Effects of Parameter Estimation on The Modified and Acceptance Control Charts

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Abstract

Control charts are powerful tools used by many industries to monitor the quality of processes and detect special cause of variations. The Shewhart $\bar{X}$ chart is one of the most popular control charts to monitor the process mean, i.e., to detect a change from an in-control (IC) to an out-of-control (OOC) situation. In its original formulation, if the actual process mean is different from the specified IC level, the process is considered to be in an OOC state. However, in many practical situations, the process might still be capable from a practical point of view even if the process mean may slightly differ from the IC level. In such situation, the Modified and the Acceptance Control Charts are the appropriate tools to monitor the process mean. To design these charts, usually, the IC process standard deviation must be estimated from reference samples. In this paper, we show that this estimation has a negative impact on the performance of the Modified and the Acceptance Charts. Solutions to this impact are investigated.

Key Words: Type I and Type II Errors, False Alarm Rate, Control Limit Adjustments, Guaranteed In-Control Performance, Average Run Length

1. Introduction

The Shewhart $\bar{X}$ chart is one of the most used tools to monitor the mean of some processes quality characteristics in many manufacturing industries. Its main purpose is sending, with great probability, an alarm to the manager every time that the in-control process (IC) becomes out-of-control (OCC), i.e., detect any shift from the in-control mean level as soon as possible so that the manager can intervene in the process in order to correct it, avoiding the production of many nonconforming items.

However, sometimes small shifts on the process, perceived over time, may be of a little or no practical importance (WOODALL, 1985). In this case, there is no necessity of any intervention, which may save some money. Mohammadian and Amari (2012) and Oprime and Mendes (2017) expose that in high capable processes, for example, some changes on the mean do not negatively affect its quality in terms of production of defective units, this occurs because the natural dispersion of the process is significantly smaller than its specification limits. Thus, the process mean might to vary inside a range, not being necessary to keep this one fixed on the target value.

Because of that, the Shewhart $\bar{X}$ chart is not the most suitable tool to monitor high capable processes, note that in these cases a lot of signals may be issued without the process should be considered rejectable, which is undesirable. Therefore, the better option is using the Modified and/or the Acceptance Charts, which allow the process mean to vary inside a specific range, determined by the specification limits, and only genuinely important changes are detected. Woodall and Faltin (2019) highlights that false alarms should be averted and, for them, even though the Modified and the Acceptance Chart have been developed a long time ago (in the 50's), they may be of a great value in practice nowadays.

The goal of the Modified and the Acceptance Chart is concerned on accept or reject a process according to the rate of nonconforming items. The main difference between the Modified Chart and the Acceptance Chart is that the first one controls the probability of the Type I error (α), and the second one is designed to control the probability of Type II error (β).

In this paper it will be provided some new insights regarding the Modified Chart, considering the known and unknown parameters cases. The Modified Chart has been chosen because it is more common, in practice, designing a control chart to monitor the probability of Type I error (α). However, if the reader wants to replicate the results of this work to the Acceptance Chart, it is completely possible and it is not complex.

It is interesting to say that we will analyze the effects of the parameter estimation on the performance of this chart (the measurement analyzed here is the False Alarm Rate – FAR) and some possible solutions to these effects are being investigated.

Finally, it should be said that the present paper is a part of a larger work that we are developing, where we also analyze another performance measure, the Average Run Length (ARL), and we examine the two main performance perspectives in the current SPC literature: The Unconditional and the Conditional Perspectives (see Jardim et al. (2019a)).

2. Modified Control Chart (case K)

As seen in the Introduction, the Modified Control Chart is useful for capable processes whose mean is allowed to vary inside a range (delimited by two extreme values) without compromising process quality (in terms of process capability). We call these values as the minimum and maximum tolerated limits for the process mean, defined respectively by μ_L (the minimum tolerated mean) and μ_U (the maximum tolerated mean).

Now, consider X the random variable of a high capable process whose mean should be monitored. Assume that X follows a normal distribution, i.e., $X \sim N(\mu, \sigma)$, where μ and σ are respectively the actual mean and variance of the process. Considering that the Modified Chart is used to control this process, we have that, instead of the in-control situation be represented just when $\mu = \mu_0$ (where μ_0 represents the target for the process mean) as occurs on the Shewhart $\bar{X}$ chart, the process is considered acceptable (or in-control) when $\mu_L < \mu < \mu_U$ (where μ_0 is the exact middle point between μ_L and μ_U). So, the process is classified as rejected (or out-of-control) when $\mu < \mu_L$ or $\mu > \mu_U$.

The μ_L and μ_U must be chosen according to the maximum tolerated rate of nonconforming units (denoted here by δ), which is defined by the specification limits. In other words, δ is the maximum allowed probability of X being smaller than the lower specification limit (LSL) or greater than the upper specification limit (USL). Figure 1 illustrates the situation when the process is running at the tolerable limits (μ_L or μ_U), which means that the process is producing the maximum percentage allowed of nonconforming items (δ). Moreover it is exposed the acceptable IC range and the rejected (or OOC) range.

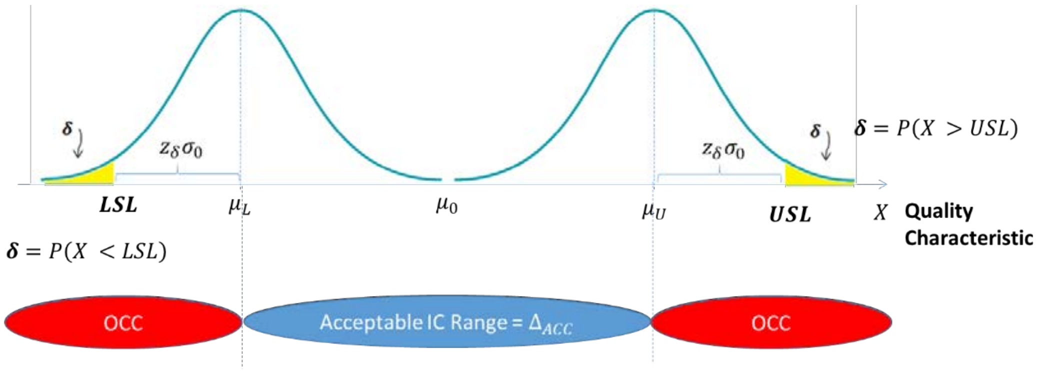

Figure 1 – The acceptable and rejectable range of Modified Chart based on the tolerable nonconforming items rate

In addition to the tolerable mean, it can be said that there are two others mean that represent a specific intolerable situation where the manager, with great interest, desires to reject: they are the lower intolerable mean ($\mu_{1,L}$) and the upper intolerable mean ($\mu_{1,U}$). Note that $\mu_{1,L} < \mu_L$ and $\mu_{1,U} > \mu_U$. These mean are defined according to a specific rate of nonconforming units (γ) that it is desired to reject with a large probability $(1-\beta)$. However, the process condition (if should be accept or reject) must be defined only by the tolerable mean, this means that even that the process mean has not exceed one of the intolerable mean, if it is outside from de acceptable range, the process must still be considered OOC.

According to Freund (1957) the tolerable and intolerable mean can be specified by some past experiences or calculated from the equations bellow.

$$\mu_U = USL - z_\delta * \sigma_0 \qquad \textbf{1)}$$

$$\mu_L = LSL + z_\delta * \sigma_0 \qquad \textbf{2)}$$

$$\mu_{1,U} = USL - z_\gamma * \sigma_0 \qquad \textbf{3)}$$

$$\mu_{1,L} = LSL + z_\gamma * \sigma_0 \qquad \textbf{4)}$$

As already said, the Modified Chart is focused on controlling the probability of the Type I error (α), i.e., controlling the probability of the process being between the tolerable mean and the chart emits erroneously a signal ($FAR = P(signal|\mu_L < \mu < \mu_U)$). Remember that a signal is given every time that a sample mean falls outside from one of the control limits (CHAKRABORTI, 2000).

It is known that there are many possible positions for the process mean inside the acceptable range and that the μ_0 is the exact middle point between μ_L and μ_U. So, to any position assumed by the mean inside this range there is a new possibility of a false alarm, in other words there are many possible FAR's (or α's) for the Modified Chart. However there is a minimum probability of a false alarm that corresponds to the moment when the process mean is centralized on the target value, i.e., $FAR_{min} = P(\bar{X} < LCL \ or\ \bar{X} > UCL\ |\mu = \mu_0)$, and there is a maximum value which refers to the moment on what the process mean equals to one of the tolerable mean, i.e., $FAR_{max} = P(\bar{X} < LCL|\mu = \mu_L)$ or $FAR_{max} = P(\bar{X} > UCL|\mu = \mu_U)$. The Panels *a* and *b* from the Figure 2 illustrate the FAR_{min} and the FAR_{max} respectively.

The control limits of the Modified Chart are calculated according to FAR_{max}, because the manager is concerned about monitoring the maximum probability of occurs an alarm when the process is still acceptable. The equations below describe the upper and lower control limits (UCL and LCL, respectively) for this chart.

$$UCL = \mu_U + z_{FAR_{max}} \frac{\sigma_0}{\sqrt{n}} = USL - z_\delta * \sigma_0 + z_{FAR_{max}} \frac{\sigma_0}{\sqrt{n}} \qquad \textbf{5)}$$

$$LCL = \mu_L - z_{FAR_{max}} \frac{\sigma_0}{\sqrt{n}} = LSL + z_\delta * \sigma_0 - z_{FAR_{max}} \frac{\sigma_0}{\sqrt{n}} \qquad \textbf{6)}$$

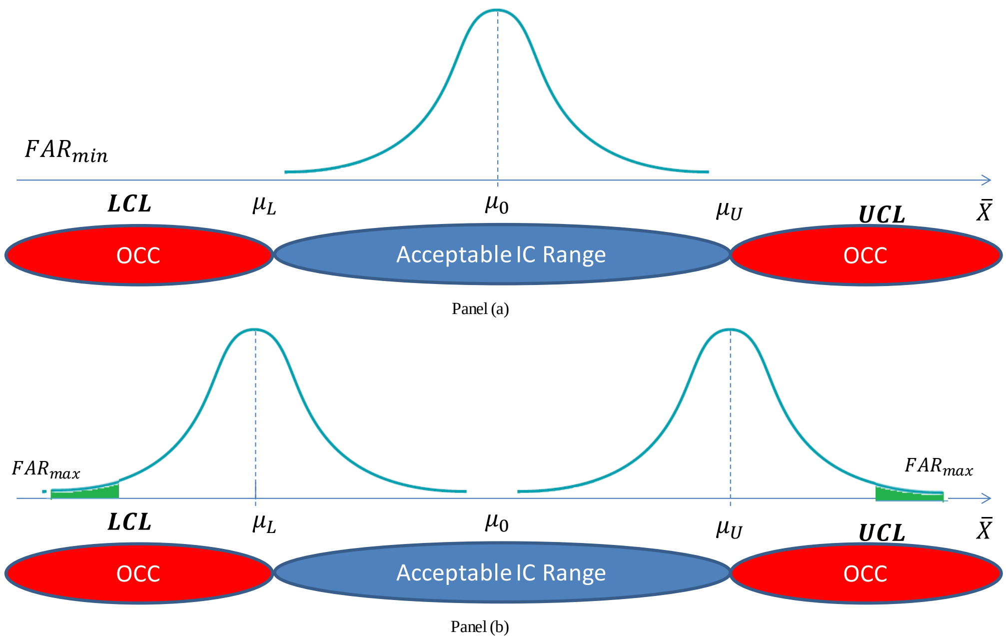

Figure 2 – The moment when FAR_{max} and FAR_{min} occurs

Let's consider $z_{FAR_{max}} = 3$, the most common 3-Sigma Limits. It is presumed that we know the standard deviation of the process (σ_0) and that its mean has moved to the lower tolerable mean (μ_L). Given a capable process, we have:

$$FAR_{max} = P\left(\frac{\bar{X} - \mu_L}{\sigma_0/\sqrt{n}} < \frac{\mu_L - z_{FAR_{max}}\frac{\sigma_0}{\sqrt{n}} - \mu_L}{\sigma_0/\sqrt{n}}\right) = P(Z < z_{FAR_{max}}) \quad \textbf{7)}$$

$$= \Phi(-3) = 0.0027$$

Note that, in this case, the Equation to find the FAR_{max} from the Modified Chart is equivalent to the Equation of the unique false alarm rate (FAR) of the one-sided $\bar{X}$ control chart (remember that the in-control process mean can assume only one value in this case).

3. Modified Control Chart (case U)

Until this moment we have supposed that the standard deviation of the process was known. However, in practice, most process have its parameters - mean and/or standard deviation - unknown, because of that it is necessary to estimate them before the charts can be set up and used (CHAKRABORTI, 2006).

The parameter estimation is made from "m" samples of size "n", which ones are taken from the process during the Phase I of the Statistical Process Control (SPC) (JARDIM; CHAKRABORTI; EPPRECHT, 2019) (SALEH, et al, 2015) (JONES-FARMER et al, 2014). Due to the fact that the data are gotten from samples and considering that each user can get different samples with different parameters values, it is reasonable to conclude that the parameter estimation is subject to some variability, which Saleh et al (2015) call by practitioner-to-practitioner variability. This variability compromise some operational properties of the control charts including the FAR (CHAKRABORTI, 2006).

At the unknown parameter case (case U) the maximum false alarm rate will be conditioned on the estimator of the in-control process standard deviation ($CFAR_{max}(\hat{\sigma})$). The $CFAR_{max}$ is a random variable which depends from estimative to estimative, because of that it can be very different from the desired value. This means that for $z_{FAR_{max}} = 3$ (the most common 3-Sigma Limits) $CFAR_{max}$ may be very different from 0.0027.

The estimator that we use and recommend in this work is the pooled sample standard deviation (S_p). This is because Mahmound et al (2010) showed that the S_p is preferable to the more traditional estimator: $\bar{S}/c_4$. The S_p is calculated by the square root of the average of the sample variance of the Phase I samples ($S_p = \sqrt{\frac{1}{m}\sum_{i=1}^{m} S_i^2}$, where $S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{i,j} - \bar{X}_i\right)^2$). So, we can write the formula of $CFAR_{max}$ as it is shown on the equation 7.

$$CFAR_{max} = 1 - P(LCL < \bar{X} < UCL | S_p \; and \; \mu = \mu_L \; or \; \mu = \mu_U) \qquad \textbf{7)}$$

Since $CFAR_{max}$ is a random variable, it is important to have knowledge of its probability distribution. The cumulative distribution function (c.d.f.) of $CFAR_{max}$ is the probability of $CFAR_{max}$ being lower than any value on its distribution (called here by t), in others words, $F_{CFAR_{max}}(t) = P(CFAR_{max} \leq t)$.

It is natural to think that the calculation of $F_{CFAR_{\max}}(t)$ is get through the distribution of the standard deviation estimator (S_p), but it is very complex. An easier way to calculate $F_{CFAR_{\max}}(t)$ is through the distribution of Y, a variable that is function of the pooled sample standard deviation ($Y = m(n-1)\, S_p^2/\sigma_0^2$), which follows a much known distribution – the central chi-square distribution with $m(n-1)$ degrees of freedom ($\chi^2_{m(n-1)}$). Thus, the c.d.f. of $CFAR_{max}$ can be written in function of Y, as shown at the next equation.

$$\begin{aligned} F_{CFAR_{max}(Y)}(t) &= P(CFAR_{max} \leq t) \qquad \textbf{8)} \\ &= 1 - F_{\chi^2_{m(n-1)}}\left(m(n-1)\left(-\frac{\Phi^{-1}(t)}{z_{FAR_{max}}}\right)^2\right) \qquad 0 \leq t \leq 1 \end{aligned}$$

Formula 8 is equivalent to the c.d.f. expression of the false alarm rate for the one-sided $\bar{X}$ chart in the case where only the in-control process standard deviation is estimated and when FAR_{max} is replaced by FAR.

From the formula 8 it is possible to draw the curve of $F_{CFAR_{max}(Y)}$ parameterized in function of "m", as presented on the Figure 3.

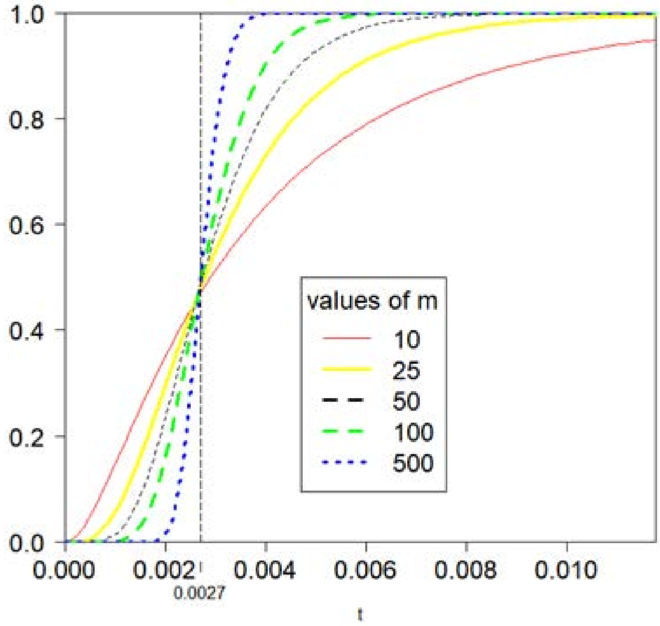

Figure 3 – C.D.F. of $CFAR_{max}$ when $n = 5$ and $FAR_{max} = 0.0027$.

Note on Figure 3 that the vertical line shows the nominal false alarm rate ($FAR_{max} = 0.0027$). It is easily to observe that the conditioned false alarm rate ($CFAR_{max}(t)$ is directly related with the number of samples collected from process on the Phase I of the SPC (m). In other words, as "m" increases, the c.d.f. curves of $CFAR_{max}$ get closer to the vertical line, meaning that the $CFAR_{max}$ is likely to be not much different from 0.0027. However, when m is small, the chances are high that $CFAR_{max}$ being greater than the nominal value (from Figure 3 we see that when $m = 10$, there is a 40% chance that the conditional false alarm rate is 48% higher than the nominal 0.0027).

To provide further insight, at Figure 4 we display the p.d.f. of $CFAR_{max}$ ($f_{CFAR_{max}}$), which was calculated taking the numerical derivatives of the corresponding c.d.f. The $f_{CFAR_{max}}$ plot shows that the distribution of $CFAR_{max}$ is extremely right skewed when "m" is small. This also shows that there is a large probability of $CFAR_{max}$ being substantially larger than the nominal value.

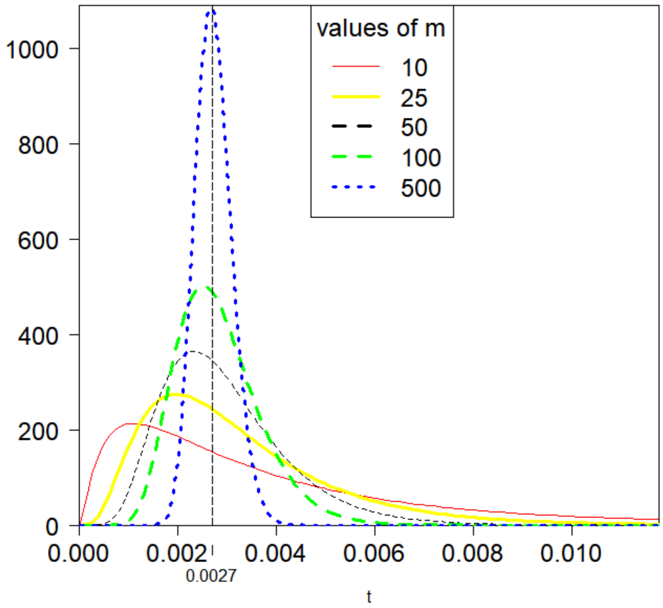

Figure 4 – P.D.F. of $CFAR_{max}$ when $n = 5$ and $FAR_{max} = 0.0027$.

Furthermore, it is interesting to analyze the expectation and the standard deviation of $CFAR_{max}$ to better understand what was presented on the Figure 4 (look Table 1).

m	$E(CFAR_{max})$	$E(CFAR_{max}) \times FAR_{max}$	$SD(CFAR_{max})$	$1 - P(CFAR_{max} < 0.0027)$
10	0.0041	52%	0.0039	53%
20	0.0032	19%	0.0020	52%
50	0.0030	11%	0.0013	51%
100	0.0028	4%	0.0009	51%
500	0.0027	0%	0.0004	50%

Table 1 – Numerical analyzes of the effects of the parameter estimation on the false alarm rate

Beyond the expectation and the standard deviation of $CFAR_{max}$, Table 1 presents two others analyzes of the effects of the parameter estimation on the false alarm rate, they are: the difference between the expected value of $CFAR_{max}$ and the nominal value (FAR_{max}); and the probability of $CFAR_{max}$ being upper than the nominal value.

It was observed that as the number of samples collected on Phase I (m) increases, the expectation of $CFAR_{max}$ gets closer for than nominal value (see the second and the third columns of Table 1). If we pay attention on the forth and fifth columns, we will observe that the dispersion of the process and the probability of $CFAR_{max}$ of being upper than the nominal value decrease with increasing of "m", what means that the p.d.f. of $CFAR_{max}$ becomes more symmetrical (less skewed) getting closer from a normal distribution.

Concluding, when the standard deviation is estimated on the Modified Chart, to increase the chances of $CFAR_{max}$ to be similar from the false alarm rate obtained when sigma is known, it is necessary to collect quite a large number of samples on Phase I (such as 500), which is unviable or even impossible. But if it is taken a small number of samples (such as 10 or 25) the probability of having more false alarms is significantly big. Because of that it is necessary to find the best way to minimize the effects of the parameter estimation on the $CFAR_{max}$ ensuring a reasonable number of samples.

More recently, several authors recommended adjusting the control limits of the Shewhart $\bar{X}$ chart by finding some adjusted limit factor ($z_{FAR_{max}}$) so that $P(CFAR_{max} \leq FAR_{max}) = p$, while $p = 85\%, 90\%\ or\ 95\%$. This is known as the Exceedance Probability Criterion (proposed by Albers in 2005) and it also relates with the theory of Tolerance Intervals. Nevertheless we are still investigating this adjustment for the Modified Control Chart.

4. Conclusions

The Modified Control Chart is an interesting tool to control high capable processes, cause it allows some variation of the process mean, which is defined so that the production of nonconforming units does not exceed the established rate by the managers.

Despite being a relevant tool, there are some concepts about the performance measurements of the Modified Chart that were still unknown, including the effects of the parameters estimation on these charts. So, our goal was to fill this gap and provide to readers and users more knowledge about this chart in practice. The present work was focused in explain the concepts of maximum and minimum false alarm rate (FAR_{max} and $FAR_{\min}$) and the effects of the standard deviation estimation (using the estimator S_P) on the conditional maximum false alarm rate ($CFAR_{max}$).

On the Modified Chart there is a multitude of possible values to the false alarm rate, by the fact that to each position assumed by the mean inside the tolerable range there is a new probability of a signal. Thus, if the process mean is equal to the target value, we have the minimum rate of false alarms ($FAR_{min} = P(signal|\mu = \mu_0)$), and if the process mean have run to the tolerable limits, we can say that exists a maximum chance of a false alarm ($FAR_{max} = P((signal|\ \mu = \mu_L\ or\ \mu = \mu_U))$).

When it is supposed that the standard deviation is known (case K), despite there are many possible values to FAR, it is not a random variable. However, when the standard deviation is unknown (case U), the false alarm rate becomes a random variable conditioned to the estimator ($CFAR(S_p)$). Thus, $CFAR(S_p)$ changes from practitioner-to-practitioner and it can assume values very different from the target one.

It was exposed that the number of samples (m) that should be collected from the process to achieve similar results between the case K and case U is much greater than 25, 30 or 50, i.e., much greater than recommendations by previous work concerning to the Shewhart $\bar{X}$ chart. In practice it is unviable, because of that it is necessary to find some solution to minimize the effects of the parameter estimation on the $CFAR$, finding a way to decrease the number of samples that are needed to be collected. A possible solution to this problem is the adjustment of the control limits, but this work is still in progress.

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