Empirical Testing of an Option Pricing Model with Memory

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Abstract
We discuss the preliminary testing of a continuous option pricing model with memory and intrinsic stochastic volatility. The stock dynamics follows a nonlinear stochastic functional differential equation with a closed-form solution and the option pricing formula is a conditional expectation that can be simulated via Monte Carlo methods. We tested the model for the S&P500 index during two time periods: during and after the 2008-2009 financial crisis. The model's performance was compared to the Black-Scholes model for different memory lengths, contract expiration times, and moneyness. We found that the option pricing model with memory was more accurate than Black-Scholes during the crisis, while the opposite was true in the post-crisis period.

Key Words: Option pricing, empirical testing, Black-Scholes, stochastic volatility, memory, stochastic functional differential equation.

1. Introduction

Since the seminal works of Black, Scholes, and Merton (1973), a significant number of option pricing (OP) models have been proposed. An assumption of the Black-Scholes (BS) model that is frequently challenged is that stock prices follow a Geometric Brownian Motion with constant volatility (e.g., Rubinstein 1994, and Scott 1987). More specifically, tests of the BS model on real market data have revealed biases on implied volatility (Bates, 1996), suggesting that volatility should not be assumed to be constant.

For this reason, several variants of the BS model with non-constant volatility have been proposed (e.g., Geske 1979, Cox & Ross 1975, Hobson & Rogers 1998). Furthermore, several authors have developed models with memory (e.g., Arriojas, Hu, Mohammed & Pap 2007, Stoica 2004, Kazmerchuk 2007, and Chang 2007). This is a reasonable consideration since decision makers take into account their knowledge of the past market behavior when selling or purchasing assets.

Our goal in the present work is to discuss the testing of the OP model with memory developed by Sancier and Mohammed (2017) against market data. This model has a hereditary structure in which the stock price follows a stochastic functional differential equation. We used data from the S&P500 index to test this model. Even though the BS model has consistently performed well on the S&P500 index (Dumas, Fleming, & Whaley, 1998), we wanted to test whether the OP model with memory would perform better during the crisis period.

The paper is outlined as follows. In section 2, we summarize the OP model with memory to be tested in this study and, in section 3, we describe the steps taken to test it. Section 4 provides a summary of our observations and a discussion about further investigations.

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2. Option pricing model with memory

Let $T > 0$ be the expiration time of an option. The option pricing model with memory proposed by Sancier and Mohammed (2017) assumes that stock prices $(S(t))_{t\in[0,T]}$ satisfy the stochastic functional differential equation (SFDE)

$$\begin{cases} dS(t) = f(t, S_t)S(t)dt + g(t, S_t)S(t)dW(t), & t \in [0, T] \\ S(t) = \theta(t), & t \in [-L, 0], \end{cases} \tag{1}$$

on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions. The initial process $\theta \in L^2(\Omega, C)$ is $\mathcal{F}_0$-measurable. The value $L > 0$ is the memory length. The process W is a 1-dimensional Brownian Motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, and the memory segment S_t is given by $S_t(s) := S(t + s)$, $s \in [-L, 0]$, for any $t \in [0, T]$. The functionals $f : [0, T] \times L^2(\Omega, C) \to \mathbb{R}$ and $g : [0, T] \times L^2(\Omega, C) \to \mathbb{R}$ are jointly continuous, globally bounded, and uniformly Lipschitz in the second variable, viz.

$$\begin{aligned} &|f(t, \psi)| \le f_{max} \quad \text{and} \quad |g(t, \psi)| \le g_{max} \quad \text{and} \\ &|f(t, \psi_1) - f(t, \psi_2)| + |g(t, \psi_1) - g(t, \psi_2)| \le \alpha\|\psi_1 - \psi_2\|_{L^2(\Omega,C)} \end{aligned} \tag{2}$$

for all $t \in [0, T]$ and ψ, $\psi_1, \psi_2 \in L^2(\Omega, C)$. The Lipschitz constant α is independent of $t \in [0, T]$. The following two theorems give the solution of the stock price model (1) and its fair option price as a conditional expectation.

Theorem 1 *The SDFE (1) has a unique solution, which is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $S \in L^2(\Omega, C([-L, T], \mathbb{R}))$, starting off at θ, given by*

$$\begin{aligned} S(t) &= \theta(0) \exp\left\{\int_0^t f(u, S_u)du + \int_0^t g(u, S_u)dW(u)\right. \\ &- \left.\frac{1}{2}\int_0^t g(u, S_u)^2 du\right\}, \quad t \in [0, T]. \end{aligned} \tag{3}$$

Theorem 2 *Let $\{B, S\}$ be a market (e.g. a bond and a stock) such that for fixed $r \ge 0$, $B(t) = e^{rt}$, $t \in [0, T]$, and such that S is described by the SFDE (1) with $\theta(t) > 0$ for all $t \in [-L, 0]$ a.s.. Let $V(t)$ be the fair price at time t of a European call option written on the stock S with exercise price K and maturity time T. Then*

$$V(t) = e^{-r(T-t)}E_Q[(S(t) - K)^+|\mathcal{F}_t^S], \quad t \in [0, T],$$

where Q is defined by $dQ = \rho(T)dP$ with $\rho(T)$ given by

$$\rho(t) := \exp\left\{-\int_0^T \frac{\{f(t, S_t) - r\}}{g(t, S_t)}dW(u) - \frac{1}{2}\int_0^T \left(\frac{\{f(t, S_t) - r\}}{g(t, S_t)}\right)^2 du\right\}.$$

That is, the model was intrinsic stochastic volatility which is calculated via past stock prices. Note that if the functionals f and g are set to be constants, then the stock price model becomes a standard Geometric Brownian Motion.

3. Methods

We obtained daily S&P500 options data from the Chicago Board Options Exchange (CBOE) for the period that ranged from January 7, 2008 to April 26, 2010. This range included a crisis period, which we considered to be from 1-7-2008 to 6-1-2009, and a post-crisis period, which we considered to be from 6-1-2009 to 4-26-2010. We selected 20 equally spaced trading days within each period and used the data from (European-style) call options that were purchased on those dates.

We evaluated the OP model's performance for 4 different choices of memory length: $L = 10, 20, 30$, and 60 trading days. We used the same length of trading days to calculate the Black-Scholes model parameters. For example, when using $L = 10$ in the OP model with memory, we estimated the BS constant drift and volatility (both historical) by using the 10 previous trading days. The option's time to expiration was taken to be the number of trading days between the option's purchase date and the expiration date. For the OP model with memory, we used a constant drift (the same one used in the BS model) and a volatility functional that behaved as a moving standard deviation of returns.

We calculated option prices for all feasible strike prices and expiration times (the corresponding ask and bid prices had to be greater than zero in order for a contract configuration to be considered). Fair option prices for the BS model were calculated using the BS formula, while fair prices for the OP model with memory were calculated via Monte Carlo Simulations. Each option price calculation was done via 10000 simulations.

To test the accuracy of the OP model with memory and to compare it with the BS model, we used two measures: the mean absolute error (MAE) and the mean relative absolute error (MRAE). The mean absolute error is an average of absolute values of errors and the mean relative absolute error is an average of relative absolute errors that use the BS model as benchmark. To better assess the OP models' performances in different scenarios, the averages were done over different moneyness groups (that varied in 5% intervals) and expiration groups (see tables 1 and 2). We defined moneyness to be the ratio $(K - S(0))/S(0)$ and expressed it as a percentage.

4. Results and Discussion

Tables 1 and 2 show a shortened version of the option configurations considered. We calculated a total of 128 MRAEs and the table shows 30 of them. The OP model with memory performed better than the BS model in 73 out of the 128 scenarios considered during the crisis period. In the time interval following the crisis, this number was 16 out of 128.

This shows that in the crisis time period, the Option Pricing Model with Memory had a better performance compared to the Black-Scholes model. This observation is consistent with well-known observations that the BS model has subpar performance during periods in which market volatility changes rapidly. On the other hand, the Black-Scholes model performed better than the OP model with memory during the post-crisis period, which is consistent with its well-known good performance during times of relative financial stability.

A future direction in testing the OP model with memory is to consider other indexes and stocks, as well as different choices of volatility functional. The volatility function used in this study was a relatively simple one, so we hope to explore more complex ones, as well as non-constant drift functionals.

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Purchase Range: 1/7/08-6/1/09								
Moneyness (%) Lower	Moneyness (%) Upper	Model	0 to 40 MRAE	0 to 40 MAE	40 to 80 MRAE	40 to 80 MAE	80 to 254 MRAE	80 to 254 MAE
-10	-5	BS M=20	1	6.2758	1	9.8400	1	13.9793
		BS M=60	1	2.6103	1	6.5962	1	14.1982
		OP M=20	0.7601	4.7707	0.6859	6.6943	0.5812	7.8990
		OP M=60	0.7669	2.0126	0.7710	4.9386	0.8797	12.7916
-5	0	BS M=20	1	5.4890	1	8.0964	1	11.8335
		BS M=60	1	6.4682	1	11.2808	1	20.3230
		OP M=20	0.7503	4.1009	0.6996	5.6206	0.5527	6.4175
		OP M=60	0.8226	5.1328	0.7458	8.3498	0.8513	17.4824
0	5	BS M=10	1	1.3540	1	2.6190	1	4.4050
		BS M=60	1	9.5128	1	14.8813	1	24.0680
		OP M=10	0.9462	1.3481	1.0099	2.6600	0.9265	3.9022
		OP M=60	0.8367	7.8159	0.7568	11.2061	0.8374	20.2597
5	10	BS M=20	1	2.6001	1	4.8610	1	8.0787
		BS M=60	1	10.1308	1	16.0785	1	27.8647
		OP M=20	0.6223	1.5634	0.6238	3.0350	0.4695	3.6976
		OP M=60	0.8151	8.2528	0.7682	12.2906	0.8352	23.4591
10	20	BS M=20	1	0.6192	1	1.6601	1	4.8610
		BS M=60	1	5.8096	1	13.9643	1	27.9170
		OP M=20	5.9527	0.6437	13.7102	1.9200	11.4747	2.2883
		OP M=60	0.7955	4.5667	0.7513	10.4352	0.8194	22.9913

Purchase Range: 6/1/09-4/26/10								
Moneyness (%) Lower	Moneyness (%) Upper	Model	0 to 40 MRAE	0 to 40 MAE	40 to 80 MRAE	40 to 80 MAE	80 to 254 MRAE	80 to 254 MAE
-10	-5	BS M=20	1	2.2368	1	17.4616	1	30.4603
		BS M=60	1	1.7659	1	13.6657	1	22.0240
		OP M=20	1.6041	2.7214	1.0607	18.5525	1.1435	34.9167
		OP M=60	1.5333	1.4979	1.0723	14.5946	1.1866	26.2939
-5	0	BS M=20	1	4.2945	1	15.6557	1	26.1604
		BS M=60	1	2.0091	1	10.1409	1	16.5409
		OP M=20	1.2079	5.3296	1.1074	17.3357	1.1817	30.9267
		OP M=60	1.2919	1.7191	1.0653	10.7615	1.2089	20.1245
0	5	BS M=20	1	2.1426	1	9.0523	1	18.7012
		BS M=60	1	1.5747	1	3.4155	1	8.4937
		OP M=20	1.5083	2.9171	1.1426	10.3254	1.2471	23.2124
		OP M=60	1.5648	1.8130	1.1861	3.5615	1.3201	11.2181
5	10	BS M=20	1	0.3816	1	2.1666	1	11.1882
		BS M=60	1	1.4059	1	2.4770	1	2.3899
		OP M=20	1.7044	0.4635	1.1807	2.5669	1.3525	14.8814
		OP M=60	1.3728	1.8603	1.2619	2.6880	4.0805	4.0388
10	20	BS M=20	1	0.3473	1	1.4131	1	2.8119
		BS M=60	1	0.7325	1	3.7872	1	6.0831
		OP M=20	0.7363	0.3536	1.3056	1.8613	2.7439	4.6045
		OP M=60	1.5778	0.9601	1.1252	4.2438	0.7116	4.6477

Figure 1: Measures of fit (MRAEs and MAEs) for the BS model and the OP model with memory during and after the 2008-2009 crisis period.

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