Time Series for Boolean Random Sets

Kofi K. Wagya * Khalil. H. Shafie †

Abstract
M. Khazae and K. Shafie[12] in 2006 introduced regression models for Boolean random sets, to be able to model the effect of explanatory variables on the distribution of a Boolean random set. However, the measurement of random sets representation of objects are mostly taken over time, which introduces correlation in the observations. One solution to deal with the correlated observations is to fit a time series to the random sets of the Boolean-model type, which is the goal of this paper. Two methods are introduced and maximum likelihood estimation of the parameters are applied to the $\log$ link function using the two methods of estimation i.e n_t, the number of points and n_t^+ , the number of lower tangent points in window W_t. A simulation study is used to analyze the behavior of this time series. The distribution of the estimates approaches approximate normality, confirming the asymptotic behavior of maximum likelihood estimators. When the β_1 and α_1 have opposite signs, multivariate normality is achieved faster (at T=1000) than when the signs are the same (at T=2500).

The model is then applied to the Mountain Pine Beetle Data for a ten-year period (2001 to 2010). Both methods produced similar results, however, method I has lower standard errors.

Key Words: Random closed set, Boolean model, generalized linear model, Poisson Time Series, Autoregression, Lower tangent points

1. Introduction

The Boolean model, perhaps the most important random set model in practice and theory is formed by placing random closed sets (RACS) at the points of a homogeneous Poisson point process (D) and taking the union of these sets. That is

$$Y_t = \bigcup_{d_t \in D} (Z_i \oplus d_t),$$

where Z_t, $i = 1, 2, ...$ are independent copies of random closed set Z_0 and $Z_t \oplus d_t$ is a realization of the (a.s) bounded RACS translated to point d_t of a homogeneous Poisson process D. The intensity parameter λ of D and the probability law of the bounded random grain, Z_0 are the sources of randomness in the Boolean model. [20],[3] and [4] discussed extensively the characteristics and some methods for estimation of the parameters for the Boolean model. The main methods of numerical parameter estimations are: the miminum contrast method which by means of the Steiner formula uses the representation of the capacity functional. (See [5] and [7]); the method of intensities, where the Boolean parameters are chosen to match the empirical values of intensities of the Minkowski measures or functionals determined for a family of expanding windows. (See [10], [24] and [26]). For an extension of the method of intensity to non-convex 'typical' grains see [22]

Recent developments include studying smoothing techniques and estimation methods for nonstationary Boolean models with applications to coverage processes by [17]. [6] studied parameter estimation in Non-Homogenous Boolean models. [13] investigated concentration inequalities for measures of a Boolean model.
[12] introduced regression models for Boolean random sets, to be able to model the effects

*University of Northern Colorado, 501 20TH ST, Greeley, CO 80639
†University of Northern Colorado, 501 20TH ST, Greeley, CO 80639

of explanatory variables on the distribution of a random set. And [23] worked on the propagation models and fitting them to set-valued observations. Missing in the literature are models for when these set-valued obsrvations are correlated over time.

Our aim in this paper is to introduce time series models for random sets (RACS) i.e. $Y = (y_1, ...y_T)^{'}$. We wish to study the behavior of this time series. Observe that D or Z_t (the grain) or both could be time series, but our focus in this paper will be treating D as a time series.

2. The Boolean Model and it's Hitting Functional

A random closed set (RACS) is uniquely determined by the corresponding *capacity functional* $T_A(K) = P\{A \cap K \neq \emptyset\}$, where K is taking over $\mathcal{K}$ of all compact sets. [15] and [11] showed that, if $T_A(K)$ (called the hitting functional of the random set A) is determined then $P_A(.)$ will be determined as well. Specfically for the Boolean RACS defined above, [3] showed the hitting functional to be of the form:

$$T_Y(K) = 1 - exp\left\{-\lambda E[|Z_0 \oplus \check{K}|]\right\},$$

where $\check{K} = \{-k : k \in K\}$, $Z_0 \oplus \check{K} = \{z - k : z \in Z_0 \, , k \in K\}$, $|L|$ is the Lebesgue measure of L and λ is the intensity of the homogenous Poisson Process D. The volume fraction of a Boolean model ([20] referred to as macroscopic parameter) is found by using the hitting functional as $p = 1 - exp\{-\lambda E[|Z_0|]\}$. However, an unbiased estimator of p when Y is observed in a window W is

$$\hat{p} = \frac{|Y \cap W|}{|W|}. \tag{1}$$

To estimate λ (a microscopic parameter) and the distribution of Z_0, [4] and [20] used a point process of tangent points to construct an estimator. Take a typical grain Z_0 that is almost surely convex. If a direction u in R^d is fixed, then the tangent point of each grain Z_0 is defined to be the lexicographical minimum among all points at which a hyperplane orthogonal to u and moving in the direction of u first touches Z_0. Some of these tangent points are covered by other grains while other points are exposed. These exposed tangent points form a point process $N^+(u)$ with intensity $\lambda(1-p)$. If u is directed upwards, then $N^+(u)$ is called a lower positive tangent point process. Thus we can simply estimate λ, by

$$\hat{\lambda} = \frac{n^+}{|W|(1-p)}, \tag{2}$$

where n^+ is the number of lower positive tangent points in window W (see [18]). It will interest the reader to note that [22] extended the method of intensity to non-convex 'typical' grains.

3. Models and Fitting Methods

As stated before, our aim in this paper is to introduce time series models for random sets (RACS) i.e, $Y = (Y_1, ..., Y_T)^{'}$ and study the behavior of the time series. Denote a count time series by $\{n_t : t \in \mathbb{N}\}$ and a time-varying r-dimensional covariate vector $\{X_t : t \in \mathbb{N}\}$, say $X_t = (X_{t,1}, ...X_{t,r})^T$. Also denote by $\mathcal{F}_{t-1}$, the history of the joint process $\{n_{t-1}, \lambda_{t-1}, X_t : t \in \mathbb{N}\}$ up to time $t-1$ including the covariate information at time t. Observe that the conditional distribution is distributed as $n_t|\mathcal{F}_{t-1} \sim Poisson(\lambda_t)$. Then we model the conditional mean $E(n_t|\mathcal{F}_{t-1})$ of the count time series by a process say $\{\lambda_t : t \in \mathbb{N}\}$ such that $E(n_t|\mathcal{F}_{t-1}) = \lambda_t$. The general form of the model (as defined by [14]) is

$$g(\lambda_t) = \beta_0 + \sum_{k=1}^{p} \beta_k \tilde{g}(n_{t-i_k}) + \sum_{l=1}^{q} \alpha_l g(\lambda_{t-j_l}) + \eta^T X_t, \tag{3}$$

where $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a link function and $\tilde{g} : \mathbb{N}_0 \longrightarrow \mathbb{R}$ is a transformation function. The parameter vector $\eta = (\eta_1, ...\eta_r)^T$ corresponds to the effects of covariates. For extensive discussion on count time series, see [14], [28] and [27].
For our study $g(m) = \log\ m$, and $\tilde{g}(x) = \log\ (x + 1)$, $p = q = 1$ and $X = 0$ (for simplicity), so that (3) reduces to

$$log\ \lambda_t = \beta_0 + \beta_1 log\ (n_{t-1} + 1) + \alpha_1 log \lambda_{t-1}, \tag{4}$$

where λ_t is the intensity of the Poisson process of the Boolean RACS. i.e $n_t|\mathcal{F}_{t-1}$. We assume all realizations of the $n_t|\mathcal{F}_{t-1}$ are observed in window W_t of Lebesgue measure of 1. Also, we assume Z_i in the Boolen RACS Y_t are independent of D_t. For our study we assume the grains are circles of constant radius 0.01. Thus we can study $n_t|\mathcal{F}_{t-1}$ by studying the relation in (4), since the λ_t controls the point process. To incorporate the information of n_{t-1} in (4), we use the suggested a bijective transformation by [14], i.e $\tilde{g}(n_{t-1}) = log\ (n_{t-1} + 1)$, where n_{t-1} is the number of points in a window W_{t-1}. This is to ensure that the n_{t-1} is transformed onto similar scale as the rates $\lambda_t' s$. In the next two sections, we will suggest two methods for studying (4).

3.1 Fitting Method I

In practice, we cannot observe n_{t-1} for overlapping grains. We use an estimate instead, as proposed by [12] i.e

$$\hat{n}_t = [|W_t|\hat{\lambda}_t] = \left[\frac{n_t^+}{1 - \hat{p}_t}\right], \tag{5}$$

where n_t^+ and $\hat{p}_t$ are respectively, the number of lower tangent points and it's estimated volume fraction obtained from (1) in window W_t. Then we can learn about $n_t|\mathcal{F}_{t-1} \sim Poisson(\lambda_t)$ by studying

$$\hat{n}_t|\mathcal{F}_{t-1} \sim Poisson(\hat{\lambda}_t).$$

We now model the conditional mean $E(\hat{n}_t|\mathcal{F}_{t-1})$ of the count time series by a process say $\{\hat{\lambda}_t : t \in \mathbb{N}\}$ such that $E(\hat{n}_t|\mathcal{F}_{t-1}) = \hat{\lambda}_t$. Let $log\ \hat{\lambda}_t = \nu_t$, then

$$\nu_t = \hat{\beta}_0 + \hat{\beta}_1 log(\hat{n}_{t-1} + 1) + \hat{\alpha}_1 \nu_{t-1}$$

We will call this Method I.

3.2 Fitting Method II

In Method I, we employed the use of exposed lower tangent points n_t^+ in the estimation of $\hat{n}_t$. However, n_t^+ has the following distrubution,

$$n_t^+ \sim Poisson(\lambda_t^+), where\ \lambda_t^+ = |W_t|\lambda_t exp[-E|Z_0|\lambda_t]$$

The estimate n_t^+ is obtained by counting the lower tangent point of the set-valued observation in window W_t. This can be achieved by Laslett's transformation implemented in the Spatstat package by [1] in [21], that returns the number of lower tangent points. We then model the conditional mean $E(n_t^+|\mathcal{F}_{t-1}) = \lambda_t^+$. Again, let $log\lambda_t^+ = \mu_t$, then

$$\mu_t = \beta_0^+ + \beta_1^+ log(n_{t-1}^+ + 1) + \alpha_1^+ \mu_{t-1}.$$

However, to get the estimates of (4) from μ_t, we use the relationship $\lambda_t^+ = |W_t|\lambda_t exp[-E|Z_0|\lambda_t]$. By the first order Taylor expansion and approximation the estimates of (4) through μ_t are as follows:

$$\beta_0 \approx \frac{\beta_0^+ + C}{1-C},\ \beta_1 \approx \frac{\beta_1^+}{1-C},\ \alpha_1 \approx \frac{\alpha_1^+}{1-C},\ where\ C = E|Z_0| = \pi r^2$$

This, we call Method II.

4. Likelihood Inference

To study the likelihood inference for (4), with the three dimensional parameter space of $\theta = (\beta_0, \beta_1, \alpha_1)$, and the initial value of λ_0 in terms of observations $n_1, ..., n_T$, the conditional likelihood function for θ is given by

$$L(\theta) = \prod_{t=1}^{T} \frac{exp(\lambda_t(\theta))\lambda_t^{n_t(\theta)}}{n_t!}, where\ \lambda_t(\theta) = \exp(\beta_0 + \beta_1 log(n_{t-1}+1) + \alpha_1 \log \lambda_{t-1})$$

Let $\log \lambda_t = \gamma_t$. Then, the log-likelihood function has the form

$$l(\theta) = \sum_{t=1}^{T}(n_t\gamma_t(\theta) - \exp(\gamma_t(\theta)) - \log n_t),\ where\ \gamma_t(\theta) = \beta_0 + \beta_1 log(n_{t-1}+1) + \alpha_1\gamma_{t-1}$$

The score function is given by

$$S_T(\theta) = \frac{\partial l(\theta)}{\partial \theta} = \sum_{t=1}^{T}(n_t - \exp(\gamma_t(\theta)))\frac{\partial \gamma_t(\theta)}{\partial \theta}$$

where $\frac{\partial \gamma_t(\theta)}{\partial \theta}$ is a vector with components

$$\frac{\partial \gamma_t}{\partial \beta_0} = 1 + \alpha_1 \frac{\partial \gamma_{t-1}}{\partial \beta_0},\ \frac{\partial \gamma_t}{\partial \beta_1} = \log(1 + n_{t-1}) + \alpha_1 \frac{\partial \gamma_{t-1}}{\partial \beta_1},\ \frac{\partial \gamma_t}{\partial \alpha_1} = \gamma_{t-1} + \alpha_1 \frac{\partial \gamma_{t-1}}{\partial \alpha_1}$$

The solution of $S_T(\theta) = 0$ i.e $\hat{\theta}$ yields the conditional maximum likelihood estimator of θ, if it exist.

The Hessian matrix of (4) is obtained from

$$H_T(\theta) = \frac{\partial^2 l(\theta)}{\partial \theta^2} = \sum_{t=1}^{T}(\exp(\gamma_t(\theta))\left(\frac{\partial \gamma_t(\theta)}{\partial \theta}\right)\left(\frac{\partial \gamma_t(\theta)}{\partial \theta}\right)' - \sum_{t=1}^{T}(n_t - \exp(\gamma_t(\theta)))\frac{\partial^2 \gamma_t(\theta)}{\partial \theta \partial \theta'}$$

[27] and [28] proved the following theorem which shows the asymptotic normality of $\hat{\theta}$:

Theorem 4.1 *Consider model(4) and that at the true value $\theta_0, |\alpha_{1_0} + \beta_{1_0}| < 1$, if both $\alpha_{1_0}, \beta_{1_0}$ have the same sign, and $\alpha_{1_0}^2 + \beta_{1_0}^2 < 1$, if both $\alpha_{1_0}, \beta_{1_0}$ have the different signs. Then, there exists a fixed open neighborhood $O = O(\theta_0) of \theta_0$- such that with probability tending to 1, as $T \to \infty$, the log-likelihood has a unique maximum point $\hat{\theta}$. Furthermore, $\hat{\theta}$. is consistent and asymptotically normal;*

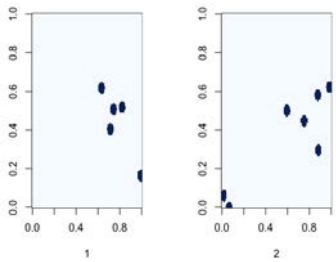
$$\sqrt{T}\left(\hat{\theta} - \theta_0\right) \xrightarrow{D} \mathcal{N}(0, G^{-1}).$$

A consistent estimator of $\mathbf{G}$ is given by $\mathbf{G_T}(\hat{\theta})$, where

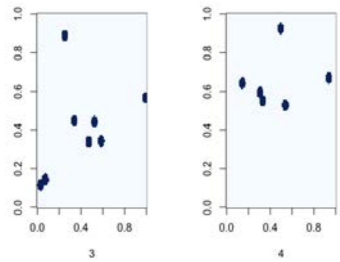
$$\mathbf{G_T}(\theta) = \sum_{t=1}^{T} (\exp(\gamma_t(\theta)) \left(\frac{\partial \gamma_t(\theta)}{\partial \theta}\right) \left(\frac{\partial \gamma_t(\theta)}{\partial \theta}\right)'$$

5. Simulation Study

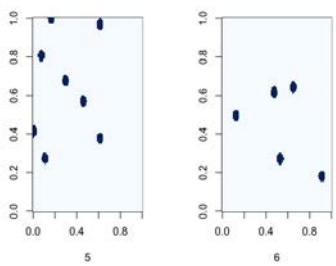
In this section, we present results for the simulation study. We assumed the grains are circles with a constant radius (known) of 0.01 units along with the Poisson point process with rates ν_t and μ_t using the time series equations under fitting methods I and II, respectively. The laslett function (found in the R package Spatstat by [1]) is used to compute the lower tangent points, then $\hat{p_t}$ and $\hat{n_t}$ are estimated using (1) and (2). Below, a realization of a Boolean time series model $Y_t,\ t = 1, 2, ..., 10$ in Figure 1 along with statistics obtained from the model found in Table 1.



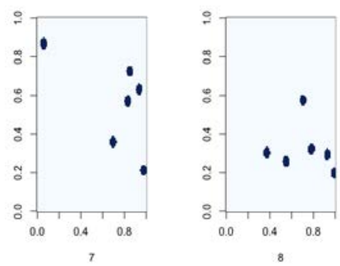
(a) T=1 and 2



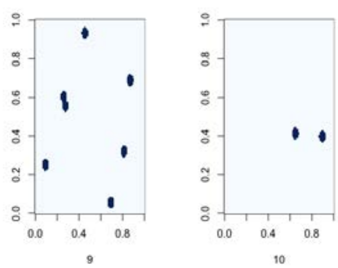
(b) T=3 and 4



(c) T=5 and 6



(d) T=7 and 8



(e) T=9 and 10

Figure 1: A ten-time period realaization of Boolean Time Series Model Y_t

Duration 5, and 9 have the most affected areas (deep blue), compared to duration 10 with the least affected areas.

Table 1: Statistics obtained from Figure 1.

T	1	2	3	4	5	6	7	8	9	10
n^+	3	4	6	5	3	5	4	5	7	9
p	0.008	0.011	0.016	0.013	0.009	0.014	0.011	0.014	0.020	0.025
n	4	5	7	5	4	5	5	6	8	9

Below, we explore $T = 10, 50, 100, 200,\ 500, \&\ 1000$ for fixed $\theta = (\beta_0 = 0.5, \beta_1 = 0.65, \alpha_1 = -0.5)$ corresponding to the condtion $\alpha_1^2 + \beta_1^2 < 1$ and $\theta = (\beta_0 = 0.5, \beta_1 = -0.35, \alpha_1 = -0.5)$ corresponding to the condtion $|\alpha_1 + \beta_1| < 1$. $\nu_0 = 1$ and substitute

n^+ in the above equation. The observations follow

$$Y_t = \bigcup_{d_t \in D_t} (Z_t \oplus d_t),$$

where D_t is a Poisson process with intensity λ_t and $Z_t s$ are circles with fixed radius of 0.01 units.. One thousand simulations are used except when stated otherwise.

The two methods generally behave similar for both methods. The results for the two methods are displayed in Table II and III below respectively. In Tables II and III, T represnts the duration, whilst the estimated MLEs are found in column 3. The last column displays the p-value for Lilliefors (Kolmogorov-Smirnov) normality test. The maximum likelihood estimmators (MLE) of both methods improves as T increases, confirming the theorem. Standard errors and biases for β_1 and α_1 decreases with increasing T. However, for β_0 the estimated MLEs deviates from the parameter of 0.5.

When α_1 and β_1 have the different signs satisfying the condition $|\alpha_{1_0} + \beta_{1_0}| < 1$, Theorem 4.1 is confirmed at T=1000, with the Royston test statistic of 0.677 and a p-value of 0.628 for method I and Royston test statistic of 5.241037 and p-value of 0.060 for method II.

However, when α_1 and β_1 have the same signs ($\alpha_{1_0}^2 + \beta_{1_0}^2 < 1$), both methods are slow to attain multivariate normaliy. At T=2500, a less expensive simulation(100), however produces the result (Royston test statistic of 1.764 and p-value 0.300 for methods I and II).

Table 2: Simulation results for θ using $\nu_t = \hat{\beta}_0 + \hat{\beta}_0 log(\hat{n}_{t-1} + 1) + \hat{\alpha}_1 \nu_{t-1}$

$\theta = (\beta_0 = 0.5, \beta_1 = 0.65, \alpha_1 = -0.5)$					
Parameter	T	MLE	Standard Error	Bias	p-value
β_0	10	1.317	0.559	-0.817	0.037
β_1		0.186	0.596	0.464	0.000
α_1		-0.236	0.664	-0.264	0.000
β_0	50	1.320	0.522	-0.820	0.019
β_1		0.421	0.178	0.229	0.087
α_1		-0.504	0.342	0.004	0.000
β_0	100	1.268	0.399	-0.768	0.949
β_1		0.418	0.129	0.232	0.193
α_1		-0.463	0.283	-0.037	0.000
β_0	200	1.280	0.290	-0.780	0.290
β_1		0.418	0.089	0.232	0.286
α_1		-0.472	0.191	-0.028	0.000
β_0	500	1.261	0.173	-0.761	0.718
β_1		0.423	0.057	0.227	0.519
α_1		-0.464	0.108	-0.036	0.229
β_0	1000	1.249	0.112	-0.749	0.117
β_1		0.471	0.039	0.179	0.179
α_1		-0.497	0.068	-0.003	0.402

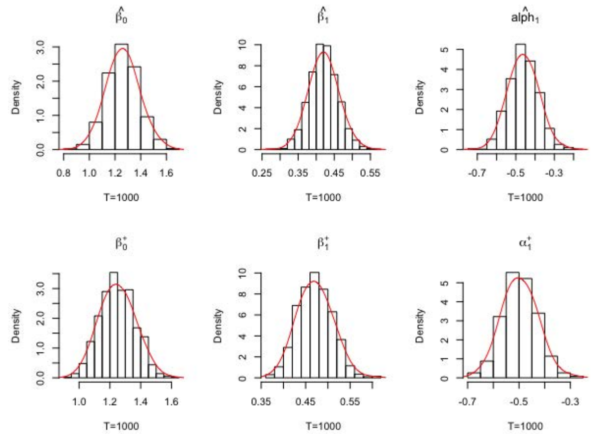
$\theta = (\beta_0 = 0.5, \beta_1 = -0.35, \alpha_1 = -0.5)$					
Parameter	T	MLE	Standard Error	Bias	p-value
β_0	10	0.987	0.487	-0.487	0.000
β_1		-0.308	0.593	-0.042	0.000
α_1		-0.001	0.692	-0.499	0.000
β_0	50	0.940	0.429	-0.440	0.000
β_1		-0.204	0.250	-0.146	0.000
α_1		-0.023	0.610	-0.477	0.000
β_0	100	1.089	0.366	-0.589	0.000
β_1		-0.169	0.148	-0.181	0.000
α_1		-0.244	0.500	-0.256	0.000
β_0	200	1.172	0.309	-0.672	0.000
β_1		-0.169	0.110	-0.181	0.014
α_1		-0.380	0.465	-0.120	0.000
β_0	500	1.203	0.230	-0.703	0.000
β_1		-0.161	0.064	-0.189	0.004
α_1		-0.430	0.331	-0.070	0.000
β_0	1000	1.243	0.142	-0.743	0.000
β_1		-0.160	0.049	-0.190	0.698
α_1		-0.486	0.219	-0.014	0.000
β_0	2500	1.254	0.073	-0.754	0.088
β_1		-0.160	0.032	-0.190	0.745
α_1		-0.505	0.119	0.005	0.387

Table 3: Simulation results for θ using $\mu_t = \beta_0^+ + \beta_1^+ log(n_{t-1}^+ + 1) + \alpha_1^+ \mu_{t-1}$.

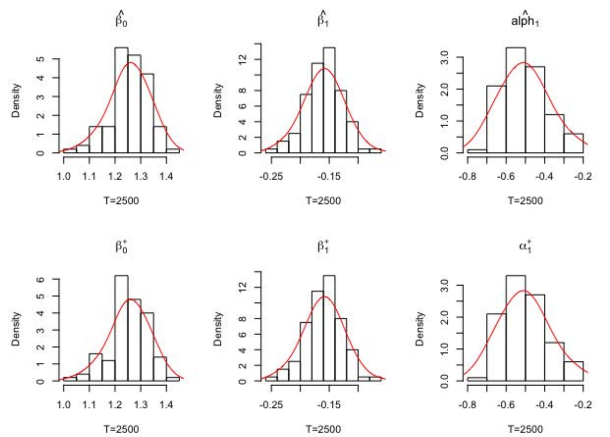
		$\theta = (\beta_0 = 0.5, \beta_1 = 0.65, \alpha_1 = -0.5)$			
Parameter	T	MLE	Standard Error	Bias	p-value
β_0	10	1.311	0.556	-0.811	0.020
β_1		0.186	0.595	0.464	0.000
α_1		-0.231	0.666	-0.269	0.000
β_0	50	1.319	0.521	-0.819	0.014
β_1		0.421	0.177	0.229	0.135
α_1		-0.504	0.340	0.004	0.000
β_0	100	1.268	0.401	-0.768	0.961
β_1		0.418	0.130	0.232	0.293
α_1		-0.464	0.285	-0.036	0.000
β_0	200	1.279	0.291	-0.779	0.407
β_1		0.418	0.089	0.232	0.433
α_1		-0.472	0.191	-0.028	0.000
β_0	500	0.971	0.141	-0.471	0.197
β_1		0.537	0.058	0.113	0.344
α_1		-0.469	0.093	-0.031	0.311
β_0	1000	1.259	0.112	-0.759	0.117
β_1		0.471	0.039	0.179	0.179
α_1		-0.497	0.068	-0.003	0.402

		$\theta = (\beta_0 = 0.5, \beta_1 = -0.35, \alpha_1 = -0.5)$			
Parameter	T	MLE	Standard Error	Bias	p-value
β_0	10	0.984	0.488	-0.484	0.000
β_1		-0.306	0.594	-0.044	0.000
α_1		-0.001	0.693	-0.499	0.000
β_0	50	0.939	0.429	-0.439	0.000
β_1		-0.204	0.250	-0.146	0.000
α_1		-0.023	0.610	-0.477	0.000
β_0	100	1.086	0.368	-0.586	0.000
β_1		-0.184	0.148	-0.166	0.000
α_1		-0.244	0.500	-0.256	0.000
β_0	200	1.172	0.307	-0.672	0.000
β_1		-0.169	0.110	-0.181	0.014
α_1		-0.380	0.463	-0.120	0.000
β_0	500	1.202	0.228	-0.702	0.000
β_1		-0.161	0.064	-0.189	0.004
α_1		-0.431	0.330	-0.069	0.000
β_0	1000	1.242	0.142	-0.742	0.000
β_1		-0.160	0.049	-0.190	0.648
α_1		-0.487	0.220	-0.013	0.000
β_0	2500	1.254	0.073	-0.754	0.085
β_1		-0.160	0.032	-0.190	0.743
α_1		-0.505	0.119	0.005	0.382

6. Some Histograms of Parameter Estimates



(a) The first row is the histogram of the $\hat{\theta}$ for Method I and row 2 displays that of Method II for T=1000 when $\beta_1 \& \alpha_1$ have opposite signs . This graphical displays confirms the results in Table II and III



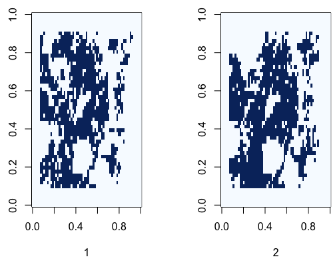
(b) The first row is the histogram of the $\hat{\theta}$ for Method I and row 2 displays that of Method II for T=2500 when $\beta_1 \& \alpha_1$ have same signs .The number of simulation used here is 100. This graphical displays confirms the results in Table II and III

Figure 2: A histogram of the $\hat{\beta}_i s$ for Methods I and II

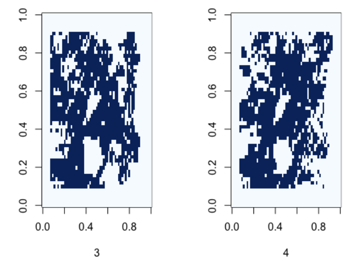
For both methods, the distribution of the estimates approaches approximate normality, confirming the asymptotic behavior of maximum likelihood estimators, these graphs agree with Tables II and III.

7. Application to Mountain Pine Beetle Data

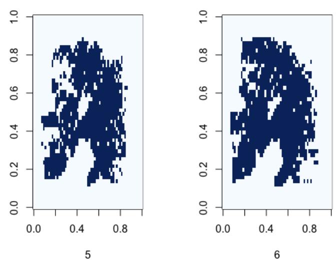
[9] applied the Generalized Method of Moments Approach for Spatial-Temporal Binary Data to the Rocky Mountain Forest Service data on mountain pine beetle from 2001 to 2010. Data was obtained from the PRISM dataset that is publicly available at http://www.prism.oregonstate.edu/. Thresholding and smoothing were applied to the data. We treat the data as a Boolean random set with the observations below in Figure 6.



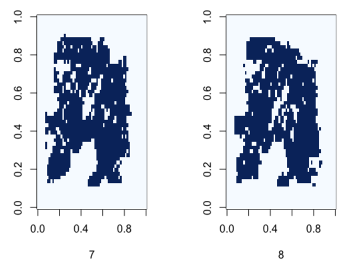
(a) T=2001 and 2002.



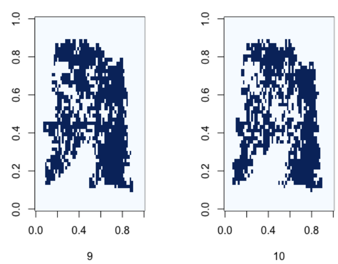
(b) T=2003 and 2004



(c) T=2005 and 2006



(d) T=2007 and 2008



(e) T=2009 and 2010

Figure 3: The Pine Beetle Data from 2001 to 2010.

The deep-blue areas in Figure 6 are the affected area, which signifies the presence of mountain pine beetle attack. The years between 2006 to 2008 saw an increase in attack, compared to 2001 and 2002. It will be interesting to be able to predict the rate of attack yearly.

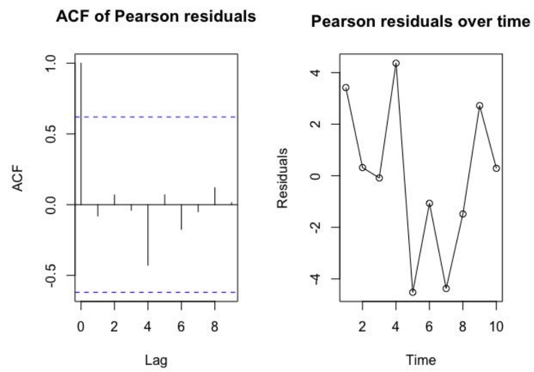
We apply the simple time series model described above to the beetle data using methods I and II. Below are the statistics.

Table 4: Statistics obtained from Figure 4.

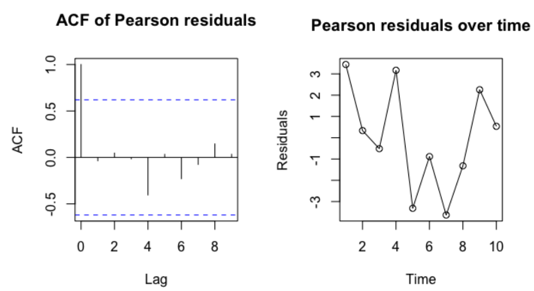
T	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
$\hat{n}^+$	55	44	34	53	20	23	11	17	41	42
$\hat{p}$	0.317	0.342	0.417	0.406	0.330	0.376	0.382	0.380	0.335	0.307
$\hat{n}$	81	67	59	90	30	37	18	28	62	61

Table 5: Summary of Method I and II

(a) Method 1			(b) Method 2		
	Coefficient	Std. Error($\hat{\beta}_i$)		Coefficient	Std. Error($\hat{\beta}_i$)
β_0	2.631	1.018	β_0	2.808	1.203
β_1	0.383	0.103	β_1	0.390	0.128
α_1	-0.038	0.274	α_1	-0.181	0.384



(a) ACF and Residual Plot for Method I



(b) ACF and Residual Plot for Method I

Figure 4: Diagnostics for the two methods

Both methods give similar results. However, method I has lower errors compared to two. Hence, we can study the Boolean time series using the above methods discussed.

References

[1] Adrian Baddeley, Ege Rubak, and Rolf Turner, *Spatial point patterns: Methodology and applications with R*, Chapman and Hall/CRC Press, 2015.

[2] Adrian Baddeley, Rolf Turner, et al., *Spatstat: an R package for analyzing spatial point patterns*, Journal of statistical software **12** (2005), no. 6, 1–42.

[3] Sung N. Chiu, Dietrich Stoyan, Wilfrid S. Kendall, and Joseph Mecke, *Stochastic geometry and its applications*, 3rd ed., John Wiley & Sons Inc, Somerset, 2013 (English).

[4] Noel Cressie, *Statistics for spatial data*, 2 ed., Wiley-Interscience, New York, 2015 (English).

[5] Noel Cressie and G. M. Laslett, *Random set theory and problems of modeling*, SIAM Review **29** (1987), no. 4, 557–574.

[6] Maria Angeles Gallego, Maria Victoria Ibanez, and Amelia Simø, *Parameter estimation in non-homogeneous Boolean models: An application to plant defense response*, Image Analysis & Stereology **34** (2014), no. 1, 27–38.

[7] Lothar Heinrich, *Asymptotic properties of minimum contrast estimators for parameters of Boolean models*, Metrika **40** (1993), no. 1, 67–94.

[8] Kimberly A Kaufeld, Matthew J Heaton, and Stephan R Sain, *A spatio-temporal model for mountain pine beetle damage*, Journal of agricultural, biological, and environmental statistics **19** (2014), no. 4, 437–450.

[9] Kimberly Ann Kaufeld, *Generalized method of moments approach for spatial-temporal binary data*, Ph.D. thesis, University of Northern Colorado, 2014.

[10] A. M. Kellerer, *Counting figures in planar random configurations*, Journal of Applied Probability **22** (1985), no. 1, 68–81.

[11] D.G Kendall, *Foundations of a theory of random sets. In: Harding, E.F. and Kendall, D.G., EDS., Stochastic Geometry*, 1974, pp. 322–376.

[12] M Khazaee and K Shafie, *Regression models for Boolean random sets*, Journal of Applied Statistics **33** (2006), no. 05, 557–567.

[13] Günter Last and Fabian Gieringer, *Concentration inequalities for measures of a Boolean model*, arXiv:1703.04971 (2017).

[14] Tobias Liboschik, Konstantinos Fokianos, and Roland Fried, *tscount: An R package for analysis of count time series following generalized linear models*, Journal of Statistical Software (2017).

[15] G Matheron, *Random sets theory and its applications to stereology*, Journal of Microscopy **95** (1972), no. 1, 15–23.

[16] Peter McCullagh, *Generalized linear models*, European Journal of Operational Research **16** (1984), no. 3, 285–292.

[17] I. S. Molchanov and S. N. Chiu, *Smoothing techniques and estimation methods for nonstationary Boolean models with applications to coverage processes*, Biometrika **87** (2000), no. 2, 265–283.

[18] Ilya Molchanov and Dietrich Stoyan, *Asymptotic properties of estimators for parameters of the Boolean model*, Advances in Applied Probability **26** (1994), no. 2, 301323.

[19] Ilya S. Molchanov, *Statistics of the Boolean model: from the estimation of means to the estimation of distributions*, Advances in Applied Probability **27** (1995), no. 1, 6386.

[20] Ilya S Molchanov et al., *Theory of random sets*, vol. 19, Springer, 2005.

[21] R Core Team, *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria, 2017.

[22] Michel Schmitt, *Estimation of the density in a stationary Boolean model*, Journal of Applied Probability **28** (1991), no. 3, 702–708.

[23] Khalil Shafie et al., *Propagation models and fitting them for the Boolean random sets*, Journal of Optimization in Industrial Engineering (2010), 45–48.

[24] Dietrich Stoyan, W. S. Kendall, and Joseph Mecke, *Stochastic geometry and its applications*, Wiley, New York;Chichester [W. Sussex];, 1987 (English).

[25] Dietrich Stoyan, Helga Stoyan, André Tscheschel, and Torsten Mattfeldt, *On the estimation of distance distribution functions for point processes and random sets*, Image Analysis & Stereology **20** (2011), no. 1, 65–69.

[26] Wolfgang Weil, *Expectation formulas and isoperimetric properties for non-isotropic Boolean models*, Journal of Microscopy **151** (1988), no. 3, 235–245.

[27] Fokianos, Konstantinos and Tjøstheim, Dag, *L*og-linear Poisson autoregression, Journal of multivariate analysis (2011), 102, 3, 563–578, Elsevier

[28] Fokianos, Konstantinos and Rahbek, Anders and Tjøstheim, Dag, *P*oisson autoregression, Journal of the American Statistical Association (2009), 104, 488, 1430–1439, Taylor & Francis