A Process Control Model with Decisions Based on Runs

William S. Griffith[1], Michelle L. DePoy Smith[2]
[1] University of Kentucky, Lexington, KY 40506
[2] Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475

Abstract
A model of process control is studied in which items might be incorrectly classified. Because of this possibility the items selected are classified multiple times before a final judgment is reached as to whether they conform to specifications. This final judgment is based on runs of conforming and nonconforming classifications of the item which in turn determines whether the process is declared to be out of control. Steady state as well as short term properties are studied.

Key Words: quality control, errors, runs

1. Introduction

In papers by Taguchi, Elsayed, and Hsiang(1989) and Taguchi, Chowdhury, and Wu (2004) a model of on-line process control by attributes is studied. Every h^{th} item produced is inspected. Initially, the process is assumed to be in control and that the fraction of items conforming to specifications denoted by p_1 is close to 1. When the process goes out of control there is a shift to p_2 ($< p_1$) for the fraction conforming. If an inspected item is judged nonconforming, the process is considered to be (possibly) out of control and is stopped and there is a search for an assignable cause.

There are several papers that have studied models along this line using a variety of assumptions. In Nayebpour and Woodall (1993) the random time until the shift from p_1 to p_2 is assumed to follow a geometric distribution. Items produced are assumed to be independent and identically distributed trials with a constant probability of π for each item to be the first item produced with the new shifted (smaller) fraction conforming in effect. Since only every h^{th} item is inspected, the first item produced under this shifted fraction conforming value might not be the one chosen to be inspected and thus it is possible that there are some number of items produced before it is possible to detect the shift.

Borges, Ho, and Turnes (2001) point out that the inspection process itself can be subject to diagnostic errors and that a classification can result in a conforming item possibly being misclassified as nonconforming. We let p_{CN} be the probability of this misclassification. In addition, a nonconforming item can be classified as conforming and let p_{NC} be the probability of this misclassification. We also define p_{CC} (p_{NN}) to be the probability of correct classification, i.e. the probability that a conforming (nonconforming) item is classified as conforming (nonconforming). This gives rise to the idea of making repeated classifications of each inspected item prior to making the final judgment of whether the item is conforming or nonconforming. When the item has been judged in this final determination to be nonconforming, the process is judged out of control and is stopped for a search for an assignable cause, and if one is found an adjustment is made to put the process back in control. Since there is the possibility of diagnostics errors in the repeated classifications process and since even when the process is in control the item selected might truly be nonconforming, there is a possibility that an item is judged to be nonconforming and the process is judged out of control, even though the process is actually not. Nonetheless, the process is paused for a search for an assignable cause and when one is not found the process is then restarted and it is assumed that the process has not somehow been put out of control by the stopping and searching for a cause. It is also possible that the process goes out of control, but the next inspected item is judged conforming and the process is not judged to be out of control at that time. The process will eventually be judged out of control at some later time when an inspected item is finally judged nonconforming. After the process has gone out of control, it is assumed that the process cannot put itself back in control. Thus, once out of control it stays out of control until finally an inspected item at some later time is judged nonconforming. At that time, a search is launched, an assignable cause is found and corrected, at which time the process goes back into control and the model starts anew.

In Trindade, Ho, and Quinino (2007), the final determination of whether the inspected item is conforming, and thus whether the process is in control, was based on a pre-specified number of repeated classifications using majority rule. In Quinino, Colin, and Ho (2009), an item was judged to be conforming and the process to be in control if and only if there were k classifications as conforming before f classifications as nonconforming, where k and f are some pre-specified positive integers. We will use the acronym TCTN because the decision is based on the total number of classifications as conforming and nonconforming. Smith and Griffith (2009, 2017) further studied this rule and another rule called CCTN.

In this paper, we continue the study of the alternative rule CCCN in which the final determination that an item is conforming, and thus the process is in control, if and only if a run of k consecutive classifications as conforming occur before a run of f consecutive classifications as nonconforming.

2. Probabilistic Analysis

Proposition 1: If the item being inspected is conforming (nonconforming), the probability that it is judged to be conforming is

$$P(judged\ conforming|\ actually\ conforming)$$
$$= CCCN(p_{CC})\frac{p_{CC}^{k-1}[1-(1-p_{CC})^{f}]}{1-(1-p_{CC}^{k-1})[1-(1-p_{CC})^{f-1}]}$$

$$P(judged\ conforming|\ actually\ nonconforming) = CCCN(p_{NC})$$

$$= \frac{p_{NC}^{k-1}\left[1-\left(1-p_{NC}\right)^{f}\right]}{1-\left(1-p_{NC}^{k-1}\right)\left[1-\left(1-p_{NC}\right)^{f-1}\right]}$$

PROOF: Consider the Markov chain $\{X_n\}$ with state space

$$\{(r,s): 0 \le r \le k, s = 0\} \cup \{(r,s): r = 0, 0 \le s \le f\}$$

where $X_n = (r,s)$ means that after the n^{th} classification there are r consecutive successes and s consecutive failures. Let p_{CC} (p_{NC}) be the probability that a conforming (nonconforming) item is classified as conforming. In the analysis below, we p will be equal to p_{CC} or p_{NC} depending on the true nature of the item. The transition probabilities when beginning in a transient state are of the form

$$P(X_n = (r + 1,0)|\ X_{n\text{-}1} = (r,s)) = p \text{ and } P(X_n = (0,s + 1)|\ X_{n\text{-}1} = (r,s)) = q.$$

The situation is depicted in Figure 1.

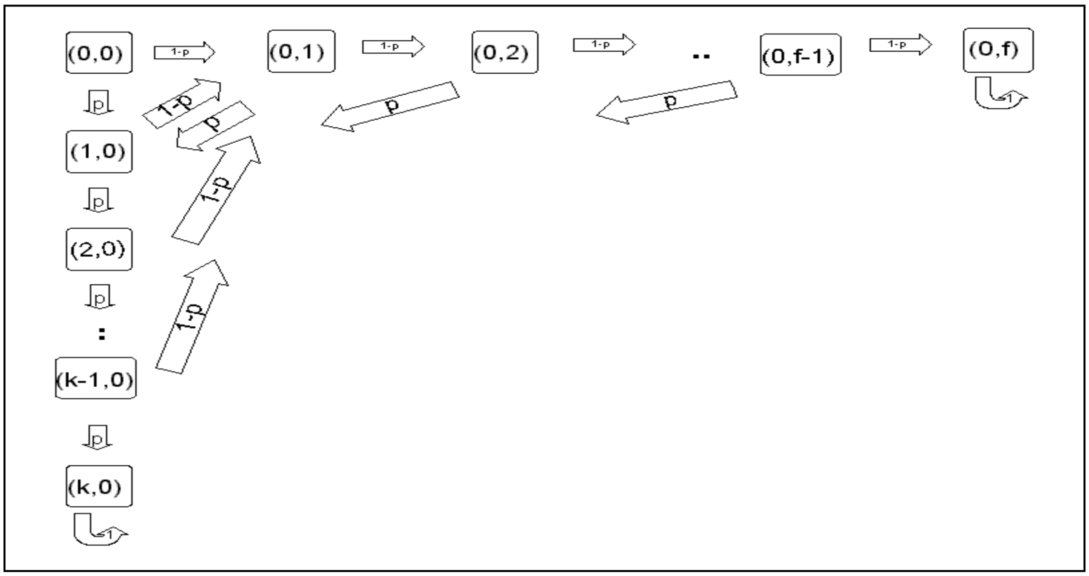

Figure 1.

Given that we are in the first column, we move down the column with probability p and move to state (0,1) with probability 1-p. Given that we are in the first row we go across the row with probability 1-p and move to state (1,0) with probability p. The state (k,0) is the absorbing state corresponding to item is judged conforming and state (0,f) is the absorbing state corresponding to item is judged nonconforming.

Consider figure 2 with a reduced state space and transition probabilities.

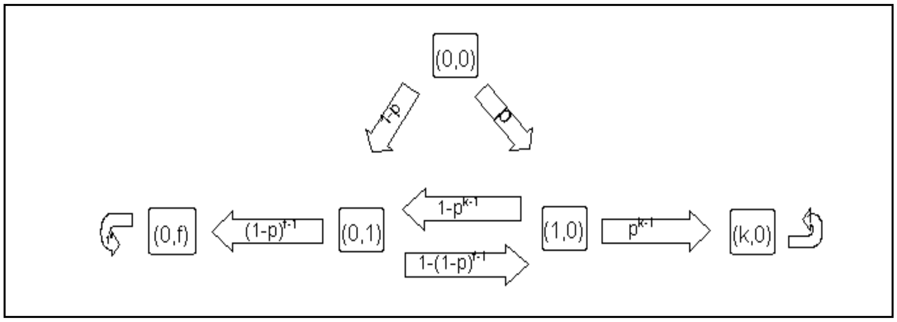

Figure 2.

For example, when the chain is in state (1,0), there are either *k*-1 consecutive successes (causing the chain to enter state (*k,* 0)) or there are not *k*-1 consecutive successes (causing the chain to enter state (0,1) upon a failure). Similarly, when the chain is in state (0,1), there are either *f*-1 consecutive failures (causing the chain to enter state (0,*f*)) or there are not *f*-1 consecutive failures (causing the chain to enter (1,0) upon a success).

To find the probability of judging the item to be conforming we may reason as follows. Starting in state (0,0), the chain enters state (1,0) with probability *p* and state (0,1) with probability 1-*p*. If the process enters (1,0), it can eventually get to (*k*,0) by going directly, or by going to (0,1) and back to (1,0) any integer number of times and then to state (*k*,0) directly. Hence the probability of reaching state (*k*,0) from state (1,0) is

$$p^{k-1} + \sum_{n=1}^{\infty}\left[\left(1-p^{k-1}\right)\left(1-(1-p)^{f-1}\right)\right]^{n} p^{k-1} = \frac{p^{k-1}}{1-\left(1-p^{k-1}\right)\left(1-(1-p)^{f-1}\right)}$$

On the other hand, if the process enters (0,1), it can eventually get to state (*k*,0) only by going to state (1,0) (rather than state (0,*f*) which is absorbing). The probability of reaching state (*k*,0) from state (1,0) has been calculated above. Hence,

$$\mathrm{P(Judged\ Conforming)} = p \cdot \frac{p^{k-1}}{1-\left(1-p^{k-1}\right)\left(1-(1-p)^{f-1}\right)} + (1-p)\left[1-(1-p)^{f-1}\right] \cdot \frac{p^{k-1}}{1-\left(1-p^{k-1}\right)\left(1-(1-p)^{f-1}\right)} = \frac{p^{k-1}\left[1-(1-p)^{f}\right]}{1-\left(1-p^{k-1}\right)\left(1-(1-p)^{f-1}\right)}$$

Proposition 2: If the process is in control, the probability that it is judged to be in control is

$$P_{II} = P(judged\ in\ control|\ actually\ control) = p_1 CCCN(p_{CC}) + (1-p_1)CCCN(p_{NC})$$

Proof: If it is in control, then the inspected item is conforming with probability p_1 and nonconforming with probability 1-p_1. In light of proposition 1 and using the law of total probability the result follows.

Proposition 3: If the process is out of control, the probability that is judged to be in control is

$$P_{OI} = P(judged\ in\ control|\ out\ of\ control) \\ = p_2 CCCN(p_{CC}) + (1 - p_2)CCCN(p_{NC})$$

Proof: If out of control, then inspected item conforms with probability p_2 and fails to conform with probability $1 - p_2$. In light of proposition 1 and using the law of total probability the result follows.

Proposition 4: When the process is out of control, the average run length is $\frac{1}{1-P_{OI}}$.

Proof: This is geometric distribution with parameter $1 - P_{OI}$.

Proposition 5: When the process is in control, the average run length is $\frac{1}{1-P_{II}}$.

Proof: This is geometric distribution with parameter $1 - P_{II}$.

3. Short Term Analysis Using Markov Chains

Markov chains will be the tool used to study the probability of judging the process to be out of control when it is in control as well as judging it to be out of control when it is out of control. We will also study the distribution of the time until the process is declared out of control using first passage probabilities in order to determine the number of items inspected until the process is finally declared out of control. We create a Markov Chain whose state space contains four ordered-pairs whose elements are 1 or 0. We use a 1 to stand for in control and a 0 to stand for out of control. The first coordinate is the actual state of the process and second coordinate is the judgment. For example, (1,1) means that at a decision point the process is in control and judged to be in control. Whereas, (0,1) means that the process is actually out of control but judged to be in control. Let $\theta = 1 - (1-\pi)^h$. So, $1 - \theta = (1-\pi)^h$ is the probability that the process has remained in control while those h items have been produced. The one-step probability matrix for the transitions of this Markov Chain is given in the following transition matrix.

$$\begin{array}{c} \\ (1,1) \\ (0,1) \\ (1,0) \\ (0,0) \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (0,1) & (1,0) & (0,0) \end{array} \\ \begin{pmatrix} (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ 0 & P_{OI} & 0 & 1-P_{OI} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

First step analysis can be used to get the probability of absorption into states (1,0) and (0,0). First-passage probabilities can be used to find the probability distribution of the time until the process is declared out of control. One can do this by finding the probability of first reaching each absorbing state in n steps and adding these probabilities to obtain the probability that it takes n steps (cycles of item inspections) to declare the process out of control. Note: P_{IO} = 1 - P_{II} and P_{OO} = 1 - P_{OI}.

4. Long Term Analysis Using Markov Chains

The long-term behavior of this process control can also be studied. When we reach state (1,0) or state (0,0) the process is judged out of control. When the cause is found and corrected or when it is determined that the process is in control and there is no cause, the process is put back online and the transitions are like the transition from state (1,1). This allows us to analyze the long term behavior of the decision process by using a one-step transition probability matrix in which the rows in the matrix that correspond to transitions out of (1,0) and (0,0) are identical to the transitions out of state (1,1). Therefore, the one-step transition probability matrix useful for long term analysis is given below.

$$\begin{array}{c} \\ (1,1) \\ (0,1) \\ (1,0) \\ (0,0) \end{array} \begin{array}{c} \begin{array}{cccc} \quad(1,1) & \qquad(0,1) & \qquad(1,0) & \qquad(0,0) \end{array} \\ \begin{pmatrix} (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ 0 & P_{OI} & 0 & 1-P_{OI} \\ (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \end{pmatrix} \end{array}$$

This one-step transition probability matrix is that of an irreducible, aperiodic, positive recurrent Markov Chain and the limiting probabilities exist and are independent of the starting state. These limiting probabilities can also be interpreted as the long-term proportion of time spent in each state and can be found by solving a system of linear equations.

References

Bhat, U.N. 1984. *Elements of Applied Stochastic Processes*, 2nd ed., John Wiley & Sons, Inc., New York, N.Y.

Borges, W., L. L. Ho, and O. Turnes. 2001. An analysis of Taguchi's on-line quality monitoring procedure for attributes with diagnosis errors. *Applied Stochastic Models in Business and Industry*, 17, 271-276.

Griffith W.S. and M. L. D. Smith. 2017. Use of Markov Chains in Process Control with the Incorporation of Possible Inspection Errors, *JSM Proceedings*, Section on Quality and Productivity. Alexandria, VA: American Statistical Association. 3167-3171.

Nayebpour, M. R., and W. H. Woodall. 1993. An analysis of Taguchi's on-line quality monitoring procedure for attributes. *Technometrics*, 35, 53-60.

Quinino, R.., E. C. Colin, and L. L. Ho. 2009. Diagnostic Errors and Repetitive sequential classifications in on-line process control by attributes. *European Journal of Operational Research*, doi:10.1016/j.ejor.2009.02.017.

Smith M. L. D. and W. S. Griffith. 2009. Repeated classifications subject to errors in process control, *JSM Proceedings*, Section on Quality and Productivity. Alexandria, VA: American Statistical Association, 1529-1534.

Taguchi, G., S. Chowdhury, and Y. Wu. 2004. *Taguchi's Quality Engineering Handbook*. John Wiley & Sons, Inc. New Jersey.

Taguchi, G., E. A. Elsayed, and T. Hsiang. 1989. *Quality Engineering in Production in Systems*. McGraw-Hill, New York.

Trindade, A., L. L. Ho, and R. Quinino. 2007. Monitoring process for attributes with quality deterioration and diagnosis errors. *Applied Stochastic Models in Business and Industry*, 23 (4), 339-358.