Trend-cycle filters comparison for real time macroeconomic data

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Abstract
Leading, coincident and lagging indicators have long been used to analyze and assess the current stage at which the economy stands. Official statistical agencies have generally applied linear filters developed by Musgrave (1964) to produce preliminary estimates of the trend-cycle of these indicators, but these estimates are subject to revisions as new observations are added to the series. To reduce the revisions size, cascade linear filters developed by Dagum and Luati (2009) have been recently used. However, only asymmetric filters related to the 13-term symmetric one are available, whereas, due to more variability in the data introduced by major financial and global changes in the economy, different filter lengths are needed to produce smoother estimates. We describe and propose a new procedure to reduce the size of the revisions and make the indicators more timely. These new filters significantly outperform their older counterpart. They offer substantial gains in real-time by providing timely and more accurate information for detecting short-term trend turning points.

Key Words: Short-term trend, economic indicators, linear filters, reproducing kernels.

1. Introduction

The purpose of short-term economic statistics is to provide a comprehensive and timely picture of economic processes, such as production, income distribution, financing and expenditure, to evaluate the stage of the cycle at which the economy stands. Economic analysts and policy makers not only require accurate and timely information on the direction and magnitude of the trend of main economic variables, but they must also have confidence that these estimates are unlikely to change significantly as more complete data become available. Therefore, the main aim of official statistical agencies is to find the best balance among three main requirements, that is accuracy, timeliness and relevance of economic statistics. Data revisions are important because they may affect policy decisions or the manner in which such decisions depend on the most recent data.

The problem of identifying the direction of the short-term trend of seasonally adjusted series contaminated by high levels of variability has become of relevant interest in recent years. Financial and economic changes of global character have introduced a large amount of noise in time series data, particularly, in socioeconomic indicators used for real time economic analysis. Official statistical agencies generally rely on asymmetric filters that were developed by Musgrave in 1964. However, the use of the latter introduces large revisions as new observations are added to the series and, from a policymaking viewpoint, they are too slow in detecting true turning points (see *e.g.* Dagum and Bianconcini, 2015, and reference therein). To overcome these main limitations, Dagum (1996) developed a new method that basically consisted of : (1) extending a smoothed seasonally adjusted series (modified by extreme values with zero weight) with ARIMA extrapolations, and (2) applying the symmetric Henderson (1916) filter to the extended series using stricter sigma limits for the identification and replacement of extreme values. The extension of the smoothed seasonally adjusted series with ARIMA extrapolations was needed to reduce the size of the revisions for the most recent estimates of the trend-cycle. A linear approximation of this

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filter has been proposed by Dagum and Luati (2009) based on the convolution of several noise suppression, trend estimation (13-term Henderson filter), and extrapolation linear filters. This cascade linear filter is nowadays used by main statistical agencies around the world to provide information on trend-cycle movements for several economic indicators. Among others, Statistics Canada, US Bureau of Census, Eurostat use it in conjunction with nonparametric seasonal adjustment software, such as the US Bureau of the Census X11 method (Shiskin, Young and Musgrave, 1967) and its variants, X11/X12ARIMA and X13 (Findley et al., 1998). However, only the symmetric filter of 13 terms and corresponding asymmetric filters are available, whereas, due to the high variability present in current seasonally adjusted data, different filter lengths are needed to produce smoother and more reliable estimates.

In this paper, we propose a generalization of the cascade linear filter based on the Reproducing Kernel Hilbert Space (RKHS) methodology that allows to derive cascade linear filters of any length. We propose a new set of symmetric and asymmetric weights that provide more reliable short-term estimates in real time, which are more useful from a policy-making viewpoint. We apply the new filters to leading. coincident and lagging indicators of the US economy, which is known to be a key player from an international macroeconomic perspective. We will concentrate on the reduction of revisions only due to filter changes. and ignore those introduced by new innovations entered with new data. In other words, the filter revisions depend on how close the asymmetric filters are with respect to the symmetric one. Specifically, as proposed by Dagum and Bianconcini (2015), we select time-varying bandwidth parameters, specific for each asymmetric filter, that minimize the distance between the gain functions of asymmetric and symmetric filters. We show that this method provides more accurate and reliable estimates of the short-term trend of major socio-economic indicators and drastically reduces the time delay to signal the upcoming of a true turning point.

2. Official statistical methods for short-term trend estimation

Research efforts by official statisticians have recently been devoted to improve existing procedures for the analysis of current economic conditions. Until the last financial and global crisis, recession and recovery analysis proposed by Moore (1961) was the main approach used to assess the stage at which the economy was, through the study of percentage changes, based on seasonally adjusted data and computed for months and quarters in chronological sequence. This was particularly effective to evaluate the behavior during incomplete phases of the business cycle, but, nowadays, the period to period movements of the seasonally adjusted series are highly influenced by the behavior of irregular fluctuations. Hence, this approach has failed in providing reliable and clear information on the main economic conditions. As a consequence, official statistical agencies have started to publish trend-cycle estimates as a complement to seasonally adjusted data to help reveal better the movements in the business cycle and the occurrence of turning points.

Linear filters developed by Henderson (1916) are the classical method to estimate the trend-cycle component of seasonally adjusted economic indicators used in conjunction with nonparametric seasonal adjustment software, such as the US Bureau of the Census X11 method (Shiskin, Young and Musgrave, 1967) and its variants, X11/X12ARIMA and X13 (Findley et al., 1998). Assuming that the input series $\{y_t, t = 1, \dots, N\}$ is seasonally adjusted where trading day variations and extreme values, if present, have been also removed, it can be decomposed into the sum of a systematic component g_t, usually called signal, plus an erratic component u_t, called the noise, such that

$$y_t = g_t + u_t, \quad t = 1, \dots, T.$$

The signal g_t represents the trend and cyclical components, usually referred to as the trend-cycle for they are estimated jointly. The noise u_t is assumed to be either a white noise, $WN(0, \sigma_u^2)$. or, more generally, to follow a stationary and invertible autoregressive moving average (ARMA) process.
The Henderson trend-cycle estimates $\hat{g}_t$ for the central observations, $t = m+1, \ldots, N-m$, are obtained through a weighted moving average as follows

$$\hat{g}_t = \sum_{j=-m}^{m} w_j^{hend} y_{t+j}, \tag{1}$$

where the weights $w_j^{hend}, j = -m, \ldots, m.$ are derived by the Henderson ideal formula:

$$w_j^{hend} = \frac{315[(m+1)^2 - j^2][(m+2)^2 - j^2][(m+3)^2 - j^2][3(m+2)^2 - 16 - 11j^2]}{8(m+2)[(m+2)^2 - 1][4(m+2)^2 - 1][4(m+2)^2 - 9][4(m+2)^2 - 25]} \tag{2}$$
$$j = -m, \ldots, m.$$

This is equivalent to assume that the trend-cycle $g_t, t = 1, \ldots, T$, is a smooth function of time that can be *locally* represented by a polynomial of degree three in a variable j, which measures the distance between y_t and its neighboring observations $y_{t+j}, j = -m, \ldots, m$, estimated by minimizing the following function

$$\sum_{j=-m}^{m} W_j [y_{t+j} - a_0 - a_1 j - a_2 j^2 - a_3 j^3]^2,$$

where the weighting penalty function is given by $W_j \propto \{(m+1)^2 - j^2\}\{(m+2)^2 - j^2\}\{(m+3)^2 - j^2\}$. The solution for the constant term $\hat{a}_0$ is the smoothed observation $\hat{g}_t$.

The Henderson weights have the property that fitted to exact cubic functions will reproduce their values, and fitted to stochastic cubic polynomials they will give smoother results than those estimated by ordinary least squares. As shown by Loader (1999), the Henderson weights can be equivalently expressed by the product of a cubic polynomial $\phi(j)$ and the weighting function W_j. For large m, Loader (1999) provided an equivalent kernel representation of the weights by showing that W_j can be approximated by the triweight function $m^6(1 - (j/m)^2)^3$, such that the weight diagram is approximately $(315/512)(3 - 11(j/m)^2)(1 - (j/m)^2)^3$.
Different kernel characterizations of the Henderson filter have been derived by Dagum and Bianconcini (2008) based on the Reproducing Kernel Hilbert Space (RKHS) methodology, according to which the symmetric filter weights are given by

$$w_j^{hend} = \frac{K_4(j/(m+1))}{\sum_{j=-m}^{m} K_4(j/(m+1))}, \qquad j = -m, \ldots, m. \tag{3}$$

K_4 is a third order kernel derived by biweight density function $f_{0B}(t) = (15/16)(1 - t^2)^2, t \in [-1.1]$, and corresponding Jacobi orthonormal polynomials $P_i, i = 0, \ldots, 3$, that is $K_4(t) = \sum_{i=0}^{3} P_i(t) P_i(0) f_{0B}(t), t \in [-1.1]$.

At the end (beginning) of the series. $t = N - m, \ldots, N$ $(t = 1, \ldots, m)$, asymmetric weights need to be applied. The asymmetric Henderson smoothers currently in use were developed by Musgrave (1964). They are based on the minimization of the mean squared revision between the final estimates (obtained by the application of the symmetric filter) and the preliminary ones (obtained by the application of an asymmetric filter) subject to the constraint that the sum of the weights is equal to 1. The assumption made is that at the end (beginning) of the series, the seasonally adjusted values follow a linear trend-cycle plus a purely random irregular ε_t, such that $\varepsilon_t \sim IID(0, \sigma^2)$.

Dagum and Bianconcini (2015) introduced a RKHS representation of them. In this framework, given the density function (in our case the biweight), once the length of the symmetric filter is chosen, say $2m + 1$, the statistical properties of the asymmetric filters are strongly affected by the bandwidth parameter of the kernel function from which the weights are derived. Dagum and Bianconcini (2008 and 2013) made the bandwidth parameters equal for all the asymmetric filters (global time-invariant bandwidth) to closely approximate the Musgrave filters, whereas Dagum and Bianconcini (2015) have proposed time-varying bandwidth parameters since the asymmetric filters are time-varying.

The selection of the length of the Henderson filters is based on the signal-to-noise ratio computed on the seasonally adjusted series and, under common economic conditions, the 13-term filter and corresponding asymmetric weights are often applied. These filters have the excellent property of fast detection of true turning points, but the limitations of producing large revisions to the most recent estimates when new observations are added to the series, and introducing a large number of 10-month cycle (unwanted ripples) in the final trend-cycle curve, that can be falsely interpreted as turning points. The problem of the unwanted ripples is specific of the 13-term Henderson filter when applied to seasonally adjusted series. The use of a longer Henderson filter, such as the 23-term, is not an alternative because it is sluggish to detect turning points. To overcome these main limitations, Dagum (1996) proposed a nonlinear semiparametric predictor to improve the 13-term Henderson filter with the advantages of: (1) reducing the number of unwanted ripples, (2) not increasing the time lag to detect turning points, and (3) reducing the size of the revisions of the most recent trend-cycle estimates. This new method basically consisted of (1) extending the seasonally adjusted series (modified by extreme values with zero weights) with ARIMA extrapolations, and (2) applying the 13-term Henderson filter to the extended series using stricter sigma limits for the identification and replacement of extreme values ($\pm 0.7\sigma$ and $\pm 1\sigma$ were recommended). The purpose of the ARIMA extrapolations was to reduce the size of the revisions of the most recent estimates, whereas that of extreme values replacement to reduce the number of unwanted ripples produced by the Henderson filter.

Dagum and Luati (2009) provided a linear approximation for both symmetric and asymmetric components by means of the convolution of several noise suppression, trend estimation, and extrapolation linear filters. This filter is currently applied by many official statistical agencies and is called cascade linear filter.

2.1 The cascade linear filter

ARIMA extrapolations and stricter replacement of extreme values represent the two sources of nonlinearity of the Dagum method. To determine the symmetric weights of the cascade linear filter, Dagum and Luati (2009) approached the replacement of extreme values as strong noise suppression in the input. The Symmetric Linear Filter (SLF) was derived by double smoothing the residuals obtained from a sequential application of the 13-term Henderson filter to the input data using the convolution of a 5-term weighted and a 7-term non weighted moving averages. The matrix representation of the symmetric cascade linear filter is given by

$$\mathbf{H}[\mathbf{H} + \mathbf{M}_{7.(0.143)}(\mathbf{I}_N - \mathbf{H})][\mathbf{H} + \mathbf{M}_{5.(0.25)}(\mathbf{I}_N - \mathbf{H})], \tag{4}$$

where $\mathbf{H}$ refers to the 13-term Henderson filter, $\mathbf{M}_{5.(0.25)}$ is the matrix representative of a 5-term moving average with weights (0.250.0.250.0.000.0.250.0.250), and $\mathbf{M}_{7.(0.143)}$ is the matrix representative of a 7-term filter with all weights equal to 0.143. The convolution (4) produces a symmetric filter of 31 terms with very small weights at both ends. Dagum and Luati truncated this filter to 13 terms, and normalized it such that the weights added up to unity. Renormalization was needed to avoid a biased mean output. To do so, the total weight discrepancy (equal to -0.065) was distributed over the 13 weights, $w_j, j = -6, \ldots, 6$, performing an ad-hoc mixed distribution given by

$$-0.103, -0.076, -0.076, -0.341, -0.127, 0.042, 0.364. \quad (5)$$

The total discrepancy was mostly allocated to w_0 ($+36\%$),w_3, and w_{-3} (-34% each). To increase the amount given to the central point, the values of w_3 and w_{-3} were reduced for it is important to maintain as much as possible the area under the positive weights without modifying the negative ones. The latter is a necessary but not sufficient condition for a filter to be unbiased respect to a second/third degree polynomial trend, which is needed to estimate properly points of maxima and minima.

Asymmetric filters were applied to the last six data points, which are crucial for current analysis. They were obtained by the convolution of the symmetric filter with linear extrapolation filters for the last six data points. The extrapolations were made linear by fixing the ARIMA model and its parameters values, chosen such as to minimize the size of revisions and phaseshifts. The model selected was the ARIMA(0, 1, 1) with $\theta = 0.40$.

2.1.1 Relationship with the Henderson filter

The mixed normalization (5) was performed in order to preserve the same area of the Henderson filter under the positive and negative weights.

Proposition 2.1 *For a Henderson filter of length $2m + 1$, the weights w_j^{hend} are negative (or null) if $j = \pm([\frac{m}{2}] + 1), \ldots, m$, and positive if $j = -[\frac{m}{2}], \ldots, [\frac{m}{2}]$.*

Proposition 2.2 *Independently on the filter length, the area under the positive Henderson weights is approximately equal to 1.1, and, consequently, that under the negative weights is almost 0.1.*

To analyze the relationship between the SLF and Henderson filter, we have to study regularities in the behavior of the SLF weight system when the ad hoc mixed normalization (5) is not necessary. Hence, we consider the 23-, and up to 39-term SLF, whose weights sum up to one (up to the third digit). The behavior of the 23-term SLF and Henderson filter is compared in Figure 1, but it is the same for all the other filter lengths, with discrepancies, especially in the central weight, that reduce as the filter length increases.

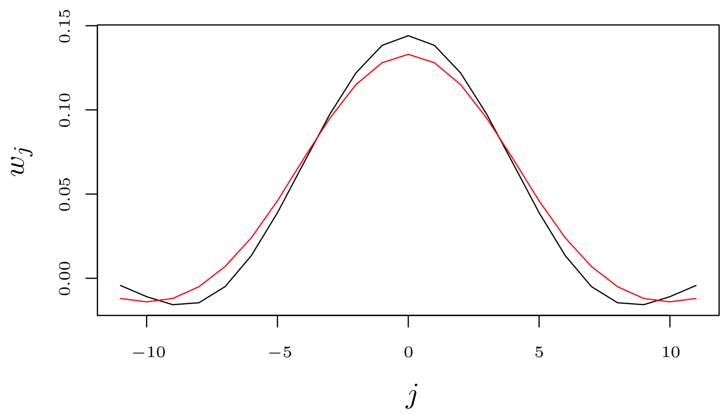
Specifically, the following regularities are observed:

Regularity 1. Both SLF and Henderson filter cover the same area under positive and negative weights, approximately equal to 1.1 and 0.1, respectively (independently on the filter length, see Proposition 2.2).

Regularity 2. The Henderson weights w_j^{hend} are negative for $j = \pm\left([\frac{m}{2}] + 1\right), \ldots, m$ (see Proposition 2.1). On the other hand, the SLF weights w_j^{SLF} are negative when $j = \pm\left([\frac{m}{2}] + 2\right), \ldots, m$.

Regularity 3. The discrepancies (especially in the central weights) between the SLF and Henderson filter reduce as the filter length increases.

Figure 1: Weight systems of the 23-term SLF (*red*) and Henderson (*black*) filter.



3. A reproducing kernel Hilbert space perspective of the cascade linear filter

To find a kernel representation of the SLF, we looked for a third order kernel within the same family of the biweight kernel from which the Henderson filter is derived (Dagum and Bianconcini, 2008 and 2015). Based on Regularity 1., it has to cover the same area under positive and negative values as the third order biweight kernel, that is approximately 1.1 and 0.1, respectively. Furthermore, based on Regularity 2., it has to be negative starting from a value greater than $\frac{1}{\sqrt{3}}$.
The third order biweight kernel belongs to the Beta kernel family based on the density

$$f_{0B}(t) = \frac{r}{2B(s+1, 1/r)}(1- \mid t \mid^r)^s, \quad t \in [-1, 1]. \tag{6}$$

The corresponding third order kernel results

$$K_3(t) = \left(\frac{\mu_4 - \mu_2 t^2}{\mu_4 - \mu_2^2}\right) f_{0B}(t), \quad t \in [-1, 1],$$

where $\mu_2 = \frac{1}{2B(s+1,1/r)} \frac{2\Gamma\left(\frac{3+r}{r}\right)\Gamma(1+s)}{3\Gamma\left(1+\frac{3}{r}+s\right)}$ and $\mu_4 = \frac{1}{2B(s+1,1/r)} \frac{2\Gamma\left(\frac{5+r}{r}\right)\Gamma(1+s)}{5\Gamma\left(1+\frac{5}{r}+s\right)}$. It assumes negative (or null) values when the polynomial $\left(\frac{\mu_4-\mu_2 t^2}{\mu_4-\mu_2^2}\right)$ is less than or equal to zero, that is when $\mid t \mid \geq \sqrt{\frac{\mu_4}{\mu_2}}$, being $\frac{\mu_4}{\mu_2} = \frac{3\Gamma\left(\frac{5+r}{r}\right)\Gamma\left(1+\frac{3}{r}+s\right)}{5\Gamma\left(\frac{3+r}{r}\right)\Gamma\left(1+\frac{5}{r}+s\right)}$. The third order biweight kernel is obtained when $r = s = 2$, such that $\frac{\mu_4}{\mu_2} = \frac{1}{3}$.
For different kernels within the Beta family, we determine the corresponding ratio $\frac{\mu_4}{\mu_2}$, looking for those combinations of r and s for which $\frac{\mu_4}{\mu_2} > \frac{1}{3}$ (see Table 1). This is satisfied when $r = 1, s = 0$ (uniform kernel), $r = 1, s = 1$ (triangle kernel), $r = 2, s = 1$ (Epanechinov kernel), and also $r = 3, s = 1$ and $r = 3, s = 2$. Among these kernels, to select the one that best resembles the behavior of the SLF, we compute the area under positive and negative values as shown in Table 2.
The third order kernel within the Beta family that, respect to the biweight ($r = s = 2$), satisfies the requirements of the SLF imposed by Regularities 1. and 2. is the third order triangle kernel ($r = s = 1$), given by

$$K_3^T(t) = \left(\frac{12}{7} - \frac{30}{7}t^2\right)(1- \mid t \mid), \quad t \in [-1, 1].$$

Table 1: Value of the ratio $\frac{\mu_4}{\mu_2}$ of different kernels within the Beta family

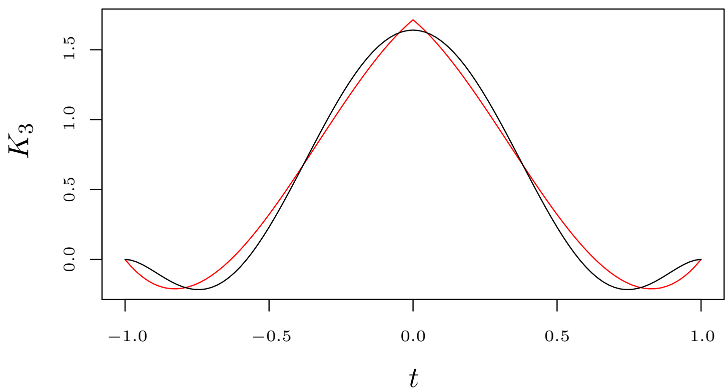
r/s	0	1	2	3
1	3/5	2/5	2/7	3/14
2	3/5	3/7	1/3	3/11
3	3/5	9/20	81/220	243/770

Table 2: Area under negative and positive values of Beta kernels for which $\frac{\mu_4}{\mu_2} > \frac{1}{3}$

r	s	$\frac{\mu_4}{\mu_2}$	$2\int_{\sqrt{\frac{\mu_4}{\mu_2}}}^{1} K_3(t)dt$	$\int_{-\sqrt{\frac{\mu_4}{\mu_2}}}^{\sqrt{\frac{\mu_4}{\mu_2}}} K_3(t)dt$
1	0	3/5	-0.16	1.16
1	1	2/5	-0.10	1.10
2	1	3/7	-0.12	1.12
2	2	1/3	-0.10	1.10
3	1	9/20	-0.14	1.14
3	2	81/220	-0.12	1.12

Figure 2 compares it with the third order biweight kernel. Differently from Figure 1, the maximum of the triangular kernel is greater than the biweight one. This means that, since the symmetric Henderson weights are derived using a bandwidth parameter equal to $m+1$, the bandwidth selected to derive the SLF weights from the triangle kernel should be greater than $m+1$.

Figure 2: Third order triangle (*red*) and biweight (*black*) kernels.



3.1 Bandwidth selection

When applied to real data, the symmetric filter weights are derived from the continuous third order kernel K_3^T as follows:

$$w_j^T = \frac{K_3^T(j/b)}{\sum_{j=-m}^{m} K_3^T(j/b)}, \quad j = -m, \ldots, m.$$

where b is a time-invariant global bandwidth parameter (same for all $t = m+1, \ldots, N-m$) selected to ensure a symmetric filter of length $2m+1$. The bandwidth parameter relates the discrete domain of the filter, that is $-m, \ldots, m$, with the continuous domain of the kernel function, that is $[-1, 1]$, and its choice is a fundamental task.

For our purposes, the bandwidth b should be selected such that the central kernel weight w_0^T reproduces the central SLF weight w_0^{SLF} in order to guarantee the kernel weights to be equal to the SLF ones at least up to the third digit, for $j = -m, \ldots, m$. In particular, based on the convolution matrix representation of the cascade filter given in eq. (4), a specific formulation for the central weight as function of m is given by

$$\begin{aligned} w_0^{SLF} &= \frac{8 \times 10^6[(m-3)(m-2)^2(m-1)(m-0.5)^2(m+0.5)(m+1)^3(m+2)]}{(m-0.5)^3(m+0.5)^3(m+1)(m+1.5)^3(m+2)^2(m+2.5)^3(m+3)(m+3.5)^3(m+4.5)^3} + \\ &+ \frac{1.5m^{15}[m+10)(m+11)(m+12)(m+15)(m^2+3m+54)]}{(m-0.5)^3(m+0.5)^3(m+1)(m+1.5)^3(m+2)^2(m+2.5)^3(m+3)(m+3.5)^3(m+4.5)^3} \end{aligned}$$

On the other hand, the central kernel weight is given by

$$\begin{aligned} w_0^T &= \frac{K_3^T(0)}{\sum_{j=-m}^{m} K_3^T(j/b)} = \frac{\frac{12}{7}}{\sum_{j=-m}^{m}\left(\frac{12}{7} - \frac{30}{7}\left(\frac{j}{b}\right)^2\right)\left(1 - \left|\frac{j}{b}\right|\right)} \\ &= \frac{12b^3}{-12m(m+1)b^2 + 15m^2(m+1)^2 + 12b^3(2m+1) - 10m(m+1)(2m+1)b}. \end{aligned} \quad (7)$$

Hence, solving the equality $w_0^T = w_0^{SLF}$ implies that the bandwidth parameter is obtained by solving the following third degree equation

$$12[1-q(2m+1)]b^3 + 12qm(m+1)b^2 + 10qm(m+1)(2m+1)b - 15qm^2(m+1)^2 = 0. \quad (8)$$

being $q = w_0^{SLF}$. The three roots of eq. (8) can be real or complex. Among them, based on the properties that the bandwidth parameter must satisfy, we select the greatest value since it is (a) real, (b) positive, and (c) greater than $m + 1$. The selected bandwidths corresponding to the different filters are given in Table 3. The relationship between the

Table 3: Bandwidth parameters.

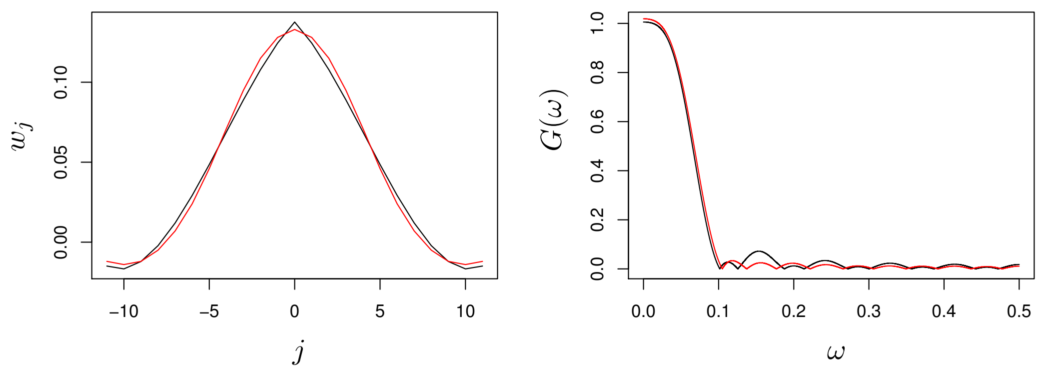
Filter length	23	25	27	29	31
m	11	12	13	14	15
b	12.37	13.28	14.20	15.11	16.02

bandwidth parameter b and the filter length m is strictly linear, such that, for all possible values of m

$$b = 2.42 + 0.91m.$$

Figure 3 illustrates the weight systems and the corresponding gain functions for the 23-term triangle filters, based on $b = 12.37$, compared with the SLF. The behavior of the other filters is similar, and not reported for space reasons.

Figure 3: Weight system (*left*) and gain function (*right*) of the 23-term triangle kernel (*black*) and 23-term SLF (*red*).



3.2 Asymmetric weights

The derivation of the symmetric triangle filter has assumed the availability of $2m + 1$ input values centered at t. However, at the end of the sample period, that is $t = N - (m+1), \dots, N$, only $2m, \dots, m+1$ observations are available, and asymmetric filters of the same length have to be considered. Hence, at the boundary, the effective domain of the kernel function K_3^T is $[-1, q^*]$, with $q^* << 1$, instead of $[-1, 1]$ as for any interior point. This implies that the symmetry of the kernel is lost, and it does not integrate to unity on the asymmetric support ($\int_{-1}^{q^*} K_3^T(t)dt \neq 1$). Furthermore, the moment conditions are not longer satisfied, that is $\int_{-1}^{q*} t^i K_3^T(t)dt \neq 0$, for $i = 1, 2, 3$. To overcome these limitations, Dagum and Bianconcini (2015) have suggested to follow the so-called "cut and normalize" method, according to which the boundary kernels $K_3^{Tq^*}$ are obtained by cutting the symmetric kernel K_3^T to omit that part of the function lying between q^* and 1, and by normalizing it on [-1, q^*]. That is,

$$K_3^{Tq^*}(t) = \frac{K_3^T(t)}{\int_{-1}^{q^*} K_3^T(t)dt}, \qquad t \in [-1, q^*]. \tag{9}$$

Applied to real data. the "cut and normalize" method yields the following formula for the asymmetric weights

$$w_{q,j} = \frac{K_3^{Tq^*}(j/b_q)}{\sum_{j=-m}^{q} K_3^{Tq^*}(j/b_q)}, \tag{10}$$

for $j = -m, \dots, q$, and $q = 0, \dots, m-1$, where $b_q, q = 0, \dots, m-1$, is the local bandwidth, specific for each asymmetric filter. As before, b_q allows us to relate the discrete domain of the filter, that is $\{-m, \dots, q\}$, for each $q = 0, \dots, m-1$, to the continuous domain of the kernel function, that is $[-1, q^*]$. Dagum and Bianconcini (2015) derive a class of optimal asymmetric filters based on bandwidth parameters $b_q, q = 0, \dots, m-1$, selected as follows

$$b_{q,G} = \min_{b_q} \sqrt{2 \int_0^{1/2} |G_q(\omega) - G(\omega)|^2 d\omega}, \tag{11}$$

where $G(\omega)$ is the gain function of the symmetric filter, whereas $G_q(\omega)$ is the one corresponding to the asymmetric filter $w_{q,j}, j = -m, \dots, q$. In this study, we consider these time-varying bandwidth parameters since they determine optimal filters that minimize revisions and time lag to detect the upcoming of a true turning point (Dagum and Bianconcini, 2015).

4. Empirical application

To analyze the performance of the proposed filters compared with the 13-term cascade one, we selected a sample of monthly series from U.S. economic and socioeconomic indicators that would give a good cross section of the various sectors as well as provide us with a representative sample of time series with regard to volatility. The volatility measure used was the noise to signal ratio (I/C) computed in the X13 program. This is the ratio of the average absolute change in the irregular and in the trend-cycle component. For the 40 analyzed series, 26 had I/C ratios below 1.0, and 14 were moderately volatile series in the I/C range between 1.0 and 3.5. The span of the series extends from January 2000 to August 2018 and thus it covers important recessionary periods. It is evident that major financial and economic global changes have introduced large variability in socio-economic indicators, and, for the majority of them, the trend-cycle has to be estimated using filters of length smaller than 13-term, as was done in the past. The data vintages are taken from the Real-Time Data Set for Macroeconomists (RTDSM, Croushore and Stark, 2001), a large dataset containing U.S. real-time data developed with cooperation between the Federal Reserve Bank of Philadelphia and the University of Richmond, and the ALFRED dataset developed by the St. Louis Federal Reserve Bank.

For the series with I/C smaller than one, we applied and compare the performance of the 9-term triangle and Musgrave filters with the 13-term cascade one, since for the latter this is the only available length. On the other hand, for series with higher I/C levels all the applied filters were of 13-terms. The comparison is performed in terms of revisions and time lag in detecting true turning points. Specifically, for the former, the reduction of the revision size in real time short-term trend estimates is evaluated by comparing the relative filter revisions between the final symmetric filter S and the last point asymmetric filter A, that is

$$R_t = \frac{S_t - A_t}{S_t}, \quad t = 1, \dots, N. \tag{12}$$

For each series and for each estimator, we calculate the ratio between the Mean Square Percentage Error (MSPE) of the revisions corresponding to the last point asymmetric filter derived following the RKHS methodology, those corresponding to the last point Musgrave filter, and those obtained by applying the last point asymmetric filter relate to the 13-term SLF. The results, illustrated in Table 4, show that the ratio, for series that required 9-term filters, is always smaller than one, indicating that the kernel last point predictor introduces smaller revisions than the Musgrave filter and the SLF. Particularly, compared with the latter, the ratios are smaller than 50%, indicating that it is necessary to select the appropriate length as suggested by the I/C ratio. This implies that the estimates obtained by the former will be more accurate than those derived by the application of the latter, when the appropriate filter length is not 13-term. However, when filters of 13-term are required, the triangle kernel still outperforms the cascade linear filter. It is important that the reduction of revisions in real time trend-cycle estimates is not achieved at the expense of increasing the time lag to detect the upcoming of a true turning point. A turning point is generally defined to occur at time t if (*downturn*):

$$y_{t-k} \leq \cdots \leq y_{t-1} > y_t \geq y_{t+1} \geq \cdots \geq y_{t+m}$$

Table 4: (Average) ratio of the MSPE of the revisions for the last point asymmetric filters based on appropriate triangle kernel and the last point Musgrave filter, and the last point filter related to the 13-term SLF, and (average) time lag in detecting true turning points for series with I/C ratio below 1.0 (9-term filter appropriate) and with I/C ratio above 1.0 (13-term filter appropriate)

	Revisions		**Lag TP detection**		
	$\frac{K_3^{T0}}{Musgrave}$	$\frac{K_3^{T0}}{SLF}$	K_3^{T0}	Musgrave	SLF
Average - series with I/C ratio below 1.0	0.423	0.504	1.346	2.25	3.048
Average - series with I/C ratio above 1.0	0.437	0.936	1.429	2.846	2.143

or (*upturn*)

$$y_{t-k} \geq \cdots \geq y_{t-1} < y_t \leq y_{t+1} \leq \cdots \leq y_{t+m}.$$

Following Zellner et al. (1991), we have chosen $k = 3$ and $m = 1$ given the smoothness of the trend-cycle data. To determine the time lag needed by an indicator to detect a true turning point we calculate the number of months it takes for the real time trend-cycle estimate to signal a turning point in the same position as in the final trend-cycle series. For the series analyzed in this paper, the average time delays are shown in Table 4 for series with I/C ratio below and above 1.0. The filters derived using the RKHS methodology always perform better than the Musgrave filters, independently on the filter length, and it outperforms the SLF also when 13-term filters are the most appropriate for the series under investigation. It generally takes less than two months, on average, to detect a true turning point.

REFERENCES

Croushore D. and Stark T. (2001). A real-time data set for macroeconomists. *Journal of Econometrics*. 105 (1), pp. 111-130.

Dagum E. B. (1996). A New Method to Reduce Unwanted Ripples and Revisions in Trend-Cycle Estimates from X11ARIMA. *Survey Methodology*. 22, pp. 77-83.

Dagum E.B. and Bianconcini S. (2008).The Henderson smoother in reproducing kernel Hilbert space. *Journal of Business and Economic Statistics*. 26, pp. 536 - 545.

Dagum E.B. and Bianconcini S. (2015). A new set of asymmetric filters for tracking the short-term trend in real time. *The Annals of Applied Statistics*. 9(3), pp. 1433-1458.

Dagum E.B. and Luati A. (2009). A cascade linear filter to reduce revisions and turning points for real time trend-cycle estimation. *Econometric Reviews*. 28(1-3), pp. 40-59.

Findley D. F., Monsell B. C., Bell W. R., Otto, M. C. and Chen, B. (1998). New capabilities and methods of the X12ARIMA seasonal adjustment program. *Journal of Business and Economic Statistics*. 16 (2), pp. 127-152.

Henderson R. (1916). Note on graduation by adjusted average. *Transaction of Actuarial Society of America*. 17, pp. 43-48.

Loader C. (1999). *Local regression and likelihood*. New York: Springer.

Moore G. H. (1961). *Business cycle indicators*. Princeton University Press. Princeton, NJ.

Musgrave J. (1964). A set of end weights to end all end weights. Working paper.*U.S. Bureau of Census*. Washington D.C.

Shiskin J., Young A. and Musgrave J. (1967). *The X-11 variant of the Census Method II seasonal adjustment program*. Technical Paper 15. US Department of Commerce, Bureau of the Census. Washington DC.

Zellner A., Hong C. and Min C. (1991). Forecasting turning points in international output growth rates using Bayesian exponentially weighted autoregression, time-varying parameter, and pooling techniques. *Journal of Econometrics*. 49(1-2), pp. 275-304.