The Inequality Process' PDF Approximation Model For Heavy-Tailed Financial Distributions

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Abstract
The Inequality Process (IP) is an interacting particle system implying many empirical invariances in personal income and wealth statistics, seven maxims of economics, and five "stylized facts" of stock price volatility. The IP's particle parameter, omega, is the fraction of wealth (a positive quantity) an IP particle loses to a competitor particle in a loss. A gamma pdf (probability density function), whose shape and scale parameters are expressed in terms of particle omegas, approximates the stationary distribution of IP particle wealth in a particular omega equivalence class of particle if its omega < .5. This approximating gamma pdf implies scalar particle wealth statistics in that omega equivalence class, expressed in terms of particle parameters, as long as that class' omega < .5. This paper introduces an approximating variance-gamma (VG) pdf with its three parameters in terms of particle omegas, for 0.0 < omega < 1.0 . This generalization converges to the approximating gamma pdf model for omegas < .5. The VG pdf is widely used to model right skewed heavy-tailed financial distributions.

Key Words: financial distributions, heavy-tailed distributions, Inequality Process, interacting particle system

1.0 Introduction: The Inequality Process (IP)

The Inequality Process (IP) (Angle, 1983-2018) is a stochastic particle system abstracted from a foundational theory of economic anthropology, the Surplus Theory of Social Stratification. Randomly paired particles exchange a positive quantity, "wealth", in encounters that can be interpreted as competitive. The IP has been shown to quantitatively reproduce many empirical invariances in statistics of personal income and wealth (Appendix A.1), seven verbal maxims of mainstream economics (Appendix A.2), and five stock market empirical invariances (Appendix A.3). The apparent ubiquity of the IP, a parsimonious mathematical model, prompts the conjecture (Angle, 2018) that where survival depends on a unidimensional positive quantity, the Inequality Process is evolutionarily optimal. Money in industrialized economies approximates such a unidimensional positive quantity.

1.1 The Inequality Process' (IP's) Transition Equations

The IP is ergodic in a population all of whose particle parameters, ω_ψ are $\omega_\psi|0 < \omega_\psi < 1$, where ω_ψ is the particle parameter in the ω_ψ equivalence class of particles. Particle ψ gives up a ω_ψ fraction of its wealth when it loses an encounter with another particle. 1(a,b) are the transition equations for the transfer of wealth from one IP particle to another. One particle, i, is in the ω_ψ equivalence class; the other particle, j, is in the ω_θ equivalence class. Pairings where $\omega_\psi = \omega_\theta$ are not excluded. These rules of wealth transfer define the Inequality Process, and, in the long run, transfer more wealth to particles that lose less when the lose an encounter. Robust losers emerge as winners in the Inequality Process. The transition equations are:

$$\begin{aligned} x_{it} &= x_{i(t-1)} + d_t \omega_\theta x_{j(t-1)} - (1-d_t)\omega_\psi x_{i(t-1)} \\ x_{jt} &= x_{j(t-1)} - d_t \omega_\theta x_{j(t-1)} + (1-d_t)\omega_\psi x_{i(t-1)} \end{aligned} \qquad 1(a,b)$$

where:

$$\begin{aligned} x_{it} &\equiv \text{particle } i\text{'s wealth at time-step } t \text{ in multiples of} \\ & \quad \mu_t \text{, the unconditional mean of wealth} \\ x_{j(t-1)} &\equiv \text{particle } j\text{'s wealth at time-step } (t-1) \\ 0 < \omega_{\theta j} &< 1.0\text{, fraction lost in loss by particle } j \\ 0 < \omega_{\psi i} &< 1.0\text{, fraction lost in loss by particle } i \\ d_t &= \text{an i.i.d. 0,1 uniform discrete r.v. equal to 1 with} \\ & \quad \text{probability .5 at time-step } t \text{ (a Bernoulli variable)} \\ \tilde{\omega}_t &\equiv \text{harmonic mean of } \omega_\psi\text{'s at time-step } t \\ \mu_t &= \text{unconditional mean of wealth at time-step } t \end{aligned} \qquad 1(c \text{ to } i)$$

Inequality Process particles have two traits:

a) a parameter, the fraction of wealth lost in a competitive encounter, that is permanent or semi-permanent, its ω, and

b) its wealth, which changes at each encounter with another particle. The Inequality Process is a pure jump process.

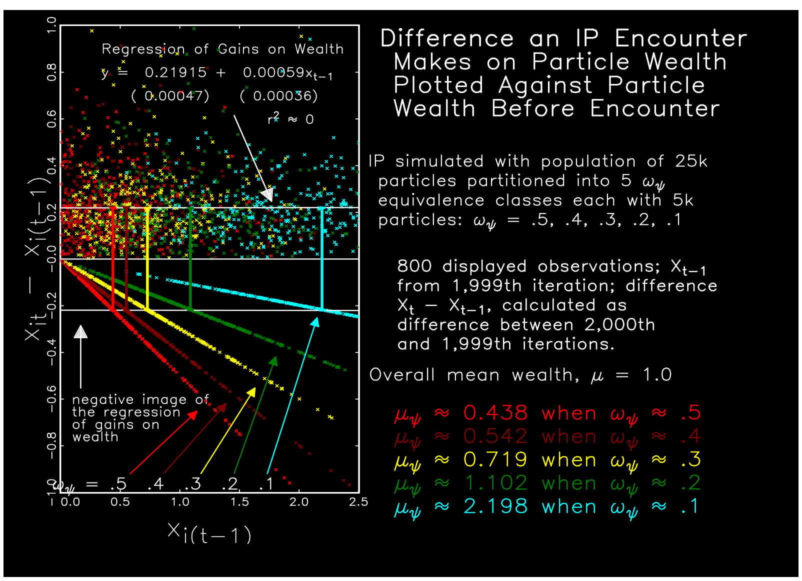

Figure 1, Plot of change in wealth in an IP encounter against particle wealth before the encounter. Without loss of generality, the unconditional mean of particle wealth, μ_t, is set to 1.0 . Note the asymmetry of gains and losses in terms of information content and the equality of losses with mean gains at mean particle wealth in each ω_ψ equivalence class.

1.2 The Gamma Probability Density Function (PDF) Model of the Inequality Process' Stationary Distribution in the ω_ψ Equivalence Class

The stationary distribution of particle wealth in the ω_ψ equivalence class of particle is well approximated by a two parameter gamma probability density function (pdf) as long as $0 < \omega_\psi < .5$. The gamma pdf's two parameters are its shape parameter, α, and its scale parameter, λ. See (2):

$$
\begin{aligned}
f(x) &\equiv \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \\
x &> 0 \\
\alpha &\equiv \text{shape parameter} \\
\lambda &\equiv \text{scale parameter}
\end{aligned}
\qquad \text{(2a,b,c,d)}
$$

where 2(a)'s shape (2c) and scale (2d) parameters, are expressed as:

$$
\begin{aligned}
x_{\psi t} &\equiv \text{wealth in the } \omega_\psi \text{ equivalence class in multiples of } \mu_t \\
x_{\psi t} &> 0 \\
\alpha_{\psi t} &\equiv \text{shape parameter} \approx \frac{1-\tilde{\omega}_t}{\omega_\psi} \\
\lambda_t &\equiv \text{scale parameter} \approx \frac{1-\tilde{\omega}_t}{\tilde{\omega}_t \mu_t} \\
\tilde{\omega}_t &\equiv \text{harmonic mean of } \omega_\psi\text{'s at time-step } t \\
\mu_t &= \text{unconditional mean of wealth at time-step } t
\end{aligned}
\qquad \text{(3a,b,c,d,e,f)}
$$

and where capital Ψ is the number of distinct ω_ψ equivalence classes and $w_{\psi t}$ the fraction of the particle population in the ω_ψ equivalence class:

$$
\tilde{\omega}_t \stackrel{\text{def}}{=} \left(\sum_{\psi=1}^{\Psi} \frac{w_{\psi t}}{\omega_\psi} \right)^{-1}
\qquad \text{(3g)}
$$

equation (2a) becomes (4) in the ω_ψ equivalence class of particles. (4) is the Gamma PDF Model of the distribution of particle wealth in the ω_ψ equivalence class of particle:

$$
f(x_{\psi t}) \equiv \frac{\lambda_t^{\alpha_{\psi t}}}{\Gamma(\alpha_{\psi t})} x_{\psi t}^{\alpha_{\psi t}-1} e^{-\lambda_t x_{\psi t}}
\qquad \text{(4)}
$$

1.3 The Estimator for Mean Particle Wealth $\mu_{\psi t}$, in the ω_ψ Equivalence Class in Terms of Particle Parameters

(5) is the estimator of mean particle wealth in the ω_ψ equivalence class, $\mu_{\psi t}$, in terms of particle parameters. (5) is derived independently of the Gamma PDF Model and is valid for the entire interval of ω_ψ, $0.0 < \omega_\psi < 1.0$. (5) is derived from the definition of the IP (1a,b) at $\mu_{\psi t}$ and the observation that the IP is ergodic when all ω_ψ s' in the population of particles are in the interval $0.0 < \omega_\psi < 1.0$.

$$\mu_{\psi t} = \left(\frac{\tilde{\omega}_t \, \mu_t}{\omega_\psi}\right) \quad (5)$$

(5) can be deduced from an examination of (1a,b) and Figure 1.

Numerically, (5) converges to estimated mean particle wealth in the ω_ψ equivalence class as the number of the particles in that and other equivalence classes increases. See Figure 2 in which the estimate based on (5) overlaps estimated particle wealth means in each of the five ω_ψ equivalence classes. Note that the accuracy of (5) does not deteriorate when ω_ψ increases past .5 .

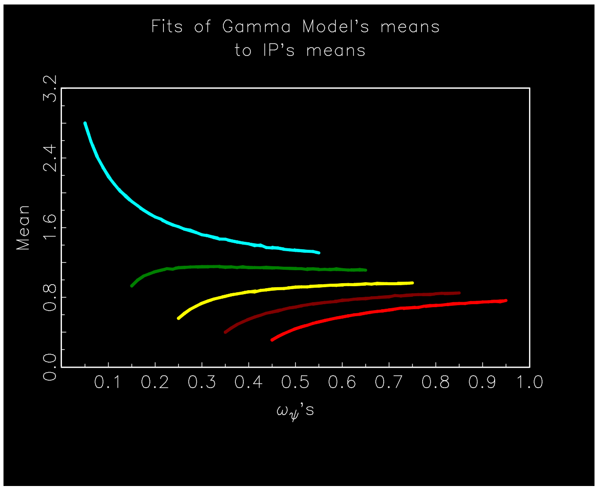

Figure 2, $\mu_t = 1$ here without loss of generality.

1.31 Inequality Process Simulations Against Which PDF Models are Tested

Simulation enables omniscience. In this paper, everything that can be known about a particle, i.e., its parameter, ω, and its current wealth, and the number of particles in each ω_ψ equivalence class is known in making calculations. Figure 2, 3, and 4 compare statistics of the wealth of particles in the ω_ψ equivalence class with the estimates of estimators derived from pdf models using knowledge of the ω's.

Five ω_ψ equivalence classes are distinguished in the 37 IP simulations generating estimates displayed in Figures 2, 3, and 4. Each of the 37 simulations of the IP has a vector of five particle parameters, ω_ψ's, one for the 1,000 particles in each ω_ψ equivalence class. The vector with the smallest particle parameters is (.05, .15, .25, .35, .45). The harmonic mean of the particle parameters in the whole population of 5,000 particles is known before an IP simulation with a particular vector of ω_ψ's begins. All 5,000 particles are involved in every IP simulation. Without loss of generality, the grand mean of particle wealth in the population of 5,000 particles is kept to 1.0 for both interpretability and computation reasons.

After 2,000 iterations of the Inequality Process with a particular vector of five ω_ψ's, the wealth of particles in all five ω_ψ equivalence classes is recorded and saved. Then after another 100 iterations of the Inequality Process, the wealth of all particles is again recorded and saved. This re-sampling of particle wealth goes on until there 10 observations on each particle's wealth. There are then 10,000 observations on particle wealth in each ω_ψ equivalence class in each IP simulation with a particular vector of particle parameters. Then the mean and variance of particle wealth each ω_ψ equivalence class are calculated from the 10,000 observations. The standard errors of estimate of these statistics are negligible.

The IP simulation with the ω_ψ vector (.05, .15, .25, .35, .45) provides the leftmost point estimates graphed in Figures 2, 3, and 4. After these point estimates have been calculated, the vector of ω_ψ equivalence classes is increased by adding .0125 to it. The next IP simulation and estimation of the mean and variance of particle wealth conditioned on ω_ψ is identical to the first simulation except that now the vector of ω_ψ's is (.0625, .1625, .2625, .3625, .4625), and so on through 35 more IP simulations, the last one of which is done with ω_ψ vector (.5, .6, .7, .8, .9). The 37 simulations of the IP were programmed in the GAUSS language (Aptech Systems, 2012).

1.4 The Gamma PDF Model Estimator of Particle Wealth Variance

Inferences made from the solution of the IP in terms of its parameters and previous stochastic events suggest approximating the IP's stationary distribution in the ω_ψ equivalence class by a gamma pdf. The gamma pdf, however, is not the stationary distribution implied by the IP, (1a,b).

1.41 Sketch of Proof that the IP Stationary Distributions are not Gamma PDF's

While Figure 3 shows that the approximation by the Gamma PDF Model to the variances of particle wealth in each ω_ψ equivalence class estimated directly from particle wealth amounts is adequate for many purposes, provided the ω_ψ's are less than .5, but Figure 3 shows this approximation is imperfect. Worse for the proposition that the IP of (1a,b)'s stationary distribution is gamma, is the proof that it is not, however close the approximation is as $\omega_\psi \to 0$.

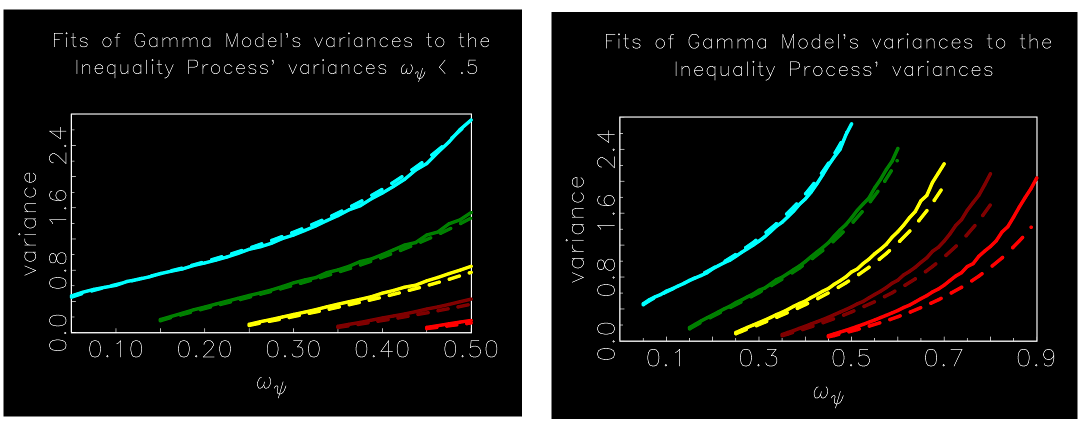

Figure 3, μ_t = 1.0 without loss of generality.

The gamma pdf is a maxentropic distribution, i.e., the result of maximizing the entropy statistic subject to two equality constraints. Boltzmann found the distribution of molecular kinetic energy in the kinetic theory of gases by maximizing the entropy statistic of kinetic energy subject to the constraint that kinetic energy is neither created nor destroyed, i.e., the sum of molecular energy after the collision of the two molecules equals their sum before. The IP of (1a,b) is a particle system quite similar (Angle, 1990) to that of the kinetic theory of gases. It has binary particle encounters in which a positive quantity is exchanged on a "zero sum" basis, i.e., no change in the sum of the positive quantities possessed by both particles. IP encounters between two IP particles preserve the positive sum, called 'wealth' in the IP, in their encounter. For the IP stationary distribution to be gamma, the IP encounters would have to preserve a second equality, i.e., the sum of the logarithm of the wealth of both particles to an encounter would have to remain unchanged by the encounter.

This second constraint does not hold in the IP. See Angle (1990) for the algebra. Thus, the stationary distribution implied by (1a,b) is not a gamma pdf. Gamma PDF Model approximations are satisfactory if $\omega_\psi < .5$ and $\tilde{\omega}_t < .5$. They become increasingly unsatisfactory if $\omega_\psi > .5$ as $\omega_\psi \rightarrow 1$ as Figure 3 shows. Its left panel shows the Gamma PDF Model's useful approximation to IP particle wealth variances when $\omega_\psi < .5$ and $\tilde{\omega}_t < .5$. The right panel of Figure 3 shows the Gamma PDF Model approximations become increasingly poor as $\omega_\psi > .5$ as $\omega_\psi \rightarrow 1$. Figure 3 is based on the same IP simulations as Figure 2.

1.42 The Gamma PDF Model Variance of Particle Wealth Expressed in Terms of Particle Parameters

Finding the gamma pdf variance of particle wealth in the ω_ψ equivalence class, $\text{variance}_{\Gamma\,\varphi}$, begins with the definition of the variance:

$$m_2 \stackrel{\text{def}}{=} \text{variance} \stackrel{\text{def}}{=} E[(x - E[x])^2] = E[x^2] - (E[x])^2 \quad (6)[1]$$

The Gamma PDF Model of the variance of particle wealth in the ω_ψ equivalence class, $m_{\Gamma\psi 2}$, is:

$$m_{\Gamma\psi 2} \stackrel{\text{def}}{=} \text{variance}_{\Gamma\psi} = \frac{\alpha_{\psi t}}{\lambda_t^2} \quad (6a)$$

When the shape and scale parameters of the gamma pdf are expressed in terms of IP particle parameters as in (2c,d):

$$m_{\mathbf{IP}\psi 2} \stackrel{\text{def}}{=} \text{variance}_{\mathbf{IP}\psi} \approx \frac{(\tilde{\omega}_t\, \mu_t)^2}{\omega_\psi\,(1-\tilde{\omega}_t)} = \frac{(\tilde{\omega}_t\, \mu_t)}{\omega_\psi} \cdot \frac{(\tilde{\omega}_t\, \mu_t)}{(1-\tilde{\omega}_t)} \quad (6b)$$

$$m_{\mathbf{IP}\psi 2} \stackrel{\text{def}}{=} \text{variance}_{\mathbf{IP}\psi} \approx \mu_{\psi t} \frac{(\tilde{\omega}_t\, \mu_t)}{(1-\tilde{\omega}_t)} \quad (6c)$$

(6c) says that the variance of particle wealth in the ω_ψ equivalence class in the Gamma PDF Model approximately equals the product of mean wealth in the ω_ψ equivalence class and the inverse of the expression for the Gamma PDF Model's scale parameter in terms of particle parameters.

2.0 A Generalization of the Gamma PDF Model for All ω_ψ| $0.0 < \omega_\psi < 1.0$

The distributions of the labor incomes of workers at particular levels of education in the U.S. and a number of other industrial countries are well modelled by the distribution of IP particle wealth in ω_ψ equivalence classes with ω_ψ varying inversely with level of worker education (Angle, 1990-2006). Because the ω_ψ 's of these IP fits are less than .5, the Gamma PDF Model can be used to both fit these distributions and estimate the ω_ψ 's of each labor income distribution conditioned on worker level of education and

[1]Finding the first and second gamma pdf moments around zero is an elementary integration. To find the first moment around zero, increment the exponent on x in (2a) by one and integrate over x from zero to positive infinity. To find the second moment around zero, increment the exponent on x in (2a) by two and integrate over x from zero to positive infinity.

all the statistics that can be expressed in terms of the gamma shape and scale parameters which, in turn, have approximations in terms of IP parameters. The Gamma PDF Model is convenient. It shows, for example, how and why time-series of the variety of statistics thought to be indicators of the concept 'inequality' covary in a particular way. The inverse relationship between ω_ψ and education level confirms the verbal theory from which the Inequality Process was abstracted that more productive workers are more sheltered in the competition for wealth (Angle, 1983, 1986).

The finding that the Inequality Process implies five aspects of stock prices (Angle, 2018, and Appendix A.3) lends urgency to finding a probability density function (pdf) for the IP when $\omega_\psi > .5$. Many financial distributions have heavier right tails than that of the gamma pdf (Resnick, 2007).

2.1 Desirability Constraints on Choice of a Generalization of the IP's Gamma PDF Model

A probability density function (pdf) as good as the Gamma PDF Model, or better, would satisfy the following six constraints:

1) The generalization of the Gamma PDF Model snugly fits IP particle wealth statistics when $\omega_\psi > .5$, including the variance and higher central moments of particle wealth as ω_ψ -> 1.0 .
2) The generalization converges to the Gamma PDF Model as ω_ψ -> 0;
3) The generalization is so closely related to the gamma pdf that the expressions for the gamma shape and scale parameters in terms of ω_ψ and $\tilde{\omega}$ are preserved as shape and scale parameters of the generalization.
4) If the generalization has a third parameter (beyond a shape and scale parameter), an expression can be found for the third pdf parameter in terms of IP particle parameters.
5) The generalization should have the same estimator for mean particle wealth in the ω_ψ equivalence class as equation (5), which is, based on numerical evidence, exact for all $\omega_\psi | 0.0 < \omega_\psi < 1.0$.
6) The generalization would be particularly welcome if it were a pdf already in use as a model of heavy-tailed financial distributions.

These six conditions might seem so tightly constraining that they prohibit finding such a generalization. There is, however, one pdf that satisfies constraints #1 – #3 and #5 and #6. It only remains to satisfy desirability constraint #4 (the expression of its parameter with no analogue in the gamma pdf in terms of IP particle parameters) to have the a completely satisfactory generalization of the Gamma PDF Model. This paper finds how constraint #4 is satisfied and, consequently, why the variance-gamma (VG) pdf is the sought after generalization of the Gamma PDF Model for all $\omega_\psi | 0.0 < \omega_\psi < 1.0$.

The variance-gamma (VG) pdf is widely used to model heavy right-tailed financial distributions. For example, Finlay (2009) constructed a variance-gamma (VG) pdf model that exhibited long range dependence in squared returns to stocks, one of the five "stylized facts" (empirical invariances) of the stock market that the Inequality Process implies (Angle, 2018). See also Filo (2009), Dilip and Seneta (1990), and Seneta (2004). No claim is made here that the variance-gamma pdf is the exact stationary distribution implied by the IP of (1a,b), only that it provides a good model of the mean and higher central moments of IP particle wealth when $\omega_\psi > .5$.

3.0 The Variance-Gamma Model of Particle Wealth Statistics in the Inequality Process for 0.0 < ω_ψ < 1.0

3.1 The Algebra of Generalizing the Gamma PDF Model of IP Statistics to the Variance-Gamma (VG) Model

Kotz et al. (2001) discuss the Laplace Distribution and its generalizations. The variance-gamma (VG) pdf is one of these generalizations. Kotz et al. (2001: 180) write: "In this book we use the terms Bessel function distribution and variance-gamma distribution interchangeably with the name generalized Laplace distribution ….". The variance-gamma (VG) pdf is also a generalization of the gamma pdf, as can be seen by comparing their characteristic functions, the $\Psi(t)$'s, (7) and (8). Equation (2) is the two parameter gamma pdf with shape and scale parameters not equated to functions of IP particle parameters. The characteristic function of the two parameter gamma pdf, $\Psi_\Gamma(t)$, is:

$$\Psi_\Gamma(t) \equiv \left(\frac{1}{1-\left(\frac{it}{\lambda}\right)}\right)^{\alpha} \quad (7)$$

where:

$\alpha \stackrel{\text{def}}{=} shape\ parameter > 0$
$\lambda \stackrel{\text{def}}{=} scale\ parameter > 0$

and $i = \sqrt{-1}$.

The characteristic function of the variance-gamma pdf (VG) equals that of the gamma pdf up to the extra (third) parameter in the denominator of the VG's characteristic function (8), σ^2. Kotz et al., (2001) gives VG's characteristic function as:

$$\Psi_{VG}(t) \equiv \left(\frac{1}{1+\frac{1}{2}\sigma^2\, t^2-i\mu t}\right)^{\tau} \quad (8)$$

where:

$\mu, \tau \in \mathbb{R}^+$(positive real numbers)

(8) is definition 4.1.1 of Kotz et al. (2001: 180). Kotz et al. (2002)'s notation is used here in all algebra quoted from Kotz et al. (2001). Their notation for the VG parameters is translated into gamma pdf parameters as:

$$\begin{aligned} \alpha &= \tau \\ 1/\lambda &= \mu \end{aligned} \quad (9)$$

σ^2 is not in (7) and so is a free VG parameter independent of τ and μ, without a formula translating it into gamma pdf parameters, as in (9) or into IP particle parameters as in (3c,d). The goal of this paper is to find an approximation to σ^2 in terms of IP particle parameters.

Since the Gamma PDF Model works well for $\omega_\psi < .5$ in the ω_ψ equivalence class of IP particles, a first guess at defining σ^2 in terms of IP particle parameters would be an expression in terms of IP particle parameters that reduces σ^2 toward zero as ω_ψ -> .5 from above, like a rheostat, a dimmer switch, unless the VG approximation is superior to that of the Gamma PDF Model for $\omega_\psi < .5$. Then the goal would be to reduce σ^2 as ω_ψ -> 0.0. If $\sigma = 0$, (8), the variance-gamma (VG) characteristic function, becomes (7), the gamma pdf characteristic function of the gamma pdf.

NB!: *the VG scale parameter μ in (8) and (9) should not be confused with the μ of equations (3) and (4), in which μ means the unconditional mean of particle wealth. The decision to keep Kotz et al. (2001)'s notation in expressions quoted from Kotz et al. (2001) necessitates using 'μ' to denote these two different quantities. Context makes clear what μ denotes in each instance.*

3.2 Fitting The Moments of Heavy-Tailed Distributions Rather Than The Distribution Itself

Although the variance-gamma (VG) pdf has a heavier right tail than the gamma pdf's, it is not practical to fit a VG pdf to a frequency distribution whose right tail is too heavy to be well approximated by a gamma pdf. Apart from the fact that the right tail of heavy-tailed distributions may be, in an extreme case, inherently difficult to fit a pdf to, the VG has the problem, noted in Kotz et al. (2001), that the VG pdf does not have a closed form for non-integer, τ, its shape parameter, the VG's analogue of the gamma pdf's shape parameter, α. The gamma pdf is defined for positive and real (i.e., continuous) α. (3c), the approximation to α in terms of IP particle parameters, is inherently non-integer.

But use of the VG as a model of distributions with heavy-right tails is not hampered by this fact, because the VG has a conveniently tractable moment generating function allowing real (continuous) τ. And, although the right tail of the VG is heavier than the corresponding gamma pdf's, the VG's right tail is not so heavy that it precludes the existence of finite higher moments. The VG's moments around zero, obtainable from its moment generating function, can be used to find central moments, the moments around its mean (e.g. variance, skew, and kurtosis). If all three of the VG's parameters have approximations in terms of IP particle parameters, the VG's central moments can be equated to the estimated central moments of IP particle wealth in each ω_ψ equivalence class. Thus estimates of the particle parameters can be estimated as easily from IP central moments by equating them to VG central moments, as they can be estimated from fitting the Gamma PDF Model to the distribution of IP particle wealth, or to an empirical relative frequency distribution.

The VG's moment generating function (Kotz et al., 2001:192) is:

$$E(Y^n) = \frac{1}{\sqrt{\pi}\,\Gamma(\tau)} \sum_{k=0}^{[[n/2]]} \binom{n}{2k} \sigma^{2k} \mu^{n-2k}\, 2^k\, \Gamma(1/2 + k)\, \Gamma(\tau + n - k) \tag{10}$$

If an approximation to σ^2 in terms of IP particle parameters can be found, (10) can generate VG moments expressed in terms of IP particle parameters. This paper announces such an expression for σ^2.

The first two variance-gamma moments around zero are:

$$E[x] = \tau\mu$$
$$E[x^2] = (\tau+1)\tau\mu^2 + \tau\sigma^2$$

(11a,b)

$$\tau \stackrel{\text{def}}{=} \text{the VG's shape parameter} = \text{the gamma pdf'shape parameter}, \alpha \approx \frac{1-\tilde{\omega}_t}{\omega_\psi}$$

(11c)

$$\mu \stackrel{\text{def}}{=} \text{the VG's scale parameter} = \text{the inverse of the gamma pdf's scale parameter}, \lambda \approx \left(\frac{\tilde{\omega}_t \ \mu_t}{1-\tilde{\omega}_t}\right)$$

(11d)

where the μ_t on the right side of (11d) is the unconditional mean of particle wealth.

The VG's estimator of mean particle wealth in the ω_ψ equivalence class (11a) equals the Gamma PDF Model's, α/λ, and are equated to (5), the estimator of mean particle wealth in the ω_ψ equivalence class. (5) is derived from properties of the IP (1a,b), and is, as far as can be told with numerical evidence, exact in a large population of IP particles for all ω_ψ, $0.0 < \omega_\psi < 1.0$.

$$\text{VG estimator of mean particle wealth}, \tau\mu = \text{gamma's estimator}, \frac{\alpha}{\lambda}$$

(11e)

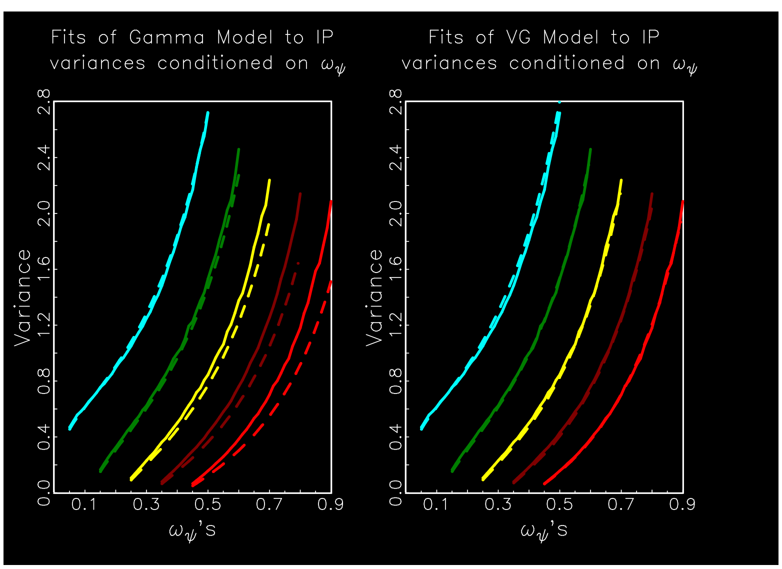

Figure 4

3.3 An Estimator For The Variance-Gamma PDF Parameter, σ^2, In Terms of ω_ψ and $\tilde{\omega}_t$

$$\sigma_\psi^2 \approx \omega_\psi^4 \left(\frac{\tilde{\omega}_t \; \mu_t}{1-\tilde{\omega}_t}\right) \tag{12}$$

(12) satisfies desirability constraint #1 (Section 2.1) on the generalization of the Gamma PDF Model, i.e., that of good fit for VG central moments which include σ^2 when $\omega_\psi > .5$. See Figure 4. Thus (12) satisfies constraint #4 on the generalization of the Gamma PDF that all parameters of the generalization have expression in terms of IP particles, constraint #4. That way IP particle parameters can be estimated from fits of VG central moments to central moments of empirical heavy-tailed distributions. (12) satisfies desirability constraint #2, convergence to the Gamma PDF Model as ω_ψ -> 0 since (12) works as a rheostat (dimmer switch) turning off σ^2 's contribution to the estimate of the variance of particle wealth in the ω_ψ equivalence class as ω_ψ -> 0 . (12) satisfies the other desirability constraints too. So, given (12), the Inequality Process (IP) now has a PDF model of statistics of competition among particles for wealth when that competition results in wealth losses of greater than 50%, i.e., $\omega_\psi > .5$, a pdf model in common use to model size distributions of stock market statistics, the variance-gamma (VG).

3.31 The VG Model's Estimator For The Variance of Particle Wealth In Terms of VG PDF Parameters τ, μ, and σ^2 in Terms of ω_ψ and $\tilde{\omega}_t$

$$\begin{aligned} m_{\mathbf{VG}\psi 2} = \text{variance}_{\mathbf{VG}\psi} &= \tau(\mu^2 + \sigma^2) \text{ [Kotz et al. notation]} \\ &\approx \frac{(\tilde{\omega}_t \, \mu_t)^2}{\omega_\psi \, (1-\tilde{\omega}_t)} + \left(\omega_\psi^3 \, \tilde{\omega}_t \, \mu_t\right) \text{ [IP notation]} \\ &= \mu_{\psi t} \frac{(\tilde{\omega}_t \, \mu_t)}{(1-\tilde{\omega}_t)} + \mu_{\psi t} \, \omega_\psi^4 \text{ [IP notation]} \end{aligned} \tag{13}$$

Appendix: The Empirical Explanandum Of the Inequality Process (IP)

Appendix	Content
A.1	The quantitative "signature" (implications) of the Inequality Process has been found on empirical invariances of personal income and wealth statistics. The IP's qualitative "signature" has been found on millennia of information about broad characterizations of personal income and wealth statistics at each stage of techno-cultural evolution.
A.2	The Inequality Process implies, mathematically and jointly, seven tenets of mainstream economics maintained verbally and not suspected in economics of being the consequence of a simple mathematical model .

A.3	The Inequality Process implies five empirical invariances of stock price volatility.

Appendix A.1: Confirmed Implications of Empirical Invariances In Statistics of Personal Income And Wealth:

1. The universal pairing (all times, all places, all cultures, all races) of the appearance of extreme social inequality (the chiefdom, society of the god-king) and concentration of wealth after egalitarian hunter/gatherers acquire a storeable food surplus (Angle, 1983, 1986).
2. The pattern of the Gini concentration ratio of personal wealth and income over the course of techno-cultural evolution beyond the chiefdom (Angle, 1983, 1986).
3. The right skew and gently tapering right tail of all distributions of income and wealth (a broad statement of the Pareto Law of income and wealth distribution) (Angle, 1983,
4. a) The sequence of shapes of the distribution of labor income by level of worker education, b) why this sequence of shapes changes little over decades, and c) why a gamma pdf model works well for fitting the distribution of labor income (Angle, 1990, 2002, 2003, 2006, 2007b);
5. How the unconditional distribution of personal income appears to be gamma distributed at the national level and in successively smaller regions although the gamma distribution is not closed under mixture, i.e., under aggregation by area (Angle, 1996);
6. Why the sequences of Gini concentration ratios of labor income by level of education from low to high recapitulates the sequence of Gini concentration ratios of labor income over the course of techno-cultural evolution (a social science analogue of "ontogeny repeats phylogeny" (Angle, 1983, 1986, 2002, 2003, 2006, 2007b);
7. Why the sequence of shapes of the distribution of labor income by level of education from low to high recapitulates the sequence of shapes of the distribution of labor income over the course techno-cultural evolution (a social science analogue of "ontogeny repeats phylogeny" (Angle, 1983, 1986, 2002, 2003, 2006, 2007b);
8. The dynamics of the distribution of labor income conditioned on education as a function of the unconditional mean of labor income and the distribution of education in the labor force (Angle, 2003a, 2006, 2007b);
9. The pattern of correlations of the relative frequency of an income smaller than the mean with relative frequencies of other income amounts (Angle, 2005; 2007a).
10. The surge in the relative frequency of large incomes in a business expansion (Angle, 2007b);
11. The "heaviness" of the far right tail of income and wealth distributions being heavy enough to account for total annual wage and salary income in the U.S. National Income and Product Accounts (Angle, 2002c; 2003a).
12. Why and how the distribution of labor income is different from the distribution of income from tangible assets; (Angle, 1997)
13. Why the IP's parameters estimated from certain statistics of the year to year labor incomes of individual workers are ordered as predicted by the IP's meta-theory and approximate estimates of the same parameters from the fit of the IP's stationary distribution to the distribution of wage income conditioned on education; (Angle, 2002)

14. The Kuznets Curve in the Gini concentration ratio of labor income during the industrialization of an agrarian economy; (Angle, Nielsen, and Scalas, 2009)
15. In an elaboration of the basic IP: if a particle in a coalition of particles has a probability different from 50% of winning a competitive encounter with a particle not in the coalition, this modified IP reproduces features of the joint distribution of personal income to African-Americans and other Americans: a) the smaller median personal income of African-Americans than other Americans; b) the difference in shapes between the African-American distribution of personal income and that of other Americans; this difference corresponds to a larger Gini concentration of the African American distribution; c) the % minority effect on discrimination (the larger the minority, the more severe discrimination on a per capita basis, as reflected in a bigger difference between the median personal incomes of African-Americans and other Americans in areas with a larger % African-American); d) the high ratio of median African-American personal income to the median of other Americans in areas where the Gini concentration ratio of the personal income of other Americans is low; e) the high ratio of median African-American to that of other Americans in areas where the median income of other Americans is high; f) the fact that relationships in d) and e) can be reduced in magnitude by controlling for a measure of economic development of an area or % African-American; g) the greater hostility of poorer other Americans to African-Americans than wealthier other Americans (Angle, 1992).

While grouped under fifteen headings, there may as many as a dozen or more individual empirical invariances under each heading.

Appendix A.2: Seven Verbal Maxims of Mainstream Economics Jointly Implied by the Inequality Process[2]

Maxim of Mainstream Economics:	Inequality Process' Implication:
1) All distributions of labor income are right skewed with tapering right tails; hence the impossibility of radical egalitarianism, the ideologically motivated findings of Vilfredo Pareto's study of income and wealth distribution.	The IP generates right skewed distributions shaped like empirical distributions of labor income or personal assets (depending on the particle parameter). The IP implies that the unconditional distribution of money income is an exponential pdf (probability density function) family shape mixture. Such a mixture has a right tail approximately as heavy as empirical right tails of money income and the Pareto pdf, the model of those right tails preferred in economics.
2) Differences of wealth and income arise easily, naturally, and inevitably via a ubiquitous stochastic process; a	In the IP, differences of wealth arise easily, naturally, and inevitably, via a

[2] Angle, 2006e, 2013a.

general statement of Gibrat's Law; hence the impossibility of radical egalitarianism. Like Pareto, Robert Gibrat's interest in income distribution was motivated by the desire to deny the possibility of a radically egalitarian income distribution.	ubiquitous stochastic process. See, equation (1a,b) below. .
3) A worker's earnings are tied to that worker's productivity [i.e., a central tenet of economics since Aesop's fable of the ant and the grasshopper was all there was to economics] but there is a wide distribution of returns to similarly productive workers.	An IP particle's expected wealth is determined by the ratio of mean productivity in the population to that particular particle's productivity. The IP implies a distribution around this expectation whose shape is determined by each particle's productivity.
4) Labor incomes small and large benefit from a business expansion strong enough to increase mean labor income, i.e., there is a community of interest between all workers regardless of their earnings in a business expansion. A conclusion encapsulated in a favorite saying of mainstream economists: "A rising tide lifts all boats."	In the IP's Macro Model, an increase in the unconditional mean of wealth increases all percentiles of the stationary distribution of wealth by an equal factor. In pithy statement form: "A rising tide lifts the logarithm of all boats equally."
5) Competition transfers wealth to the more productive of wealth via transactions without central direction, i.e., via parallel processing.	In the IP, competition between particles causes wealth to flow via transactions from particles that are by hypothesis and empirical analogue less productive of wealth to those that are more productive of wealth, enabling the more productive to create more wealth, explaining economic growth without a) requiring knowledge of how wealth is produced or b) central direction, i.e., with a minimum of information, two reasons for hypothesizing that the IP would arise to allocate wealth in every economy. These features enable the IP to operate homogeneously over the entire course of techno-cultural evolution independently of wealth level.
6) The ratio of mean labor income in two different occupations remains roughly constant as long as the skill levels in the two occupations remain roughly constant. This phenomenon is evidence of equilibrium in labor markets.	This conclusion falls out of the ratio of expected wealth in subsets of the population with two distinct values of wealth productivity.

7) Competition and transactions maximize societal gross product and over the long run drive techno-cultural evolution.	The Inequality Process operates as an evolutionary wealth maximizer in the whole population of particles, given a relaxation of the zero-sum constraint on wealth transfers within the model (equation 1a,b) , by transferring wealth to the more productive and doing so more efficiently as mean wealth productivity in the population of particles increases.

Appendix A.3: Five "Stylized Facts" Empirical Invariances) of Stock Market Statistics

1.	Association between greater corporate market capitalization and a lower mean absolute value of the logarithm of its daily stock returns. Volatility is defined here as the mean absolute logarithm of daily returns. Source: Malkiel (2015:124)
2.	Big stock price movements down are associated with greater volatility, while big stock price movements up are associated with lower volatility. In finance this phenomenon is terms "leverage effect". Source: Tsay (2013:177).
3.	(t+k) autocorrelations of daily log returns to stocks of a particular corporation converge to near zero for k small beyond k = 1. Sources: Georgakopoulos (2015:115), Resnick (2007:6), Tsay (2013:178).
4.	t+k autocorrelations of squared daily log returns to stocks of a particular corporation show long term memory (i.e., do not converge to zero) as k increases. Sources: Georgakopoulos (2015:115), Resnick (2007:6).
5.	Bollinger Band-like bounded volatility of particle wealth. Source: Kaufman (2005: 294).

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