On the Maximum Likelihood Estimation for Progressively Censored Lifetimes from Constant-stress and Step-stress Accelerated Tests

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Abstract
In order to gather the information about the lifetime distribution of a product, a standard life testing method at normal operating conditions is not practical when the product has an extremely long lifespan. Accelerated life testing solves this difficult issue by subjecting the test units at higher stress levels than normal for quicker and more failure data. The lifetime at the design stress is then estimated through extrapolation using an appropriate regression model. Estimation of the regression parameters based on exponentially distributed lifetimes from accelerated life tests has been considered by a number of authors using numerical methods but without systematic or analytical validation. In this article, we propose an alternative approach based on a simple and easy-to-apply graphical method, which also establishes the existence and uniqueness of the maximum likelihood estimates for constant-stress and step-stress accelerated life tests under progressive censorings.

Key Words: accelerated life tests, constant-stress loading, exponential distribution, maximum likelihood estimation, progressive censoring, step-stress loading

1. Introduction

With ever increasing reliability and substantially long life-spans of products, it is often very difficult for standard life testing methods under normal operating conditions to obtain sufficient information about the failure time distribution of the products. This practical difficulty is overcome by accelerated life test (ALT). By subjecting test units to higher stress levels than normal, the ALT collects more failure data in a shorter period of time. By applying more severe stresses, ALT collects information on the parameters of lifetime distributions more quickly. The lifetime at the normal operating stress can be estimated through extrapolation using an appropriate stress-response regression model. Some key references in the area of ALT include Nelson (1980), Meeker and Escobar (1998), and Bagdonavicius and Nikulin (2002).

The parameter estimation and design optimization for the ALT models have been discussed by numerous authors over the decades; see, for instance, Miller and Nelson (1983), Bai et al. (1989), Leemis et al. (1990), Bagdonavicius and Nikulin (1997), Han et al. (2006), Balakrishnan and Han (2008, 2009), Balakrishnan et al. (2010), Laronde et al. (2010), Han and Balakrishnan (2010), Wu and Huang (2010), Han and Ng (2013), Sha and Pan (2014), Han and Kundu (2015), Ismail (2016), and Han (2015, 2017). In the literature as noted by Balakrishnan and Kateri (2008), the estimation problem has been approached by different techniques including probability plotting, method of moments, and maximum likelihood estimation (MLE). In particular, the MLE requires solving a series of likelihood equations computationally. Since the solution is numerical in nature, one needs to address the issues of existence and uniqueness of the estimates, which get quite involved in the case of progressive censoring. Studying the existence and uniqueness of the estimates are not only theoretically but also practically important in order to guarantee the estimability under general settings as well as to develop and implement an efficient computational estimation algorithm. In this article, a simple graphical method is proposed for determination of the MLE of the regression slope parameter for the general k-level constant-stress and

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step-stress ALT under progressive Type-I and Type-II censorings. This approach ensures the existence and uniqueness of the MLE as well.

It is assumed that the physical relationship between the mean lifetime parameter and stress level is log-linear along with the accelerated failure time (AFT) model for the effect of changing stress in step-stress ALT. For deriving the analytical tractable results, it is further assumed that the lifetimes are exponentially distributed at each stress level. Although simple, the exponential distribution is a very good approximate model for many practical applications, including the decay time of a radioactive particle, the waiting time for service calls, the default time in credit risk modeling, and the distance between mutations on a DNA strand. In electrical and mechanical engineering, it has been successfully used to model the lifetime of an electric circuit and a semiconductor. Reliability theory and reliability engineering also make extensive use of the exponential distribution since its memoryless property renders it well-suited for modeling the constant hazard rate portion of the bathtub curve. More importantly, its statistical property serves as a theoretical proof of concept for other popular lifetime distributions such as gamma and Weibull, which is also the case based on the research outcomes of this study.

Here we also consider a generalized form of censoring known as *progressive censoring*, which has attracted considerable attention in the reliability literature for its efficient exploitation of the available resources in comparison to traditional designs. There are two fundamental censoring schemes: Type-I and Type-II. Progressive Type-I censoring occurs when a prefixed number of surviving units are continuously removed during the experiment at the end of each pre-specified time interval. On the other hand, progressive Type-II censoring corresponds to the situation where a prefixed number of surviving units are continuously withdrawn from the experiment at each observed failure time until the pre-specified number of units have failed; see Balakrishnan et al. (2010) for more details. Both censoring schemes provide greater flexibility to the experimenter in the design stage by allowing removal of test units at non-terminal time points. Those withdrawn unfailed test units could be used in other experiments in the same or at a different facility. As special cases, when no intermediate censoring takes place but the censoring is allowed only at the terminal time point of an experiment, it reduces to the conventional Type-I and Type-II censorings, respectively.

The rest of the paper is organized as follows. Section 2 presents the model descriptions and formulations for k-level constant-stress ALT and step-stress ALT under progressive Type-I and Type-II censorings. The MLEs of the model parameters are then derived, and the proposed estimation procedure is described in Section 3 under the unified structure of the likelihoods. Section 4 illustrates the proposed method using a real dataset. Finally, Section 5 is devoted to some concluding remarks.

2. Model Descriptions and MLE

Let $s(t)$ be the given stress loading (a deterministic function of time) for ALT. Also, let s_H be an upper bound of stress level and s_U be the normal use-stress level. The standardized stress loading is then defined as

$$x(t) = \frac{s(t) - s_U}{s_H - s_U}, \qquad t \geq 0$$

so that the range of $x(t)$ is $[0, 1]$. Now, let us define $0 \equiv x_0 \leq x_1 < x_2 < \cdots < x_k \leq 1$ to be the ordered k standardized stress levels to be used in the test. It is further assumed that under any stress level x_i, the lifetime of a test unit follows an exponential distribution whose probability density function (PDF) and cumulative distribution function (CDF) are

$$f_i(t) = \frac{1}{\theta_i}\exp\left(-\frac{t}{\theta_i}\right), \qquad 0 < t < \infty, \tag{1}$$

$$F_i(t) = 1 - S_i(t) = 1 - \exp\left(-\frac{t}{\theta_i}\right), \qquad 0 < t < \infty, \tag{2}$$

respectively. Also, it is assumed that under any stress level x_i, the mean time to failure (MTTF) of a test unit, θ_i, is a log-linear function of stress given by

$$\log \theta_i = \alpha + \beta x_i, \tag{3}$$

where the regression parameters α and β need to be estimated. The log-linear relationship is a commonly used and well-studied model for the accelerated exponential distribution model. Along with its simplicity, the log-linear link represents several significant life-stress relationships built from physical principles such as Arrhenius, inverse power law, Eyring, temperature-humidity, and temperature-non-thermal; see Miller and Nelson (1983).

Here we consider two popular classes of ALT: constant-stress and step-stress. In constant-stress testing, a unit is tested at a fixed stress level until failure occurs or the life test is terminated, whichever comes first. On the other hand, (step-up) step-stress testing allows the experimenter to gradually increase the stress levels at some prefixed time points during the test. The following subsections present the likelihoods and the MLEs of α and β for general k-level constant-stress ALT and step-stress ALT under (progressive) Type-I and Type-II censorings. For simplicity, no notational distinction is made in this article between the random variables and their corresponding realizations. Also, we adopt the usual conventions that $\sum_{j=m}^{m-1} a_j \equiv 0$ and $\prod_{j=m}^{m-1} a_j \equiv 1$.

2.1 *k-level step-stress test under progressive Type-I censoring*

For $i = 1, 2, \dots, k$, let n_i denote the (random) number of units failed at stress level x_i in time interval $[\tau_{i-1}, \tau_i))$. Let $y_{i,l}$ denote the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \dots, n_i$ while c_i denotes the number of units censored at time τ_i. Furthermore, let N_i denote the number of units operating and remaining on test at the start of stress level x_i. That is, $N_i = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} c_j$. Then, a step-stress ALT under progressive Type-I censoring proceeds as follows. A total of $N_1 \equiv n$ test units is initially placed at stress level x_1 and tested until time τ_1 at which point c_1 live items are arbitrarily withdrawn from the test and the stress is changed to x_2. The test is continued on $N_2 = n - n_1 - c_1$ units until time τ_2, when c_2 items are withdrawn from the test and the stress is changed to x_3, and so on. Finally, at time τ_k, all the surviving items are withdrawn, thereby terminating the life test. Note that since $n \equiv \sum_{i=1}^{k}(n_i + c_i)$, the number of surviving items at time τ_k is $c_k = n - \sum_{i=1}^{k} n_i - \sum_{i=1}^{k-1} c_i = N_k - n_k$. Obviously, when there is no intermediate censoring ($viz.$, $c_1 = c_2 = \cdots = c_{k-1} = 0$), this situation corresponds to the k-level step-stress ALT under conventional Type-I right censoring as a special case. When there is no right censoring ($viz.$, $\tau_k = \infty$ and $n_k = N_k$), this situation corresponds to the k-level step-stress testing under complete sampling as a special case.

It is noted that unlike progressive Type-II censoring, prefixing the progressive Type-I censoring scheme $(c_1, c_2, \dots, c_{k-1})$ has an inherent mathematical lapse due to a non-zero probability that all the units could fail before reaching the last stress level x_k, resulting in an early termination of the test as well as failing to fully implement the censoring scheme. To ensure the feasibility of progressive Type-I censoring, Balakrishnan and Han (2009) proposed a simple adjustment, which is to determine a sequence of a fixed proportion of surviving items to be censored at the end of each stress level x_i, denoted by $(\pi_1^*, \pi_2^*, \dots, \pi_{k-1}^*)$

with $0 \leq \pi_i^* < 1$. Then, the actual number of items withdrawn at the end of x_i is determined by $c_i = \Upsilon((N_i - n_i)\pi_i^*)$, where $\Upsilon(\cdot)$ is a discretizing function of choice to transform its argument to a whole number. It could be $round(\cdot)$, $trunc(\cdot)$, $floor(\cdot)$, or $ceiling(\cdot)$, for example. This adjustment essentially allows the ALT to terminate prior to reaching the final level x_k. As the number of surviving items at the end of each level before censoring occurs is random, the actual censoring scheme $(c_1, c_2, \ldots, c_{k-1})$ is also random through this modification. Another practical modification suggested is first to determine a sequence of a fixed number of units to be censored at the end of each stress level x_i, say $(c_1^*, c_2^*, \ldots, c_{k-1}^*)$ with $c_i^* \geq 0$ and $\sum_{i=1}^{k-1} c_i^* < n$. Then, the actual number of units withdrawn at the end of stress level x_i is determined by $c_i = \min\left\{c_i^*, N_i - n_i\right\}$. In case the number of remaining units at any time point of censoring is at most the prefixed number of items to be withdrawn at that time point, every surviving and operating item is withdrawn and the ALT is terminated. Hence, this modification also allows an earlier termination of the ALT whenever the number of the items remaining on the ALT is insufficient. Again, as the number of the functioning items at the end of each stress level prior to censoring is random, the actual censoring scheme $(c_1, c_2, \ldots, c_{k-1})$ is essentially random as well.

Since the step-stress loading is non-constant stress loading, an additional assumption is required to represent the effect of changing stress. The AFT model, also referred to as the additive accumulative damage model, is often appropriate as it generalizes several well-known models in reliability engineering for the exponential distribution, including the basic (linear) cumulative exposure model and the PH model; see Leo and Mikhail (2007). Now, under the AFT model along with the assumption of exponentiality, the PDF and CDF of a test unit are

$$f(t) = \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] f_i(t-\tau_{i-1}) \qquad \text{if} \begin{cases} \tau_{i-1} \leq t \leq \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \leq t < \infty & \text{for } i = k \end{cases} \tag{4}$$

$$F(t) = 1 - \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] S_i(t-\tau_{i-1}) \qquad \text{if} \begin{cases} \tau_{i-1} \leq t \leq \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \leq t < \infty & \text{for } i = k \end{cases} \tag{5}$$

where $\Delta_j = \tau_j - \tau_{j-1}$ is the step duration at stress level x_j, and $f_i(t)$ and $F_i(t)$ are as given in (1) and (2), respectively. Then, using (4) and (5), the joint distribution function of $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k})$ with $\mathbf{y_i} = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ is obtained as

$$f_J(\mathbf{y}, \mathbf{n}) = \mathcal{C}\left[\prod_{i=1}^{k} \theta_i^{-n_i}\right] \exp\left(-\sum_{i=1}^{k} \frac{U_i}{\theta_i}\right), \tag{6}$$

where

$$\mathcal{C} = \prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!},$$

$$U_i = \sum_{l=1}^{n_i} (y_{i,l} - \tau_{i-1}) + (N_i - n_i)\Delta_i \tag{7}$$

for $i = 1, 2, \ldots, k$. The detailed derivation of (6) is similar to Balakrishnan and Han (2009). Note that U_i in (7) is the *Total Time on Test* statistic at stress level x_i. Now, using (6) and the log-linear link given in (3), the log-likelihood function of (α, β) can be written as

$$l(\alpha,\beta) = -\alpha\sum_{i=1}^{k} n_i - \beta\sum_{i=1}^{k} n_i x_i - \sum_{i=1}^{k} U_i \exp\big[-(\alpha+\beta x_i)\big]. \tag{8}$$

Upon differentiating (8) with respect to α and β, the MLEs $\hat{\alpha}$ and $\hat{\beta}$ are obtained as simultaneous solutions to the following two equations:

$$\left[\sum_{i=1}^{k} n_i\right]\left[\sum_{i=1}^{k} U_i x_i \exp(-\hat{\beta}x_i)\right] = \left[\sum_{i=1}^{k} n_i x_i\right]\left[\sum_{i=1}^{k} U_i \exp(-\hat{\beta}x_i)\right], \tag{9}$$

$$\hat{\alpha} = \log\left(\frac{\sum_{i=1}^{k} U_i \exp(-\hat{\beta}x_i)}{\sum_{i=1}^{k} n_i}\right). \tag{10}$$

2.2 *k-level constant-stress test under Type-I censoring*

For illustrative simplicity, let us consider the procedure of a constant-stress ALT under Type-I censoring. A constant-stress ALT under progressive Type-I censoring can be described in a similar manner like in the previous subsection by introducing a set of time points for intermediate censoring. For $i = 1, 2, \ldots, k$, N_i units are allocated on test at stress level x_i such that $\sum_{i=1}^{k} N_i = n$. The allocated units are then tested until time τ_i at which point all the surviving items are withdrawn, thereby terminating the life test. Let n_i denote the (random) number of units failed at stress level x_i in time interval $[0, \tau_i)$ and $y_{i,l}$ denote the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \ldots, n_i$ while $N_i - n_i$ denotes the number of units censored at time τ_i. Obviously, when there is no right censoring ($viz.$, $\tau_i = \infty$ and $n_i = N_i$), this situation corresponds to the k-level constant-stress ALT under complete sampling as a special case.

Then, using (1) and (2), the joint distribution function of $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k})$ with $\mathbf{y_i} = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ is obtained as in (6) where

$$U_i = \sum_{l=1}^{n_i} y_{i,l} + (N_i - n_i)\tau_i, \qquad i = 1, 2, \ldots, k. \tag{11}$$

Again, note that U_i in (11) is the *Total Time on Test* statistic at stress level x_i. Using (6) and the log-linear link in (3), the log-likelihood function of (α, β) can be written as in (8) and as a result, we obtain the MLEs $\hat{\alpha}$ and $\hat{\beta}$ as simultaneous solutions to (9) and (10) with U_i given in (11).

2.3 *k-level constant-stress test under progressive Type-II censoring*

Let us now describe the procedure of a constant-stress ALT under progressive Type-II censoring. For $i = 1, 2, \ldots, k$, N_i units are allocated on test at stress level x_i such that $\sum_{i=1}^{k} N_i = n$. Let n_i denote the prefixed number of failure times to be observed at x_i along with the progressive censoring scheme given by $\mathbf{R_i} = (R_{i,1}, R_{i,2}, \ldots, R_{i,n_i})$. Also, let $y_{i,l}$ denote the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \ldots, n_i$. Then, a constant-stress ALT under progressive Type-II censoring proceeds as follows. At stress level x_i, N_i units are tested until the first failure time $y_{i,1}$ at which $R_{i,1}$ live items are arbitrarily withdrawn from the test. The test continues until the second failure time $y_{i,2}$ at which $R_{i,2}$ items are withdrawn from the test, and so on. Finally, at the n_i-th failure time y_{i,n_i}, all the surviving items are withdrawn, thereby terminating the life test. Note that since $N_i \equiv n_i + \sum_{l=1}^{n_i} R_{i,l}$, the number of items censored at the n_i-th failure time is $R_{i,n_i} = N_i - n_i - \sum_{l=1}^{n_i-1} R_{i,l}$. Obviously, when there is no intermediate censoring ($viz.$,

$R_{i,1} = R_{i,2} = \cdots = R_{i,n_i-1} = 0$), this situation corresponds to the k-level constant-stress ALT under conventional Type-II right censoring as a special case. When there is no right censoring ($viz.$, $n_i = N_i$), this situation corresponds to the k-level constant-stress testing under complete sampling as a special case.

Then, using (1) and (2), the joint distribution function of $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k})$ with $\mathbf{y_i} = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ is obtained as

$$f_J(\mathbf{y}) = \mathcal{C}\left[\prod_{i=1}^{k} \theta_i^{-n_i}\right] \exp\left(-\sum_{i=1}^{k} \frac{U_i}{\theta_i}\right), \tag{12}$$

where

$$\begin{aligned} \mathcal{C} &= \prod_{i=1}^{k} \prod_{j=0}^{n_i-1} \left(N_i - j - \sum_{l=1}^{j} R_{i,l}\right), \\ U_i &= \sum_{l=1}^{n_i} y_{i,l}(1 + R_{i,l}) \end{aligned} \tag{13}$$

for $i = 1, 2, \ldots, k$. Note that the structure of (12) is identical to (6). Also, U_i in (13) is the *Total Time on Test* statistic at stress level x_i. Using (12) and the log-linear link in (3), the log-likelihood function of (α, β) can be written as in (8) and as a result, we obtain the MLEs $\hat{\alpha}$ and $\hat{\beta}$ as simultaneous solutions to (9) and (10) with U_i given in (13).

2.4 *k-level step-stress test under progressive Type-II censoring*

For $i = 1, 2, \ldots, k$, let n_i denote the (random) number of units failed at stress level x_i in time interval $[\tau_{i-1}, \tau_i))$ such that the total number of failure observations is fixed at $n_T \leq n$ ($viz.$, $n_T = \sum_{i=1}^{k} n_i$) along with the progressive censoring scheme specified by $\mathbf{R} = (R_1, R_2, \ldots, R_{n_T})$. Also, let $y_{i,l}$ denote the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \ldots, n_i$ while N_i denotes the number of units operating and remaining on test at the start of stress level x_i. That is, $N_i = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} \sum_{l=1}^{n_i} R_{j,l}$ where $\mathbf{R}^* = (\mathbf{R_1}, \mathbf{R_2}, \ldots, \mathbf{R_k})$ with $\mathbf{R_i} = (R_{i,1}, R_{i,2}, \ldots, R_{i,n_i})$ such that $\mathbf{R}$ matches with the first n_T elements of $\mathbf{R}^*$. Then, a step-stress ALT under progressive Type-II censoring proceeds as follows. A total of $N_1 \equiv n$ test units is initially placed at stress level x_1 and tested until the first failure time $y_{1,1}$ at which $R_{1,1} \equiv R_1$ live items are arbitrarily withdrawn from the test. The test continues until the second failure time $y_{1,2}$ at which $R_{1,2} \equiv R_2$ items are withdrawn, and so on. During this process, if the testing time reaches τ_1, the stress is changed to x_2. The test continues until time τ_2 at which the stress is changed to x_3, and so on. Finally, at the n_T-th failure time, all the surviving items are withdrawn, thereby terminating the life test. Note that since $n \equiv n_T + \sum_{i=1}^{n_T} R_i$, the number of items censored at the n_T-th failure time is $R_{n_T} = n - n_T - \sum_{i=1}^{n_T-1} R_i$. When there is no intermediate censoring ($viz.$, $R_1 = R_2 = \cdots = R_{n_T-1} = 0$), this situation corresponds to the k-level step-stress ALT under conventional Type-II right censoring as a special case. When there is no right censoring ($viz.$, $n_T = n$), this situation corresponds to the k-level step-stress testing under complete sampling as a special case.

Using (4) and (5), the joint distribution function of $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k})$ with $\mathbf{y_i} =$

$(y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ is obtained as in (12) where

$$\begin{aligned} \mathcal{C} &= \prod_{j=0}^{n_T-1} \left(n - j - \sum_{i=1}^{j} R_i\right), \\ U_i &= \sum_{l=1}^{n_i} (y_{i,l} - \tau_{i-1})(1 + R_{i,l}) + N_{i+1}\Delta_i \end{aligned} \tag{14}$$

for $i = 1, 2, \ldots, k$ with $\Delta_i = \tau_i - \tau_{i-1}$ being the step duration at stress level x_i. Again, U_i in (14) is the *Total Time on Test* statistic at stress level x_i. Using (12) and the log-linear link in (3), the log-likelihood function of (α, β) can be written as in (8) and as a result, we obtain the MLEs $\hat{\alpha}$ and $\hat{\beta}$ as simultaneous solutions to (9) and (10) with U_i given in (14).

3. Determination of the MLE

We can see from (9) and (10) that for the existence of $\hat{\beta}$ and also of $\hat{\alpha}$, at least one failure has to be observed from at least two different stress levels. Otherwise, the parameters are not estimable. Under such a condition, to prove the existence and uniqueness of the MLEs of α and β, let us express (9) alternatively as

$$\frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i} = \frac{\sum_{i=1}^{k} U_i x_i \exp\left(-\beta x_i\right)}{\sum_{i=1}^{k} U_i \exp\left(-\beta x_i\right)}, \tag{15}$$

whose RHS is denoted by $H(\beta; \mathbf{x}, \mathbf{U})$. We will show that for given $\mathbf{x}$ and $\mathbf{U}$, $H(\beta; \mathbf{x}, \mathbf{U})$ is a monotone decreasing function of β with a limit smaller than LHS of (15) as $\beta \to +\infty$ and with a limit greater than LHS of (15) as $\beta \to -\infty$. Since LHS of (15) is a constant, it then follows that the plots of $\sum_{i=1}^{k} n_i x_i / \sum_{i=1}^{k} n_i$ and $H(\beta; \mathbf{x}, \mathbf{U})$ would intersect exactly once, at the MLE of β. This intersection guarantees the unique existence of $\hat{\beta}$ and also of $\hat{\alpha}$ from (10).

For this purpose, we have to ensure that

$$\frac{\partial}{\partial \beta} H(\beta; \mathbf{x}, \mathbf{U}) = \frac{h(\beta; \mathbf{x}, \mathbf{U})}{\left[\sum_{i=1}^{k} U_i \exp\left(-\beta x_i\right)\right]^2} \leq 0,$$

or equivalently that $h(\beta; \mathbf{x}, \mathbf{U}) \leq 0$ where

$$h(\beta; \mathbf{x}, \mathbf{U}) = -\left[\sum_{i=1}^{k} U_i x_i^2 \exp(-\beta x_i)\right]\left[\sum_{i=1}^{k} U_i \exp(-\beta x_i)\right] + \left[\sum_{i=1}^{k} U_i x_i \exp(-\beta x_i)\right]^2. \tag{16}$$

Setting $a_i = x_i \sqrt{U_i \exp(-\beta x_i)}$ and $b_i = \sqrt{U_i \exp(-\beta x_i)}$ for $i = 1, 2, \ldots, k$, (16) can be expressed as

$$h(\beta; \mathbf{x}, \mathbf{U}) = -\sum_{i=1}^{k} a_i^2 \sum_{i=1}^{k} b_i^2 + \left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq 0$$

by the Cauchy-Schwarz inequality, which establishes the required property that $H(\beta; \mathbf{x}, \mathbf{U})$ is indeed a monotone decreasing function of β. It is also observed that the limits for $H(\beta; \mathbf{x}, \mathbf{U})$ are

$$\begin{aligned} \lim_{\beta \to +\infty} H(\beta; \mathbf{x}, \mathbf{U}) &= x_1 \leq \frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i}, \\ \lim_{\beta \to -\infty} H(\beta; \mathbf{x}, \mathbf{U}) &= x_{k^*} \geq \frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i}, \end{aligned}$$

Table 1: Progressively Type-I censored dataset from $n = 30$ prototypes of a solar lighting device on a three-level step-stress ALT with $\tau_1 = 15$, $\tau_2 = 20$, and $\tau_3 = 25$

Failure Times at Temperature Level 1 ($x_1 = 0.1$)	Failure Times at Temperature Level 2 ($x_2 = 0.5$)	Failure Times at Temperature Level 3 ($x_3 = 0.9$)
1.515	15.164	20.318
2.225	15.355	21.228
4.629	15.953	21.543
4.654	16.735	24.541
6.349	18.796	
8.003	19.248	
8.262	19.295	
10.416		
11.381		
12.433		
14.755		
$n_1 = 11$ $c_1 = 4$	$n_2 = 7$ $c_2 = 1$	$n_3 = 4$ $c_3 = 3$
$n_{\oplus} = 22,\ c_{\oplus} = 8$		

where x_{k^*} is the observed last stress level when the life test is terminated. Thus, a plot of the LHS and RHS of (15) gives a simple graphical method of determining the MLE of the parameter β; see Figure 1. The proposed method is advantageous compared to the traditional methods for obtaining the MLE of the model parameters. The Newton-Raphson algorithm has been one of the standard procedures for the parameter estimation. In order to implement the Newton-Raphson procedure, however, it is necessary to acquire the second-order derivatives of the log-likelihood function, and this might be complicated under progressive censorings. This is a clear benefit of the proposed method since its simplicity does not require such derivations.

4. Illustrative Example

The graphical estimation method proposed here is illustrated with a real engineering case study. A three-level step-stress ALT was conducted under progressive Type-I censoring in order to assess the reliability characteristics of a solar lighting device, whose dominant failure mode is controller failure. Here, temperature is the stress factor whose level was changed during the test in the range of 293K to 353K with the normal operating temperature at 293K. The standardized stress loading was $x_1 = 0.1$, $x_2 = 0.5$, and $x_3 = 0.9$. The stress change time points were $\tau_1 = 15$ (in hundred hours) and $\tau_2 = 20$ (in hundred hours) with the censoring time point at $\tau_3 = 25$ (in hundred hours). The number of devices censored at $\tau_1 = 15$ and $\tau_2 = 20$ were $c_1 = 4$ and $c_2 = 1$, respectively, in order to utilize them for further engineering analyses and in other tests. The dataset obtained is presented in Table 1 and it consists of total $n_{\oplus} = 22$ failure times from the initial sample size of $n = 30$ prototypes ($i.e.$, 26.7% right censoring).

Initially, Weibull models with a constant shape parameter across different stress levels were fitted under the power law relationship but the inference for the shape parameter supported an exponential lifetime of the device at any constant temperature. Consistent

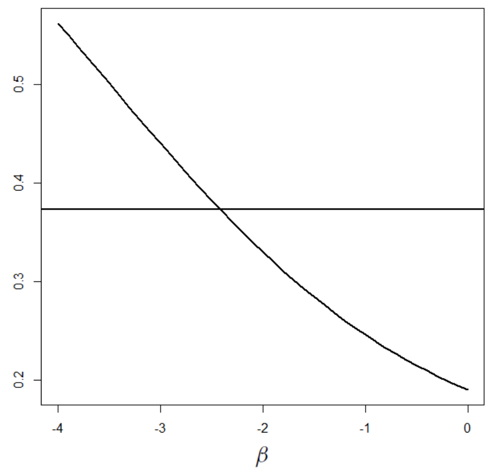

Figure 1: Plot of $H(\beta; \mathbf{x}, \mathbf{U})$ and LHS functions of (15) for the progressively Type-I censored data in Table 1

with our model assumption, fitting exponential distribution to the data with the log-linear parameter-stress relationship in (3), the estimation procedure described in Section 3 leads to a simple graphical solution of $\hat{\beta} = -2.41309$ with no need to use the Newton-Raphson method; see Figure 1. This in turn produces $\hat{\alpha} = 3.659685$ from (10).

5. Conclusion

In this work, a simpler estimation method was proposed for determination of the MLE of the regression slope parameter for the general k-level constant-stress and step-stress ALT under progressive Type-I and Type-II censorings. The unified structure of the likelihoods was provided upon using the popular physics-based log-linear link function between the mean lifetime parameter and the (transformed) stress level along with the AFT model for explaining the effect of changing stress levels in step-stress ALT. It was demonstrated that this proposed approach ensures the existence and uniqueness of the MLE. For analytical tractability, the derivations and numerical results presented in this work are based on the exponentially distributed lifetimes at each stress level. It is of practical interest to extend the results of this research to other types of censoring schemes and the failure data from other popular lifetime distributions containing non-scale parameters such as Weibull, extreme value, gamma, and lognormal. With added distributional parameters, it is challenging to assess the nature of the likelihood equations analytically but luckily, the existence and uniqueness of the MLE can be inferred based on the results reported in this work. For instance, if the failure times follow Weibull distributions with a common shape parameter across stress levels, a simple power transformation converts the lifetime distribution to an exponential, which has been discussed in this paper; see Balakrishnan and Kateri (2008) for example. Research in these direction is under progress and it is hoped to report these findings in future communications.

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