(\Delta_i^*) = 0$, or equivalently, $\Delta_i^* = \theta_i \log(\delta_{i-1}/\delta_i)$, we see that $A_i''(\Delta_i^*) = -A_i(\Delta_i^*)\delta_{i-1}^{-1}\delta_i^{-1} < 0$, meaning that $A_i(\Delta)$ achieves the unique maximum at $\Delta_i^* = \theta_i \log(\delta_{i-1}/\delta_i)$. It is also observed that $\delta_i < \Delta_i^* < \delta_{i-1}$ since the basic analysis reveals that $\dfrac{z}{z+1} < \log(z+1) < z$ for all $z > 0$. Since $\delta_k < \delta_{k-1} < \cdots < \delta_1 = \theta_1$ with $A_i(\Delta) > 0$, this result implies that excluding $A_1(\Delta)$, any linear combination of (the cross products of) $A_i(\Delta)$'s with non-negative coefficients achieves its global maximum when Δ is in the range of $\left[\min_{i=2,\ldots,k} \Delta_i^*, \max_{i=2,\ldots,k} \Delta_i^*\right] = [\Delta_k^*, \Delta_2^*]$.

If this linear combination of (the cross products of) $A_i(\Delta)$'s with non-negative coefficients includes $A_1(\Delta)$, it is a little more complicated to analyze since $A_1(\Delta) = F_1(\Delta)$ strictly increases to 1 asymptotically. Let's say, the derivative of this linear combination is expressed in the form of $w_1 A_1'(\Delta) + \sum_{i=2}^{k} w_i A_i'(\Delta)$ with w_i being a non-negative coefficient, which can be a function of Δ, for $i = 1, 2, \ldots, k$. Then, it can be shown that the maximum for this combination exists for $\Delta \geq \Delta_k^*$ if $w_1 A_1'(\Delta) + \sum_{i=2}^k w_i A_i'(\Delta) < 0$ for all $\Delta > \Delta_U$, where Δ_U is some $\Delta \geq \Delta_2^*$. Otherwise, the linear combination is non-decreasing in the end and the maximum may not exist. Similarly, it can be shown that under the same condition, the reciprocal of a linear combination of (the cross products of) $A_i(\Delta)$'s, including $A_1(\Delta)$, with non-negative coefficients attains its minimum for $\Delta \geq \Delta_k^*$. As a special case, let us define a linear combination in the form of $\sum_{i=1}^k w_i A_i(\Delta)$ with w_i being a non-negative coefficient, independent of Δ. This combination can be decomposed into $\sum_{i=1}^2 w_i A_i(\Delta) + \sum_{i=3}^k w_i A_i(\Delta)$. Based on the previous argument, it is clear to see that the second sum achieves its maximum when Δ is in the range of $\left[\min_{i=3,\ldots,k} \Delta_i^*, \max_{i=3,\ldots,k} \Delta_i^*\right] = [\Delta_k^*, \Delta_3^*] \subset [\Delta_k^*, \Delta_2^*]$. Besides, it is apparent that the first sum of this linear combination is an increasing function of Δ in the beginning. If its first derivative turns negative when $\Delta > \Delta_U$ for some $\Delta_U \geq \Delta_2^*$, it ensures that the first sum also has a maximum, and therefore, the entire linear combination $\sum_{i=1}^k w_i A_i(\Delta)$ has a maximum for $\Delta \in [\Delta_k^*, \Delta_U]$. Solving for Δ_U, it is observed that

$$w_1 A_1'(\Delta) + w_2 A_2'(\Delta) < 0 \qquad \Longleftrightarrow \qquad \Delta > \Delta_U,$$

where $\Delta_U = \Delta_2^* - \theta_2 \log(1 - W)$ is a non-sharp bound with $W = \dfrac{w_1}{w_2} \dfrac{1}{1 - \pi_1^*}$. If $0 \leq W < 1$, or equivalently, $0 \leq w_1/w_2 < 1 - \pi_1^*$, there exists $\Delta_U \geq \Delta_2^*$ and thus, the entire linear combination is guaranteed to have a maximum at $\Delta \in [\Delta_k^*, \Delta_U]$. Otherwise, the linear combination may not have a maximum point. On the other hand, if a linear combination is composed of the reciprocals of (the cross products of) $A_i(\Delta)$'s, including $A_1(\Delta)$, with non-negative coefficients, it is assured that the global minimum for this combination exists for $\Delta \geq \Delta_k^*$ since $1/A_1(\Delta) = 1/F_1(\Delta)$ decreases from ∞ to 1 asymptotically while $1/A_i(\Delta)$ decreases from ∞ to its minimum and then increases back to ∞ for $i = 2, 3, \ldots, k$. This result is particularly useful when searching for the optimal step duration Δ^* under certain optimality criteria using a numerical procedure.

4. Design Criteria & Optimal Step Duration

Various design criteria were considered in this study for determining the optimal stress duration Δ^*. These objective functions are formulated based on the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$ presented in the preceding section. Unlike $A_i(\Delta)$ in Gouno *et al.* [27], $A_i(\Delta)$ in (9) is positive for all $\Delta > 0$, ensuring that $\mathbf{I}_n(\alpha, \beta)$ has a positive determinant and the variance functions are also positive when $\mathbf{I}_n(\alpha, \beta)$ is inverted. With the suggested change in the censoring scheme, progressive censoring is performed based on the number of units remaining at the end of each stress level, and thus, censoring beyond what is available on the ALT is prohibited. As a result, there is no restriction on the search region for the optimal step duration $\Delta^* > 0$. Under each design criterion, we formally discuss the existence of the optimal step duration Δ^* for a general k-level step-stress ALT under progressive Type-I censoring. Studying the existence of these optimal stress durations is both theoretically and practically important to ensure the design feasibility under general conditions as well as to develop and implement an efficient computational search algorithm. Every design optimality criterion considered in this study, as well as some other information-based criteria, have been applied extensively in the design selection process for linearly designed experiments. In the practitioner's point of view, the choice of the optimality criterion is guided by the objective of the ALT. For further elaboration on the advantages and disadvantages of each design criterion, interested readers may refer to Wu and Hamad [33], Montgomery [34].

4.1 D-optimality

A design optimality criterion often used in planning the ALT is based on the reciprocal of the determinant of the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$, or equivalently, the determinant of the asymptotic variance-covariance matrix. It is well-known that at a fixed level of confidence, the overall volume of the Wald-type joint confidence region of (α, β) is proportional to $|\mathbf{I}_n^{-1}(\alpha, \beta)|^{1/2}$, or inversely proportional to $|\mathbf{I}_n(\alpha, \beta)|^{1/2}$. Accordingly, the larger the determinant of $\mathbf{I}_n(\alpha, \beta)$ is, the smaller the asymptotic joint confidence ellipsoid of (α, β) is and the higher the joint precision of the estimators of α and β would be. Under the D-optimality criterion, the objective function is formulated based on this as

$$\phi_{_D}(\Delta) \quad = \quad n^2|\mathbf{I}_n(\alpha, \beta)|^{-1} = 2\left[\sum_{i=1}^{k}\sum_{j=1}^{k} A_i(\Delta)A_j(\Delta)(x_i - x_j)^2\right]^{-1}, \qquad (10)$$

and the D-optimal stress duration $\Delta^*_{_D}$ is obtained by minimizing (10) for the maximal joint precision of $\big(\hat{\alpha}, \hat{\beta}\big)$. In the case of a simple step-stress ALT ($viz.$, $k = 2$), the objective function in (10) reduces to

$$\phi_{_D}(\Delta) = \big[A_1(\Delta)A_2(\Delta)(x_2 - x_1)^2\big]^{-1}.$$

Theorem 1. *In the case of a general k-level step-stress ALT under progressive or conventional Type-I censoring, there exists the D-optimal step duration $\Delta^*_{_D}$ in the range of $[\Delta^*_k, \Delta_U]$, where $\Delta^*_k = \theta_k \log(\delta_{k-1}/\delta_k)$ and $\Delta_U = \theta_1 \log(1 + \delta_1/\delta_k)$. It is the solution to the equation $\sum_{i=1}^{k}\sum_{j=1}^{k} A_i'(\Delta)A_j(\Delta)(x_i - x_j)^2 = 0$.*

4.2 T-optimality

This design optimality criterion is based on the total marginal Fisher information terms of the model parameters, which is identical to the sum of the diagonal elements or trace of $\mathbf{I}_n(\alpha, \beta)$. Like the D-optimality, the T-optimality criterion is a general measure of the size of the Fisher information $\mathbf{I}_n(\alpha, \beta)$. Based on (8), the T-optimal step duration $\Delta^*_{_T}$ minimizes the objective function defined by

$$\begin{aligned}\phi_{_T}(\Delta) \quad &= \quad n\,\mathrm{tr}^{-1}\big(\mathbf{I}_n(\alpha, \beta)\big) = \left[\sum_{i=1}^{k} A_i(\Delta) + \sum_{i=1}^{k} A_i(\Delta)x_i^2\right]^{-1} \\ &= \quad \left[I_\alpha(\Delta) + \sum_{i=1}^{k} A_i(\Delta)x_i^2\right]^{-1} = \left[\sum_{i=1}^{k} A_i(\Delta)(1 + x_i^2)\right]^{-1}. \qquad (11)\end{aligned}$$

In the case of a simple step-stress ALT, the objective function in (11) simply becomes

$$\phi_{_T}(\Delta) = \big[A_1(\Delta)(1 + x_1^2) + A_2(\Delta)(1 + x_2^2)\big]^{-1}.$$

Theorem 2. *In the case of a general k-level step-stress ALT under progressive Type-I censoring, if $0 \leq \pi_1^* < (x_2^2 - x_1^2)/(1 + x_2^2)$, the T-optimal step duration $\Delta^*_{_T}$ exists in the range of $[\Delta^*_k, \Delta_U]$, where $\Delta^*_i = \theta_i \log(\delta_{i-1}/\delta_i)$ and $\Delta_U = \Delta^*_2 - \theta_2 \log\left(1 - \dfrac{1 + x_1^2}{1 + x_2^2}\dfrac{1}{1 - \pi_1^*}\right)$. It is the solution to the equation $\sum_{i=1}^{k} A_i'(\Delta)(1 + x_i^2) = 0$. Otherwise, $\Delta^*_{_T}$ may not exist. In the case of a general k-level step-stress ALT under the conventional Type-I censoring, the T-optimal step duration $\Delta^*_{_T}$ is guaranteed to exist in the range of $[\Delta^*_k, \Delta_U]$, where $\Delta_U = \Delta^*_2 - \theta_2 \log\left(\dfrac{x_2^2 - x_1^2}{1 + x_2^2}\right)$.*

4.3 C-optimality

An aim of the ALT is often to estimate the parameters of interest with maximum precision and minimum variability as possible. For the step-stress ALT, such a parameter of interest is the MTTF of a test unit under the normal usage condition ($viz.$, θ_0 at stress level x_0). Based on (8), the objective function to serve this purpose is defined as

$$\begin{aligned}\phi_{_C}(\Delta) &= n\ \mathrm{AVar}\big(\log\hat{\theta}_0\big) = n\ \mathrm{AVar}\big(\hat{\alpha}+\hat{\beta}x_0\big)\\ &= n\ \mathrm{AVar}\big(\hat{\alpha}\big) \qquad (\because\ x_0\equiv 0)\\ &= n\,(1\ 0)\mathbf{I}_n^{-1}(\alpha,\beta)\begin{pmatrix}1\\0\end{pmatrix} = \phi_{_D}(\Delta)\big[\phi_{_T}^{-1}(\Delta)-I_\alpha(\Delta)\big]\\ &= 2\left[\sum_{i=1}^{k}A_i(\Delta)x_i^2\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{k}A_i(\Delta)A_j(\Delta)(x_i-x_j)^2\right]^{-1}, \qquad (12)\end{aligned}$$

where AVar stands for the asymptotic variance. The C-optimal step duration $\Delta^*_{_C}$ is the one that minimizes the objective function in (12). It is worthwhile to mention that when the interest lies in estimation of the p-th lifetime quantile under the normal usage condition ($viz.$, $t_p = -\theta_0\log(1-p)$ with $0 < p < 1$), the objective function to minimize is identical to (12). It is due to the fact that the MLE of the p-th quantile is $\hat{t}_p = -\hat{\theta}_0\log(1-p)$ by the invariance property, and thus, the objective function defined as $\phi_{_p}(\Delta) = n\ \mathrm{AVar}\big(\log\hat{t}_p\big)$ eventually becomes $\phi_{_p}(\Delta) = \phi_{_C}(\Delta)$ with the property of the variance operator. Consequently, the C-optimal results based on (12) are applicable to any parameter of interest which can be expressed as a scalized function of θ_0 ($i.e.$, $w\theta_0$ with $w\in\Re$). In the case of a simple step-stress ALT, the objective function in (12) gets simplified to

$$\phi_{_C}(\Delta) = \frac{A_1(\Delta)x_1^2 + A_2(\Delta)x_2^2}{A_1(\Delta)A_2(\Delta)(x_2-x_1)^2} = \frac{(1+\xi_0)^2}{A_1(\Delta)} + \frac{\xi_0^2}{A_2(\Delta)},$$

where $\xi_0 = x_1/(x_2-x_1)$.

Theorem 3. *In the case of a general k-level step-stress ALT under progressive or conventional Type-I censoring, there exists the C-optimal step duration $\Delta^*_{_C}$, which is the solution to the equation* $\left[\sum_{i=1}^{k}A_i'(\Delta)x_i^2\right]\left[\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}A_i(\Delta)A_j(\Delta)(x_i-x_j)^2\right] = \left[\sum_{i=1}^{k}A_i(\Delta)x_i^2\right]$ $\left[\sum_{i=1}^{k}\sum_{j=1}^{k}A_i'(\Delta)A_j(\Delta)(x_i-x_j)^2\right]$.

4.4 A-optimality

Another design optimality criterion considered in this study is based on the trace of the first-order approximation of the variance-covariance matrix of the MLE, or the sum of the diagonal entries of $\mathbf{I}_n^{-1}(\alpha,\beta)$. This A-optimality criterion provides an overall measure of the average variance of the parameter estimates and gives the sum of the eigenvalues of the inverse of the Fisher information matrix $\mathbf{I}_n(\alpha,\beta)$. The A-optimal stress duration $\Delta^*_{_A}$ is the one that minimizes the objective function defined as

$$\begin{aligned}\phi_{_A}(\Delta) &= n\ \mathrm{tr}\big(\mathbf{I}_n^{-1}(\alpha,\beta)\big) = \phi_{_D}(\Delta)/\phi_{_T}(\Delta)\\ &= 2\left[\sum_{i=1}^{k}A_i(\Delta)(1+x_i^2)\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{k}A_i(\Delta)A_j(\Delta)(x_i-x_j)^2\right]^{-1}. \qquad (13)\end{aligned}$$

In the case of a simple step-stress ALT ($k = 2$), the objective function in (13) reduces to

$$\phi_{A}(\Delta) = \frac{A_1(\Delta)(1 + x_1^2) + A_2(\Delta)(1 + x_2^2)}{A_1(\Delta)A_2(\Delta)(x_2 - x_1)^2} = \frac{\xi_2^2}{A_1(\Delta)} + \frac{\xi_1^2}{A_2(\Delta)},$$

where $\xi_i = \sqrt{1 + x_i^2}/(x_2 - x_1)$ for $i = 1, 2$.

Theorem 4. *In the case of a general k-level step-stress ALT under progressive or conventional Type-I censoring, the A-optimal stress duration Δ_{A}^* exists as the solution to the equation* $\left[\sum_{i=1}^{k} A_i'(\Delta)(1 + x_i^2)\right]\left[\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} A_i(\Delta)A_j(\Delta)(x_i - x_j)^2\right] = \left[\sum_{i=1}^{k} A_i(\Delta)(1 + x_i^2)\right]$ $\left[\sum_{i=1}^{k}\sum_{j=1}^{k} A_i'(\Delta)A_j(\Delta)(x_i - x_j)^2\right]$.

4.5 E-optimality

This design optimality criterion aims to minimize the maximum variance of all possible normalized linear combinations of the parameter estimates. The E-optimal design point maximizes the minimum eigenvalue of the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$, or equivalently, minimizes the maximum eigenvalue of the asymptotic variance-covariance matrix $\mathbf{I}_n^{-1}(\alpha, \beta)$. Hence, the E-optimal step duration Δ_{E}^* minimizes the objective function defined by

$$\begin{aligned}
\phi_{E}(\Delta) &= n\,\lambda_{\max}\big(\mathbf{I}_n^{-1}(\alpha, \beta)\big) \\
&= \frac{1}{2}\left[\phi_{A}(\Delta) + \sqrt{\phi_{A}^2(\Delta) - 4\phi_{D}(\Delta)}\right] \\
&= \left[\sum_{i=1}^{k} A_i(\Delta)(1 + x_i^2) + \sqrt{Q(\Delta)}\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{k} A_i(\Delta)A_j(\Delta)(x_i - x_j)^2\right]^{-1} \qquad (14)
\end{aligned}$$

where $\lambda_{\max}$ stands for the maximum eigenvalue, and

$$Q(\Delta) = \sum_{i=1}^{k}\sum_{j=1}^{k} A_i(\Delta)A_j(\Delta)(1 + x_i - x_j + x_ix_j)(1 - x_i + x_j + x_ix_j).$$

In the case of a simple step-stress ALT, the objective function in (14) reduces to

$$\begin{aligned}
\phi_{E}(\Delta) &= \frac{A_1(\Delta)(1 + x_1^2) + A_2(\Delta)(1 + x_2^2) + \sqrt{Q(\Delta)}}{2A_1(\Delta)A_2(\Delta)(x_2 - x_1)^2} \\
&= \frac{1}{2}\left[\frac{\xi_2^2}{A_1(\Delta)} + \frac{\xi_1^2}{A_2(\Delta)} + \sqrt{\frac{\xi_2^4}{A_1^2(\Delta)} + \frac{\xi_1^4}{A_2^2(\Delta)} + 2\frac{\xi_{21}^2}{A_1(\Delta)}\frac{\xi_{12}^2}{A_2(\Delta)}}\right],
\end{aligned}$$

where

$$\begin{aligned}
Q(\Delta) &= A_1^2(\Delta)(1 + x_1^2)^2 + A_2^2(\Delta)(1 + x_2^2)^2 \\
&\quad +2A_1(\Delta)A_2(\Delta)(1 + x_1 - x_2 + x_1x_2)(1 - x_1 + x_2 + x_1x_2)
\end{aligned}$$

and $\xi_{ij} = \sqrt{1 + x_i - x_j + x_1x_2}/(x_2 - x_1)$ for $i = 1, 2$.

Theorem 5. *In the case of a general k-level step-stress ALT under progressive or conventional Type-I censoring, there exists the E-optimal step duration Δ^*_E, which is the solution to the equation* $\left[\sum_{i=1}^{k} A'_i(\Delta)(1+x_i^2)+\frac{1}{2}\frac{Q'(\Delta)}{\sqrt{Q(\Delta)}}\right]\left[\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} A_i(\Delta)A_j(\Delta)(x_i-x_j)^2\right] = \left[\sum_{i=1}^{k} A_i(\Delta)(1+x_i^2)+\sqrt{Q(\Delta)}\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{k} A'_i(\Delta)A_j(\Delta)(x_i-x_j)^2\right]$, *where* $Q'(\Delta) = 2\sum_{i=1}^{k}\sum_{j=1}^{k} A'_i(\Delta)A_j(\Delta)(1+x_i-x_j+x_ix_j)(1-x_i+x_j+x_ix_j)$.

References

[1] H. Chernoff, "Optimal accelerated life designs for estimation," *Technometrics*, vol. 4, pp. 381–408, 1962.

[2] W. Nelson and W.Q. Meeker, "Theory for optimum accelerated censored life tests for Weibull and extreme value distributions," *Technometrics*, vol. 20, pp. 171–177, 1978.

[3] W. Nelson, *Accelerated Testing: Statistical Models, Test Plans, and Data Analyses*. New York, NY: Wiley & Sons, 1990.

[4] W.Q. Meeker and M.J. LuValle, "An accelerated life test model based on reliability kinetics," *Technometrics*, vol. 37, pp. 133–146, 1995.

[5] W.Q. Meeker and L.A. Escobar, *Statistical Methods for Reliability Data*. New York, NY: Wiley & Sons, 1998.

[6] V. Bagdonavicius and M. Nikulin, *Accelerated Life Models: Modeling and Statistical Analysis*. Boca Raton, FL: Chapman Hall, 2002.

[7] R. Miller and W. Nelson, "Optimum simple step-stress plans for accelerated life testing," *IEEE Trans. Rel.*, vol. 32, pp. 59–65, 1983.

[8] D.S. Bai, M.S. Kim, and S.H. Lee, "Optimum simple step-stress accelerated life tests with censoring," *IEEE Trans. Rel.*, vol. 38, pp. 528–532, 1989.

[9] I.H. Khamis and J.J. Higgins, "Optimum three-step step-stress tests," *IEEE Trans. Rel.*, vol. 45, pp. 341–345, 1996.

[10] I.H. Khamis and J.J. Higgins, "A new model for step-stress testing," *IEEE Trans. Rel.*, vol. 47, pp. 131–134, 1998.

[11] W.Q. Meeker and G.J. Hahn, "A comparison of accelerated test plans to estimate the survival probability at a design stress," *Technometrics*, vol. 20, pp. 245–247, 1978.

[12] W.Q. Meeker, "A comparison of accelerated life test plans for Weibull and lognormal distributions and Type I censoring," *Technometrics*, vol. 26, pp. 157–171, 1984.

[13] C.A. Meeter and W.Q. Meeker, "Optimum accelerated life tests with a nonconstant scale parameter," *Technometrics*, vol. 36, pp. 71–83, 1994.

[14] L.A. Escobar and W.Q. Meeker, "Planning accelerated life tests with two or more experimental factors," *Technometrics*, vol. 37, pp. 411–427, 1995.

[15] K.P. Yeo and L.C. Tang, "Planning step-stress life-test with a target acceleration-factor," *IEEE Trans. Rel.*, vol. 48, pp. 61–67, 1999.

[16] N. Balakrishnan and D. Han, "Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring," *J. Statist. Plann. Inference*, vol. 138, pp. 4172–4186, 2008.

[17] N. Balakrishnan and D. Han, "Optimal step-stress testing for progressively Type-I censored data from exponential distribution," *J. Statist. Plann. Inference*, vol. 139, pp. 1782–1798, 2009.

[18] D. Han and N. Balakrishnan, "Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint," *Comput. Statist. Data Anal.*, vol. 54, pp. 2066–2081, 2010.

[19] X. Liu and W.S. Qiu, "Modeling and planning of step-stress accelerated life tests with independent competing risks," *IEEE Trans. Rel.*, vol. 60, pp. 712–720, 2011.

[20] D. Han and D. Kundu, “Inference for a step-stress model with competing risks from the generalized exponential distribution under Type-I censoring,” *IEEE Trans. Rel.*, vol. 64, pp. 31–43, 2015.

[21] Y. Zhang and W.Q. Meeker, “Bayesian methods for planning accelerated life tests,” *Technometrics*, vol. 48, pp. 49–60, 2006.

[22] W.Q. Meeker, L.A. Escobar, and Y. Hong, “Using accelerated life tests results to predict product field reliability,” *Technometrics*, vol. 51, pp. 146–161, 2009.

[23] H.M. Lee, J.W. Wu, and C.L. Lei, “Assessing the lifetime performance index of exponential products with step-stress accelerated life-testing data,” *IEEE Trans. Rel.*, vol. 62, pp. 296–304, 2013.

[24] D. Han and H.K.T. Ng, “Comparison between constant-stress and step-stress accelerated life tests under time constraint,” *Naval Res. Logist.*, vol. 60, pp. 541–556, 2013.

[25] D. Han, “Time and cost constrained optimal designs of constant-stress and step-stress accelerated life tests,” *Rel. Engrg. Syst. Safety*, vol. 140, pp. 1–14, 2015.

[26] D. Han, “Optimal accelerated life tests under a cost constraint with non-uniform stress durations,” *Qual. Engrg*, vol. 29, pp. 409–430, 2017.

[27] E. Gouno, A. Sen, and N. Balakrishnan “Optimal step-stress test under progressive Type-I censoring,” *IEEE Trans. Rel.*, vol. 53, pp. 383–393, 2004.

[28] D. Han, N. Balakrishnan, A. Sen, and E. Gouno, “Corrections on Optimal step-stress test under progressive Type-I censoring,” *IEEE Trans. Rel.*, vol. 55, pp. 613–614, 2006.

[29] N. Balakrishnan, D. Han, and G. Iliopoulos, “Exact inference for progressively Type-I censored exponential failure data,” *Metrika*, vol. 73, pp. 335–358, 2010.

[30] A.C. Cohen, “Progressively censored samples in life testing,” *Technometrics*, vol. 5, pp. 327–339, 1963.

[31] J.F. Lawless, *Statistical Models and Methods for Lifetime Data*. New York, NY: Wiley & Sons, 1982.

[32] D. Han and T. Bai, “On the maximum likelihood estimation for progressively censored exponential lifetimes from constant-stress and step-stress accelerated life tests,” submitted, 2018.

[33] C.F. Wu and M. Hamad, *Experiments: Planning, Analysis, and Parameter Design Optimization*. New York, NY: Wiley & Sons, 2000.

[34] D.C. Montgomery, *Design and Analysis of Experiments*. New York, NY: Wiley & Sons, 2017.