Expected Termination Times of Progressively Type-I Censored Step-stress Accelerated Life Tests under Continuous and Interval Inspections

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Abstract

A step-stress accelerated life test is a special life test where test units are subjected to higher stress levels than normal operating conditions so that the information on the lifetime parameters of a test unit can be obtained more quickly in a shorter period of time. Also, progressive Type-I censoring is a generalized form of time censoring where functional test units are withdrawn successively from the life test at some prefixed non-terminal time points. Despite its flexibility and efficient utilization of the available resources, progressively censored sampling has not gained much popularity due to its analytical complexity compared to the conventional censoring schemes. In particular, understanding the mean completion time of a life test is of great practical interest in order to design and manage the life test optimally under frequent budgetary and time constraints. In this work, the expected termination time of a general k-level step-stress accelerated life test under progressive Type-I censoring is derived using a recursive relationship of the stochastic termination time based on the conditional block independence. To be comprehensive, two different modes of failure inspections are considered: continuous inspection where the exact failure times are observed, and interval inspection where the exact failure times are not available but only the number of failures that occurred.

Key Words: accelerated life tests, continuous inspection, interval inspection, order statistics, progressive Type-I censoring, step-stress loading

1. Introduction

Thanks to the continual improvement in manufacturing process and technology, products are becoming increasingly reliable with substantially long life-spans, which makes the standard life tests under normal usage conditions very difficult if not impossible. This difficulty is overcome by accelerated life test (ALT) where the units are subjected to higher stress levels than normal operating conditions so that more failures can be collected in a shorter period of time. The lifetime at the normal operating condition is then estimated through extrapolation using a stress-response regression model. The (step-up) step-stress test is a special class of ALT where the stress levels are gradually increased at some fixed time points during the experiment. During the past decades, the inference and design optimization for the step-stress ALT have attracted great attention in the reliability literature; see, for example, Miller and Nelson (1983), Bai et al. (1989), Nelson (1990), Meeker and Escobar (1998), Bagdonavicius and Nikulin (2002), Wu et al. (2006), Balakrishnan and Han (2008, 2009), Han and Balakrishnan (2010), Kateri et al. (2010), Han and Ng (2013), Han and Kundu (2015), and Han (2015).

Moreover, due to time and cost constraints, censored sampling is usually unavoidable in practice, and in particular, a generalized censoring scheme known as progressive Type-I censoring allows functional test units to be withdrawn successively from the life test at some prefixed non-terminal time points. Withdrawn unfailed units can be used in other tests in the same or at a different facility; see, for instance, Gouno et al. (2004), Han et al. (2006), and Balakrishnan et al. (2010). Despite its flexibility and efficient utilization of the available resources, progressively censored sampling has not gained much popularity

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in ALT due to its analytical complexity compared to the conventional censoring schemes; see Cohen (1963) and Lawless (1982). In particular, understanding the mean completion time of a life test under progressive Type-I censoring is of great practical interest in order to design and manage the life test optimally under frequent budgetary and time constraints.

The research presented here is motivated by the following engineering case study. A three-level step-stress ALT under progressive Type-I censoring is being planned with the sample size of $n = 30$ prototypes in order to assess the reliability characteristics of a solar lighting device in the second phase. The dominant failure mode of the device is controller failure, and the stress factor is temperature whose level is increased during the test in the range of 293K to 353K with the normal operating temperature at 293K. The standardized stress loading of the testing chamber is determined to be $x_1 = 0.1$, $x_2 = 0.5$, and $x_3 = 0.9$. One of the objectives of the study is to report the mean termination time in order to manage the future tests more efficiently. Due to technical limitations and budgetary constraints, it is also considered that the exact failure times of test units may not be observable ($i.e.$, interval inspection). To answer these questions, this work formulates the expected termination time of a general k-level step-stress ALT under progressive Type-I censoring by using a recursive relationship of the stochastic termination time. It is assumed that without changes in the failure mechanism, the lifetimes of test units follow an exponential distribution at each stress level, along with the accelerated failure time (AFT) model for the effect of changing stress. This results in the conditional block independence of the ordered failure time data as discussed by Iliopoulos and Balakrishnan (2009), Balakrishnan and Cramer (2014). Allowing the intermediate censoring to take place at each stress change time point ($viz.$, τ_i, $i = 1, 2, \ldots, k$), two different modes of failure inspections are considered: continuous inspection where the exact failure times are observed, and interval inspection where the exact failure times are not available but only the number of failures that occurred.

The rest of the paper is organized as follows. Section 2 presents the model descriptions for the progressively Type-I censored k-level step-stress ALT under continuous and interval inspections. The expected termination time under continuous inspection is then derived recursively in Section 3 while the expected termination time under interval inspection is derived in Section 4. For simplicity, no notational distinction is made in this article between the random variables and their corresponding realizations. Also, we adopt the usual conventions that $\sum_{j=m}^{m-1} a_j \equiv 0$ and $\prod_{j=m}^{m-1} a_j \equiv 1$.

2. Progressively Type-I Censored Step-stress ALT

To describe the procedure of a general k-level step-stress ALT under progressive Type-I censoring, let us first denote $s(t)$ to be the given stress loading (a deterministic function of time) for ALT. Also, let s_H be an upper bound of stress level and s_U be the normal use-stress level. The standardized stress loading is then defined as

$$x(t) = \frac{s(t) - s_U}{s_H - s_U}, \qquad t \geq 0$$

so that the range of $x(t)$ is $[0, 1]$. Now, let us define $0 \equiv x_0 \leq x_1 < x_2 < \cdots < x_k \leq 1$ to be the ordered k standardized stress levels to be used in the test. Then, for $i = 1, 2, \ldots, k$, let n_i denote the (random) number of units failed at stress level x_i in time interval $[\tau_{i-1}, \tau_i))$. Let $y_{i,l}$ denote the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \ldots, n_i$ while c_i denotes the number of units censored at time τ_i. Furthermore, let N_i denote the number of units operating and remaining on test at the start of stress level x_i. That is, $N_{i+1} = N_i - n_i - c_i = n - \sum_{j=1}^{i} n_j - \sum_{j=1}^{i} c_j$. Then, the step-stress ALT under progressive Type-I censoring proceeds as follows. A total of $N_1 \equiv n$ test units is initially placed at stress level x_1 and tested until time τ_1 at which point c_1 live items are arbitrarily withdrawn from the test and the stress is changed to x_2. The test is continued on $N_2 = n - n_1 - c_1$ units until time τ_2, when c_2 items are withdrawn from the test and the stress is changed to x_3, and so on. Finally, at time τ_k, all the surviving items are withdrawn, thereby terminating the life test. Note that since $n \equiv \sum_{i=1}^{k}(n_i + c_i)$, the number of surviving items at time τ_k is $c_k = n - \sum_{i=1}^{k} n_i - \sum_{i=1}^{k-1} c_i = N_k - n_k$. Obviously, when there is no intermediate censoring ($viz.$, $c_1 = c_2 = \cdots = c_{k-1} = 0$), this situation corresponds to the k-level step-stress ALT under conventional Type-I right censoring as a special case. When there is no right censoring ($viz.$, $\tau_k = \infty$ and $n_k = N_k$), this situation corresponds to the k-level step-stress testing under complete sampling as a special case. It is also noted that under the continuous failure inspection, a step-stress ALT produces the observed values of $\boldsymbol{n} = (n_1, n_2, \ldots, n_k)$ and $\boldsymbol{y} = (\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_k})$ with $\boldsymbol{y_i} = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ while only $\boldsymbol{n}$ is observed under the interval (or group) inspection since the exact failure times are not available.

As noted in Balakrishnan and Han (2009), Balakrishnan et al. (2010), unlike progressive Type-II censoring scheme, prefixing the progressive Type-I censoring scheme $\boldsymbol{c} = (c_1, c_2, \ldots, c_{k-1})$ bears an inherent mathematical issue since there is a positive probability that all the units could fail before reaching the last stress level x_k, resulting in an early termination of the test as well as failing to fully implement $\boldsymbol{c}$. For this reason, Gouno et al. (2004) had to assume a large sample size, small global censoring proportions, and a small number of stress levels for an approximate/asymptotic analysis of progressively Type-I censored data so that the prefixed number of units could be successfully removed at the end of each interval. However, in a reliability test, the sample size is usually small and there might be severe censoring due to budgetary constraints and facility requirements. Under such conditions, the assumption of a large sample is violated and consequently, the progressive censoring scheme needs to be modified to assure its feasibility. One simple modification which can be entertained in practice is to first decide on a fixed number of units to be censored at the end of each stress level x_i, say $\boldsymbol{c^*} = (c_1^*, c_2^*, \ldots, c_{k-1}^*)$ where $c_i^* \geq 0$ and $\sum_{i=1}^{k-1} c_i^* < n$. Then, the actual number of units censored at the end of x_i is determined by $c_i = \min\left\{c_i^*, N_i - n_i\right\}$. If the number of remaining units at any censoring time point is less than or equal to the prefixed number of units to be censored at that point, all the remaining units are withdrawn and the life test is terminated. Hence, this modification allows the life test to terminate earlier than scheduled whenever there are insufficient live units remaining on the test. Since the number of surviving units at the end of each stress level before censoring takes place is random, the actual censoring scheme $\boldsymbol{c}$ is essentially random in this case. When $\boldsymbol{c^*} = (0, 0, \ldots, 0) = \boldsymbol{0}_{k-1}$, $\boldsymbol{c} = \boldsymbol{0}_{k-1}$ and it corresponds to the k-level step-stress ALT under conventional Type-I right censoring as a special case. Another practical modification is to decide on the fixed proportions of surviving units to be censored at the end of each stress level x_i, say $\boldsymbol{\pi^*} = (\pi_1^*, \pi_2^*, \ldots, \pi_{k-1}^*)$ where $0 \leq \pi_i^* < 1$. Since all the remaining units are withdrawn from the test at time τ_k, one could define $\pi_k^* = 1$. Then, the actual number of units censored at the end of x_i is determined by $c_i = \Upsilon((N_i - n_i)\pi_i^*)$ where $\Upsilon(\cdot)$ is a discretizing function of choice, mapping its argument to a non-negative integer. It could be $round(\cdot)$, $floor(\cdot)$, $ceiling(\cdot)$, or $trunc(\cdot)$, for example. This modification again allows the life test to terminate before reaching the last stress level x_k without any mathematical inconsistency. Since the number of surviving units at the end of each stress level before censoring takes place is random, the actual censoring scheme $\mathbf{c}$ is essentially random as well. When $\boldsymbol{\pi^*} = (0, 0, \ldots, 0) = \boldsymbol{0}_{k-1}$, $\boldsymbol{c} = \boldsymbol{0}_{k-1}$ and it corresponds to the k-level step-stress ALT under conventional Type-I right censoring as a special case.

In order to derive the distributional properties for the progressively Type-I censored step-stress ALT, here we assume that under any stress level x_i, the lifetime of a test unit follows an exponential distribution whose probability density function (PDF) and cumulative distribution function (CDF) are given by

$$f_i(t) = \frac{1}{\theta_i}\exp\left(-\frac{t}{\theta_i}\right), \qquad 0 < t < \infty, \tag{1}$$

$$F_i(t) = 1 - S_i(t) = 1 - \exp\left(-\frac{t}{\theta_i}\right), \qquad 0 < t < \infty, \tag{2}$$

respectively. The parameter θ_i is the mean time to failure (MTTF) of a test unit at stress level x_i. Since this is non-constant stress loading, an additional assumption is required to represent the effect of changing stress. The AFT model, also referred to as the additive accumulative damage model, is often appropriate as it generalizes several well-known models in reliability engineering for the exponential distribution, including the basic (linear) cumulative exposure model and the PH model. Under the AFT model along with (1) and (2), the PDF and CDF of a test unit for the step-stress ALT are

$$f(t) = \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] f_i(t-\tau_{i-1}) \qquad \text{if } \begin{cases} \tau_{i-1} \le t \le \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases} \tag{3}$$

$$F(t) = 1 - \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] S_i(t-\tau_{i-1}) \qquad \text{if } \begin{cases} \tau_{i-1} \le t \le \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases} \tag{4}$$

where $\tau_0 \equiv 0$ and $\Delta_j = \tau_j - \tau_{j-1}$ is the step duration at stress level x_j. Of course, $f_i(t)$ and $F_i(t)$ are as given in (1) and (2), respectively. It is worth mentioning that under the assumption of exponentiality, the AFT model coincides with the cumulative exposure model, which produces the conditional block independence of the ordered failure time data as discussed by Iliopoulos and Balakrishnan (2009), Balakrishnan and Cramer (2014). This is a critical property for deriving the expected termination time of a k-level step-stress ALT under progressive Type-I censoring as shown in the following sections.

3. Expected Termination Time under Continuous Inspection

For both continuous and interval inspections, let T_i denote the duration (or the time until termination) of a progressively Type-I censored step-stress ALT starting from the i-th stage ($viz.$, stress level x_i) with N_i surviving units. It can be shown that for $i = 1, 2, \ldots, k$, a general k-level progressively Type-I censored step-stress ALT, starting from the i-th stage forms a sub- $(k-i+1)$-level progressively Type-I censored step-stress ALT with the sample size N_i, the ordered stress levels $x_i < x_{i+1} < \cdots < x_k$, and the sub-censoring scheme $\boldsymbol{c_i} = (c_i, c_{i+1}, \ldots, c_{k-1})$. Also, given N_i, the lifetime of a remaining test unit follows a left-truncated distribution at τ_{i-1}. Hence, the termination time T_1 of a progressively Type-I censored step-stress ALT can be expressed recursively using the duration of a sub-step-stress ALT. Under the continuous inspection in particular, for $i = 1, 2, \ldots, k$, T_i can be defined recursively as

$$T_i = \begin{cases} \min\{Y_{i,N_i} - \tau_{i-1}, \Delta_i\} + T_{i+1} & \text{if } N_i > 0; \\ 0 & \text{if } N_i = 0 \text{ or } i = k+1 \end{cases},$$

where Y_{i,N_i} is the largest order statistic (failure time) from a sample of size N_i starting at time τ_{i-1}. Then, the conditional mean of T_i given N_i is expressed as

$$E[T_i|N_i] = \begin{cases} E[\min\{Y_{i,N_i} - \tau_{i-1}, \Delta_i\}|N_i] + E[T_{i+1}|N_i] & \text{if } N_i > 0; \\ 0 & \text{if } N_i = 0 \text{ or } i = k+1 \end{cases}. \tag{5}$$

In order to derive more explicit formula of (5), let $\boldsymbol{c}^* = (c_1^*, c_2^*, \ldots, c_{k-1}^*)$ denote the vector of a prefixed number of units to be censored at the end of each stress level x_i. The actual number of units censored at the end of x_i is then determined by $c_i = \min\left\{c_i^*, N_i - n_i\right\}$, which guarantees a feasible progressive Type-I censoring as discussed in Section 2. When $N_i > 0$, it is derived that

$$\begin{aligned} E[T_i|N_i] &= E[\min\{Y_{i,N_i} - \tau_{i-1}, \Delta_i\}|N_i] + E[T_{i+1}|N_i] \\ &= E[\min\{Y_{i,N_i} - \tau_{i-1}, \Delta_i\}|N_i] + E_{N_{i+1}}\left[E_{T_{i+1}}[T_{i+1}|N_i, N_{i+1}]\middle|N_i\right] \\ &= E[Y_{i,N_i} - \tau_{i-1}|N_i, Y_{i,N_i} < \tau_i]P(Y_{i,N_i} < \tau_i|N_i) + \Delta_i P(Y_{i,N_i} \geq \tau_i|N_i) \\ &\qquad + \sum_{N_{i+1}=0}^{\max\{0, N_i - c_i^*\}} E[T_{i+1}|N_i, N_{i+1}]P(N_{i+1}|N_i) \\ &= E[Y_{i,N_i} - \tau_{i-1}|N_i, Y_{i,N_i} < \tau_i]P(Y_{i,N_i} < \tau_i|N_i) \\ &\quad + \left[\Delta_i + \sum_{N_{i+1}=1}^{\max\{0, N_i - c_i^*\}} E[T_{i+1}|N_i, N_{i+1}]P(N_{i+1}|N_i, Y_{i,N_i} \geq \tau_i)\right] \\ &\quad \times P(Y_{i,N_i} \geq \tau_i|N_i), \end{aligned}$$

where the last equality is due to the fact that $E[T_{i+1}|N_i, N_{i+1}] = 0$ when $N_{i+1} = 0$. Also, given $N_i > 0$, $N_{i+1} > 0$ implies $Y_{i,N_i} > \tau_i$ since continuation of the step-stress ALT at the $(i+1)$-th stage with a positive number of surviving units indicates that the test has successfully gone through all the preceding stress levels $x_1, x_2, \ldots, x_i$ with a positive number of functioning units. Thus, $N_{i+1} > 0$ suggests that the largest lifetime out of N_i surviving units in the beginning of the i-th stage must have passed τ_i. This also means that there were enough surviving units to be censored at each preceding stress change time point, and the desired number of units $c_1^*, c_2^*, \ldots, c_i^*$ could be successfully withdrawn.

Now, using the Markovian property, $E[T_{i+1}|N_i, N_{i+1}] = E[T_{i+1}|N_{i+1}]$ and hence, for $i = 1, 2, \ldots, k-1$, (5) becomes

$$E[T_i|N_i] = \begin{cases} E[Y_{i,N_i} - \tau_{i-1}|N_i, Y_{i,N_i} < \tau_i]P(Y_{i,N_i} < \tau_i|N_i) \; + \; \Delta_i P(Y_{i,N_i} \geq \tau_i|N_i) & \\ \quad + \displaystyle\sum_{N_{i+1}=1}^{\max\{0, N_i - c_i^*\}} E[T_{i+1}|N_{i+1}]P(N_{i+1}|N_i) & \text{if } N_i > 0; \\ 0 & \text{if } N_i = 0 \end{cases} \tag{6}$$

with the stopping condition specified by

$$E[T_k|N_k] = \begin{cases} E[Y_{k,N_k} - \tau_{k-1}|N_k, Y_{k,N_k} < \tau_k]P(Y_{k,N_k} < \tau_k|N_k) + \Delta_k P(Y_{k,N_k} \geq \tau_k|N_k) & \\ \qquad\qquad\qquad\qquad \text{if } N_k > 0; & \\ 0 \qquad\qquad\qquad\qquad\quad \text{if } N_k = 0 & \end{cases}.$$

It can be shown that given N_i, Y_{i,N_i} is conditionally distributed as the largest order statistic from a random sample of size N_i from a left-truncated distribution at τ_{i-1}. Using (1)-(4), this truncated PDF is expressed as $f_{i;trL}(t) = f(t)/\left[1 - F(\tau_{i-1})\right] = f_i(t - \tau_{i-1})$ for $t \geq \tau_{i-1}$, $i = 1, 2, \ldots, k$. It is then easy to see that given N_i, $(Y_{i,N_i} - \tau_{i-1})$ is conditionally distributed as the largest order statistic from a random sample of size N_i from the exponential distribution whose PDF and CDF are as given in (1) and (2). Therefore, $P(Y_{i,N_i} < \tau_i|N_i) = P(Y_{i,N_i} - \tau_{i-1} < \Delta_i|N_i) = [F_i(\Delta_i)]^{N_i}$ in (6) using the property of the order statistics.

Moreover, given N_i and $Y_{i,N_i} < \tau_i$, Y_{i,N_i} is conditionally distributed as the largest order statistic from a random sample of size N_i from a left- and right-truncated distribution at τ_{i-1} and τ_i, respectively. Using (1)-(4), this doubly truncated PDF is given by $f_{i;trLR}(t) = f(t)/\big[F(\tau_i) - F(\tau_{i-1})\big] = f_i(t - \tau_{i-1})/F_i(\Delta_i)$ for $\tau_{i-1} \le t \le \tau_i$, $i = 1, 2, \ldots, k$. It is then obvious that given N_i and $Y_{i,N_i} < \tau_i$, $(Y_{i,N_i} - \tau_{i-1})$ is conditionally distributed as the largest order statistic from a random sample of size N_i from a right-truncated distribution at Δ_i, whose PDF is given as $f_{i;trR}(t) = f_i(t)/F_i(\Delta_i)$ for $0 \le t \le \Delta_i$, $i = 1, 2, \ldots, k$. Therefore, the first conditional expectation in (6) is derived as

$$\begin{aligned} E[Y_{i,N_i} - \tau_{i-1}|N_i, Y_{i,N_i} < \tau_i] &= N_i \int_0^{\Delta_i} t \left[\frac{F_i(t)}{F_i(\Delta_i)}\right]^{N_i-1} \left[\frac{f_i(t)}{F_i(\Delta_i)}\right] dt \\ &= \frac{N_i}{[F_i(\Delta_i)]^{N_i}} \sum_{l=0}^{N_i-1} \binom{N_i-1}{l} (-1)^l \int_0^{\Delta_i} t[S_i(t)]^l f_i(t)\, dt \\ &= \frac{\theta_i}{[F_i(\Delta_i)]^{N_i}} \sum_{l=1}^{N_i} \binom{N_i}{l} (-1)^l \left[\frac{\Delta_i}{\theta_i} S_i(l\Delta_i) - \frac{1}{l} F_i(l\Delta_i)\right]. \end{aligned}$$

Analyzing $c_i = \min\{c_i^*, N_i - n_i\}$, the conditional probability mass function (PMF) of N_{i+1} given N_i is also obtained as

$$\begin{aligned} f_{N_{i+1}|N_i}(m|N_i) &= P(N_{i+1} = m|N_i) = P(N_i - n_i - c_i = m|N_i) \\ &= \begin{cases} P(n_i \ge N_i - c_i^*|N_i), & m = 0; \\ P(n_i = N_i - m - c_i^*|N_i), & m = 1, 2, \ldots, \max\{0, N_i - c_i^*\} \end{cases} \\ &= \begin{cases} \displaystyle\sum_{n_i=\max\{0,N_i-c_i^*\}}^{N_i} \binom{N_i}{n_i} [F_i(\Delta_i)]^{n_i} [S_i(\Delta_i)]^{N_i-n_i}, & \\ & m = 0; \\ \displaystyle\binom{N_i}{N_i - m - c_i^*} [F_i(\Delta_i)]^{N_i-m-c_i^*} [S_i(\Delta_i)]^{m+c_i^*}, & \\ & m = 1, 2, \ldots, \max\{0, N_i - c_i^*\} \end{cases} \end{aligned}$$

since given N_i, n_i follows a binomial distribution with parameters N_i and $\dfrac{F(\tau_i) - F(\tau_{i-1})}{1 - F(\tau_{i-1})} = F_i(\Delta_i)$.

Upon utilizing these distributional results, the conditional expectation in (6) is simplified as

$$E[T_i|N_i] = \begin{cases} \displaystyle\theta_i \sum_{l=1}^{N_i} \binom{N_i}{l} \frac{(-1)^{l+1}}{l} F_i(l\Delta_i) + [F_i(\Delta_i)]^{N_i} & \\ \displaystyle\quad \times \sum_{N_{i+1}=1}^{\max\{0,N_i-c_i^*\}} E[T_{i+1}|N_{i+1}] \binom{N_i}{N_{i+1} + c_i^*} \left[\frac{S_i(\Delta_i)}{F_i(\Delta_i)}\right]^{N_{i+1}+c_i^*} & \text{if } N_i > 0; \\ 0 & \text{if } N_i = 0 \end{cases} \tag{7}$$

for $i = 1, 2, \ldots, k-1$, with the stopping condition specified by

$$E[T_k|N_k] = \begin{cases} \theta_k \sum_{l=1}^{N_k} \binom{N_k}{l} \frac{(-1)^{l+1}}{l} F_k(l\Delta_k) & \text{if } N_k > 0; \\ 0 & \text{if } N_k = 0 \end{cases}.$$

Using the recursive relation in (7), the expected termination time of a progressively Type-I censored step-stress ALT can be computed as $E[T_1] = E[T_1|N_1 = n]$ under the continuous inspection mode. As a special case when $\boldsymbol{c}^* = \mathbf{0}_{k-1}$, the expected termination time of a general k-level step-stress ALT under Type-I censoring can be derived explicitly as

$$E[T_1] = \sum_{i=1}^{k} \theta_i \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l+1}}{l} \left[\prod_{j=1}^{i-1} S_j(\Delta_j) \right]^l F_i(l\Delta_i)$$

using the induction on (7). The above formula is still valid under complete sampling by letting $\tau_k \to \infty$ or equivalently, $\Delta_k \to \infty$. As a corollary, in the case of a simple step-stress ALT ($viz.$, $k = 2$) under Type-I censoring, the expected termination time of test is

$$E[T_1] = \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l+1}}{l} \big[\theta_1 F_1(l\Delta_1) + \theta_2 S_1(l\Delta_1) F_2(l\Delta_2) \big]$$

while the expected termination time of a simple step-stress ALT under complete sampling is

$$E[T_1] = \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l+1}}{l} \big[\theta_1 F_1(l\Delta_1) + \theta_2 S_1(l\Delta_1) \big].$$

4. Expected Termination Time under Interval Inspection

Similar to the progressively Type-I censored step-stress ALT under the continuous inspection, the termination time T_1 of a progressively Type-I censored step-stress ALT under the interval inspection can be expressed recursively using the duration of a sub-step-stress ALT. Assuming that the failure inspection occurs at the end of each stress level prior to censoring (at each stress change time point), T_i can be defined recursively as

$$T_i = \begin{cases} \Delta_i + T_{i+1} & \text{if } N_i > 0; \\ 0 & \text{if } N_i = 0 \text{ or } i = k+1 \end{cases}$$

for $i = 1, 2, \ldots, k$. Then, the conditional mean of T_i given N_i is expressed as

$$E[T_i|N_i] = \begin{cases} \Delta_i + E[T_{i+1}|N_i] & \text{if } N_i > 0; \\ 0 & \text{if } N_i = 0 \text{ or } i = k+1 \end{cases}. \quad (8)$$

Again, to derive more explicit formula of (8), let $c_i = \min\{c_i^*, N_i - n_i\}$ be the actual number of units censored at the end of x_i, where $\boldsymbol{c}^* = (c_1^*, c_2^*, \ldots, c_{k-1}^*)$ is the vector of a prefixed number of units to be censored. When $N_i > 0$, it is derived that

$$\begin{aligned} E[T_i|N_i] &= \Delta_i + E[T_{i+1}|N_i] = \Delta_i + E_{N_{i+1}}\big[E_{T_{i+1}}[T_{i+1}|N_i, N_{i+1}]\big|N_i\big] \\ &= \Delta_i + \sum_{N_{i+1}=0}^{\max\{0, N_i - c_i^*\}} E[T_{i+1}|N_i, N_{i+1}] P(N_{i+1}|N_i) \\ &= \Delta_i + \sum_{N_{i+1}=1}^{\max\{0, N_i - c_i^*\}} E[T_{i+1}|N_{i+1}] P(N_{i+1}|N_i), \end{aligned}$$

where the last equality is due to the fact that $E[T_{i+1}|N_i, N_{i+1}] = 0$ when $N_{i+1} = 0$, and the Markovian property stating $E[T_{i+1}|N_i, N_{i+1}] = E[T_{i+1}|N_{i+1}]$. Applying the conditional PMF of N_{i+1} given N_i provided in Section 3, the conditional expectation in (8) can be expressed as

$$E[T_i|N_i] = \begin{cases} \Delta_i + [F_i(\Delta_i)]^{N_i} \displaystyle\sum_{N_{i+1}=1}^{\max\{0, N_i - c_i^*\}} E[T_{i+1}|N_{i+1}] \binom{N_i}{N_{i+1} + c_i^*} \left[\frac{S_i(\Delta_i)}{F_i(\Delta_i)}\right]^{N_{i+1}+c_i^*} & \text{if } N_i > 0; \\ 0 & \text{if } N_i = 0 \end{cases} \tag{9}$$

for $i = 1, 2, \ldots, k-1$, with the stopping condition specified by

$$E[T_k|N_k] = \begin{cases} \Delta_k & \text{if } N_k > 0; \\ 0 & \text{if } N_k = 0 \end{cases}.$$

Using the recursive relation in (9), the expected termination time of a progressively Type-I censored step-stress ALT can be computed as $E[T_1] = E[T_1|N_1 = n]$ under the interval inspection mode.

For the interval inspection mode, there is another way to derive the expected termination time of a step-stress ALT. It is via the conditional mean of T_i given $N_i > 0$, which is expressed as

$$\begin{aligned} E[T_i|N_i > 0] &= \Delta_i + E[T_{i+1}|N_i > 0] \\ &= \Delta_i + E[T_{i+1}|N_i > 0, N_{i+1} = 0]P(N_{i+1} = 0|N_i > 0) \\ &\quad + E[T_{i+1}|N_i > 0, N_{i+1} > 0]P(N_{i+1} > 0|N_i > 0) \\ &= \Delta_i + E[T_{i+1}|N_{i+1} > 0]P(N_{i+1} > 0|N_i > 0) \\ &= \Delta_i + E[T_{i+1}|N_{i+1} > 0]\frac{P(N_{i+1} > 0)}{P(N_i > 0)} \end{aligned}$$

for $i = 1, 2, \ldots, k-1$, with the stopping condition $E[T_k|N_k > 0] = \Delta_k$. The third equality is due to the fact that $T_{i+1} = 0$ when $N_{i+1} = 0$. Also, $N_{i+1} > 0$ implies $Y_{i,N_i} > \Delta_i$, which implies $N_i > 0$ in turn, since continuation of the step-stress ALT at the $(i+1)$-th stage with a positive number of surviving units indicates that the test has successfully gone through all the preceding stress levels $x_1, x_2, \ldots, x_i$ with a positive number of functioning units. Since $P(N_1 \equiv n > 0) = 1$, using the above recursive relation sequentially, one can obtain that

$$E[T_1] = E[T_1|N_1 > 0] = \sum_{i=1}^{k} \Delta_i P(N_i > 0) = \Delta_1 + \sum_{i=2}^{k} \Delta_i P(N_i > 0). \tag{10}$$

In order to derive more concrete expression of (10), it is necessary to formulate the probability of a step-stress ALT proceeding to stress level x_i or $P(N_i > 0)$. For this purpose, let $\boldsymbol{\pi}^* = (\pi_1^*, \pi_2^*, \ldots, \pi_{k-1}^*)$ denote the vector of a prefixed proportion of surviving units to be censored at the end of each stress level x_i. The actual number of units censored at the end of x_i is then determined by $c_i = \Upsilon((N_i - n_i)\pi_i^*)$ with a discretizing function of choice $\Upsilon(\cdot)$. This definition of c_i nevertheless complicates the derivation of the distributional characteristics of the associated random quantities. For simplicity, $c_i = (N_i - n_i)\pi_i^*$ is assumed for $i = 1, 2, \ldots, k-1$ as $\Upsilon((N_i - n_i)\pi_i^*) \approx (N_i - n_i)\pi_i^*$. Under this setup, the following lemma specifies the recursive nature of the conditional probability of N_i.

Lemma 1. *For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, i-1$, we have*

$$P(N_i = 0 | n_1, n_2, \ldots, n_{j-1}) = \big[H_j^{(i)}\big]^{N_j},$$

where

$$H_j^{(i)} = \begin{cases} F_j(\Delta_j) + S_j(\Delta_j)\big[H_{j+1}^{(i)}\big]^{1-\pi_j^*} & \text{for } j = 1, 2, \ldots, i-1 \\ 0 & \text{for } j = i \end{cases}.$$

It is also observed that the following property holds for the recursive function $H_j^{(i)}$ defined above. That is, it is always bounded between 0 and 1.

Corollary 1. *For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, i$, we have $0 \leq H_j^{(i)} < 1$.*

Now, it follows naturally that

$$P(N_i > 0) = 1 - P(N_i = 0) = 1 - \big[H_1^{(i)}\big]^{N_1} = 1 - \big[H_1^{(i)}\big]^{n}.$$

Upon applying this result to (10), the expected termination time of a progressively Type-I censored step-stress ALT under the interval inspection mode is obtained as

$$E[T_1] = \tau_k - \sum_{i=2}^{k} \Delta_i \big[H_1^{(i)}\big]^{n}. \tag{11}$$

As a special case when $\boldsymbol{\pi}^* = \mathbf{0}_{k-1}$, $H_j^{(i)}$ can be explicitly expressed as

$$H_j^{(i)} = \sum_{l=j}^{i-1} F_l(\Delta_l) \prod_{l'=j}^{l-1} S_{l'}(\Delta_{l'}) = 1 - \prod_{l=j}^{i-1} S_l(\Delta_l) = \frac{F(\tau_{i-1}) - F(\tau_{j-1})}{1 - F(\tau_{j-1})}$$

by sequentially expanding its recursive relation. Consequently, $P(N_i > 0) = 1 - [F(\tau_{i-1})]^n$ in this case, and thus, the expected termination time of a general k-level step-stress ALT under Type-I censoring is explicitly obtained as

$$E[T_1] = \tau_k - \sum_{i=2}^{k} \Delta_i [F(\tau_{i-1})]^n.$$

As a corollary, in the case of a simple step-stress ALT ($viz.$, $k = 2$) under Type-I censoring, the expected termination time of test is simply

$$E[T_1] = \tau_2 - \Delta_2 [F_1(\Delta_1)]^n.$$

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