A Repairable System with Two Spare Units and Two Repair Facilities Serviced by Two Types of Repairers

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Abstract
We study a one-unit repairable system with two identical, cold standby spare units in terms of limiting availability A_∞ and limiting profit per unit time ω, using semi-Markov processes, when life- and repair time distributions are exponential. The failed unit undergoes repair either by an in-house repairer within a randomly chosen exponential patience time T, or else by a visiting expert repairer, who works faster but charges at a higher rate. Since there are two repair facilities, we allow the regular repairer to begin repair or to continue repair beyond T, if the expert is busy. Two models arise according as the expert repairs one or all failed units during each visit. We show that (1) adding a second spare unit to a one-unit system backed by only one spare unit increases both A_∞ and ω; (2) thereafter adding a second repair facility improves both criteria further. Finally, we determine whether the expert must repair one or all failed units in order to maximize these criteria. This optimal strategy fulfills the maintenance management objectives better than those in previously studied models.

Key Words: Cold standby, Perfect repair, Patience time, Semi-Markov process, Sojourn time, Busy time

1. Introduction

We consider a continuously monitored, one-unit repairable system supported by two other identical units, and serviced by two types of repairers in order to reduce maintenance cost. Also, there are two repair facilities to accommodate both repairers at a time. A regular in-house repairer may have limited maintenance knowledge, but he is paid less per hour and his continual presence eliminates the overhead expense payable to a visiting expert repairer. Generally, the regular repairer can do minor repairs within a given patience time, and is either incapable of performing more complicated repairs, or is unable to do so within the patience time. The visiting expert repairer, on the other hand, can fix any problem with the failed unit, and she performs the repair faster than the regular repairer. However, her hourly charges are comparatively higher, and she must be paid also a trip charge for each visit.

This is how the system operates: Initially, one unit is put on operation and the other two units are on cold standby. Consequently, the system differs from a 1-out-of-3 system. Upon failure of the operating unit, immediately a spare unit is placed on operation, and the failed unit undergoes repair—first by the regular repair person, and if it is not repaired within the patience time T, the visiting expert repair person is called in. We allow a random patience time (RPT). We also call in the expert repairer when the system goes down because all

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three units are down; that is, the regular repairer is busy fixing a previously failed unit, the patience time is not over yet, but the other two units have successively failed.

Since the two repairers can work simultaneously, the regular repairer works on the failed unit until his patience time is over or until the expert is freed up to take over, whichever comes later. Also, we assume that the benefit of any partial repair done by the regular repair person is forfeited when the expert takes over the job. We also assume that when repair is complete by either repairer, the repaired unit becomes as good as new.

How long will the expert remain at the repair facility? We consider two possibilities before the expert leaves the system: Either she repairs all failed units before leaving one unit on operation and the other two on cold standby or one unit on operation, one unit on standby and another one under repair by regular person while she is visiting, which we call the multiple repair by expert (MRE) policy. Or, she fixes only one failed unit during each visit; and she lets the regular repairer attend to the waiting failed unit(s), if any. This second possibility we call the single repair by expert (SRE) policy.

Depending on the number of repairs done by the expert—single or multiple—two possible models arise: (1) MRE-RPT, (2) SRE-RPT. We evaluate the performance of these two models in terms of limiting availability A_∞ and limiting profit per unit time ω. Under the assumption of continuous monitoring and continuous life- and repair times, the limiting availability exists; and it is defined as the long-run proportion of time the system is up [1]. Likewise, the limiting profit per unit time is defined as the long-run difference between the net revenue earned and the repair cost paid to the repair persons, including a trip charge payable to the expert, all expressed per unit time.

[2] studies Models (1) and (2), and also those under deterministic patience time policy (DPT)–(3) MRE-DPT and (4) SRE-DPT–when there is only one spare unit and one repair facility. Assuming exponential life- and repair times, they obtain A_∞ and ω using the technique of semi-Markov processes (SMP). [3] extends their results to the case of two spare units. Such an extension is desirable if, for example, A_∞ with only one spare unit falls below an acceptable threshold even when the units are state-of-the-art. Assuming that the engineering side has already done its best to manufacture such crucial units, on behalf of the maintenance team we can further improve A_∞ to exceed the acceptable threshold by utilizing one more spare unit. In this paper, we extend their results to a system with two repair facilities under RPT policy. It can be seen that the Markovian property fails under the DPT policy, since the transition out of some states may depend not only on the current state but also on the history of the process.

We demonstrate that under RPT policy, the system with two spare units and two repair facilities has higher A_∞ and ω compared to a system with one spare unit or two spare units, with only one repair facility.

The rest of the paper is organized as follow: In Section 2, we give a literature review. In Section 3, we formulate the stochastic behavior of the repairable system as an SMP; and we describe the analytic techniques for deriving the limiting availability and the limiting profit per unit time. In Section 4, we provide detailed analytic derivations for our two repair models. Section 5 compares the two models against those when there is one spare unit or two spare units, with only one repair facility. Finally, Section 6 concludes the paper with a summary and several directions for future research.

2. Literature Review

In this section, we review some latest developments in modeling repairable systems to address various reliability characteristics.

[4] considers a one-unit repairable system, supported by r identical repair facilities and s cold standby spare units, $r \leq s + 1$, which fails when all units are down and are undergoing or awaiting repair. They obtain limiting average availability under a perfect repair policy when lifetime is arbitrary and repair time is exponential. [5] studies a similar model, but they obtain the instantaneous availability function under both life- and repair times exponentially distributed.

[6] deals with reliability and sensitivity analysis of a repairable system with several operating- and warm standby units, and several unreliable service stations. Failure times and service times are exponentially distributed, and the service station is subject to breakdowns according to a Poisson process. They determine the mean time to failure (MTTF) and system reliability; and study how these characteristics change with the model parameters.

[7] studies a cold standby repairable system consisting of two dissimilar components—with Component 1 having priority in use—and one repairman. Component 2 is as good as new after repair, while Component 1 follows a geometric process repair. Assuming exponential life- and repair times, they derive some important reliability indices such as the system availability, reliability, mean time to first failure (MTTFF), rate of occurrence of failure and the probability the repairman remains idle. For Component 1, they determine an optimal replacement policy which minimizes the long-run average cost per unit time.

[8] designs a maintainable cold standby system which minimizes the system cost rate subject to availability constraint. [9] investigates the cost-benefit analysis of a two-unit cold standby system with two-stage repair with waiting time in between. They use regenerative point processes to obtain time dependent availability, steady state availability, reliability, MTTF and profit function.

[10] proposes two interval availability indexes for Markov repairable systems which measure the probability that the system is working during a given time window containing either a specified point or an interval. [11] studies a discrete-time semi-Markovian repairable system where the state space of the process includes three subsets—working, changeable and failed. They apply Z-transform to derive reliability, point availability and interval availability. They also discuss for their system the two new reliability measures introduced in [10].

[12] describes repairable systems in which defects are detected before failure, triggering repair. The system is either perfectly repaired within a time period, and the process renews; or it is not repaired within the time period, causing fatal failure. The authors derive the survival function of these systems assuming exponential time to defect, deterministic time period and arbitrary repair time; though they illustrate the results only under exponential repair time. They also obtain asymptotic survival probability under the assumption of fast repair when distributions are arbitrary.

[13] employs cost analysis approach in the redundancy-allocation problem to obtain the optimal number of allocated cold redundant units in a one-unit repairable system. In this

system, the main component is put on operation first, and as soon as the failure happens, the redundant component is replaced and the failed unit undergoes repair. They develop a model using continuous-time Markov chain to analyze system reliability assuming that both failure- and repair time are exponentially distributed.

Repairable systems with two types of repairers have not been studied extensively. [14] studies Model (2) with only one spare unit. They allow an expert to take over the repair only after the patience time of the regular repairer is exhausted without completing the repair, even if the system fails during this time. [15] calls in the expert as soon as the patience time is over or the system fails. Although they claim to allow arbitrary life-, repair- and patience time distributions, their results are correct only under exponential life- and exponential repair times, as pointed out in [2]. [16] allows a random pre-inspection time for the regular repairer to determine whether he is able to repair a failed unit or not. If he is capable of repairing, he starts the repair; otherwise, the expert is called immediately. [2] studies Models (1)-(4), when there is only one spare unit. They obtain limiting availability and limiting profit per unit time using the SMP technique under exponential life- and repair times. They also extend the technique to allow arbitrary life- and repair times.

[17] allows only one repair person but permits two types of failures and hence two types of repair. They find MTTF, limiting availability and limiting profit using the Laplace transformation technique.

[18] studies a one-unit system backed by a hot standby spare unit in a master-slave relationship. Initially, the master unit is operating and the slave unit is on hot standby. There are three types of failures: minor, major-repairable and major-irreparable (which requires replacement). The regular repairer repairs only minor failures. They claim to derive the system MTTF, steady-state availability and limiting profit per unit time assuming repair- and replacement times are arbitrary but lifetime is exponential; however, no analytic solutions are given. In fact, their theoretical results are valid only under exponential life-, repair- and replacement times.

The papers discussed above utilize the Laplace transform technique to obtain various system reliability indices including, but not limited to, availability, busy periods for the two repairers and profit. None of those papers actually invert the Laplace transform except in the case of exponential distribution. Therefore, we prefer to use the relatively more straight-forward and simpler method of semi-Markov processes (SMP).

[3] extends the results of [2] to the system with two spare units and one repair facility using SMP technique. For any choice of parameter values, they determine a range of values of T for which Model (3) performs the best in terms of both A_∞ and ω. Furthermore, they obtain a threshold value for the cost per unit time payable to the expert repairer such that so long as the expert charges less than this threshold value the MRE policy yields higher profit than the SRE policy, and vice versa.

3. System Description and Mathematical Framework

For the two models (1) and (2) discussed in Section 1, we study the system limiting availability and limiting profit per unit time under the following assumptions:

1. A one-unit system has three identical units. At the very beginning, one unit is put on

operation, and the other two spare units remain on cold standby.

2. There are two repair facilities attended by regular and the expert repairer.

3. Failure of the operating unit is immediately detected; the failed unit is sent for repair, and if a standby unit is available, it is put on operation immediately.

4. The regular repair person has to finish repair within a maximum allowable patience time T which is random (RPT).

5. The system fails when all three units are down.

6. When either the patience time for the regular repair person is over or the system fails, whichever happens first, the expert is called; and she arrives immediately.

7. The regular repairer works on the failed unit until his patience time isover or until the expert is freed up to take over, whichever comes later.

8. Life-, repair- and patience times are exponentially distributed with arbitrary parameters, and are independent of one another. Admittedly, this is a restrictive assumption, which we intend to remove in subsequent research.

9. When the expert repairer takes over the job, the benefits of partial repair done by the regular repairer is forfeited. In fact, this assumption follows from the previous assumption.

10. We consider two options for the expert repairer: She may leave the repair facility after repairing all failed units before leaving one unit on operation and the other two on cold standby or one unit on operation, one unit on standby and another one under repair by regular person, which is called the MRE model. Or, she may leave the facility after repairing only one failed unit and letting the regular repairer attend to the other failed unit(s), if any. This alternative model is called the SRE model.

11. We assume a perfect repair policy under which a repaired unit becomes as good as new.

At any time, a unit exhibits one of five possible features: s (on standby), p (operating), r (undergoing repair by regular repairer), $\bar{r}$ (undergoing repair by regular repairer beyond T when the expert is busy with repairing another failed one) e (undergoing repair by expert repairer) or w (awaiting repair). Since the units are identical, it suffices to record how many units are exhibiting each feature. Accordingly, the system is in one of the nine possible states: $1 = (p, s, s)$, $2 = (r, p, s)$, $3 = (e, p, s)$, $4 = (r, w, p)$, $5 = (e, r, p)$, $6 = (e, \bar{r}, p)$, $7 = (e, r, w)$, $8 = (e, r, w)$, $9 = (e, \bar{r}, w)$. The system is down in States 7, 8 and 9, and is up in all other states. States 7 and 8 represents the same features of units but we separate those because the system enters in the state with units' features (e, r, w) from two different paths.

Figure 1 shows the transitions under SRE and MRE models, along with random variables that determine the sojourn times and transition probabilities.

Let us first explain the random variables. Let X, Y and Z denote the lifetime of the unit, the repair time by the regular repairer and the repair time by the expert respectively. Some additional random variables shown in the diagram have the following interpretations: The variable X' is another lifetime which has the same distribution as X, but is independent of

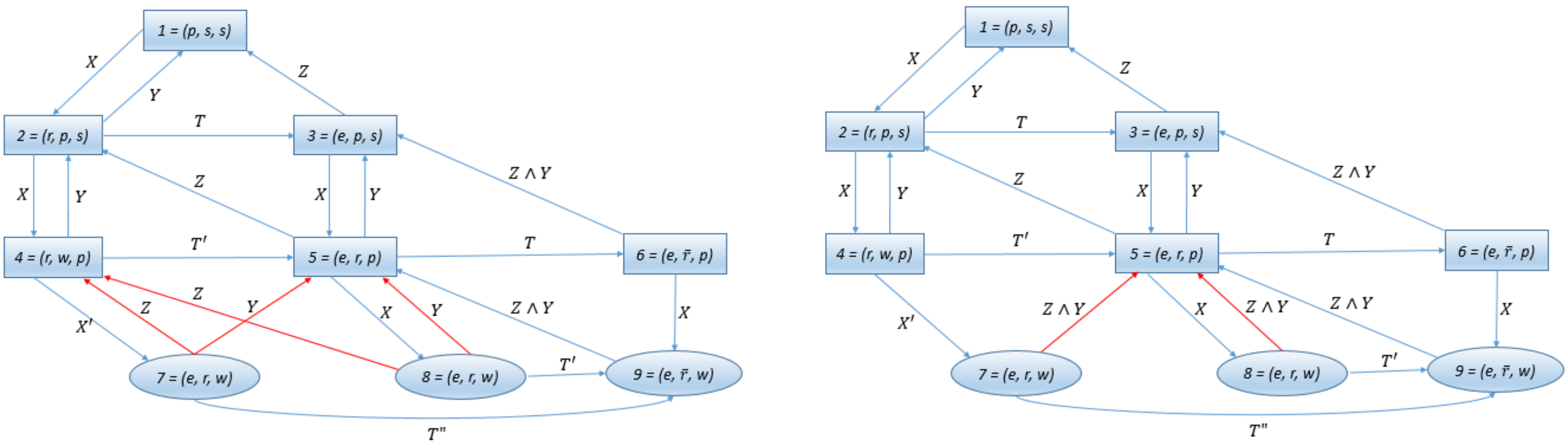

Figure 1: Transition diagrams for SRE (left) and MRE (right) models

X. The variable T' and T'' are the remaining patience times. They reduce to $T' = T - X$ and $T'' = T - X - X'$ under the DPT policy; but under the RPT policy, in view of the memoryless property of exponential distribution, T' and T'' has the same distribution as T, but they are all independent.

Next, let us explain the sojourn times in each state and the transitions out of them. The system starts in State 1 at time $t = 0$; it stays there for a random duration X; and then it moves to State 2. The sojourn time in State 2 is $min(X, Y, T)$; and the system returns to State 1 if Y is the smallest, to State 3 if T is the smallest, or to State 4 if X is the smallest. The sojourn time in State 3 is $min(X, Z)$; and the system moves to State 1 if $Z < X$, or to State 5 otherwise. The sojourn time in State 4 is $min(X', Y, T')$; and the system moves to State 2 if Y is the smallest, to State 5 if T' is the smallest, or to State 7 if X' is the smallest. The sojourn time in State 5 is $min(X, Y, Z, T)$. The system moves to State 2 if Z is the smallest, to State 3 if Y is the smallest, to State 6 if T is the smallest, or to State 8 if X is the smallest. The sojourn time in State 6 is $min(X, Y, Z)$. The system moves to State 3 if either Y or Z is the smallest, or to State 9 if X is the smallest. The sojourn time in State 7 is $min(Z, Y, T'')$. The system moves to State 9 if T'' is the smallest, to State 5 (under MRE policy) if either Y or Z is the smallest. However, under SRE policy, the system moves to State 4 if Z is the smallest, or to State 5 if Y is the smallest. The sojourn time in State 8 is $min(Z, Y, T')$. The system moves to State 9 if T' is the smallest. Transitions from this state to State 4 and State 5 are the sames as those from State 7 under both SRE and MRE policies. Finally, as soon as either the expert or the regular repairer repairs one of the failed units in State 9, the system moves to State 5 under both SRE and MRE policies. The red arrows emphasize the transitions exclusive to each model, while the solid arrows are common to both models. The transition probabilities out of each state are determined based on whichever associated random variable attains the minimum.

Let θ_k be the proportion of time the system spends in State k $(k = 1, \ldots, 9)$. Since the system is down in States 7, 8 and 9, the limiting availability of the system is,

$$A_\infty = 1 - \theta_7 - \theta_8 - \theta_9. \qquad (1)$$

Having obtained A_∞, we can now derive ω, the limiting profit per unit time. We need the following parameters: The proportion of busy time for the regular repairer is $\Theta_r = \theta_2 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8 + \theta_9$, and that for the expert is $\Theta_e = \theta_3 + \theta_5 + \theta_6 + \theta_7 + \theta_8 + \theta_9$.

Let R_p, C_p, C_r, C_e denote respectively the net revenue, the operation cost, the payment to the regular repairer and the payment to the expert—all expressed per unit time. Also, let C_l denote the trip charge paid to the expert per trip (not per unit time). Then the limiting profit per unit time is given by

$$\omega = A_\infty(R_p - C_p) - [\Theta_r C_r + \Theta_e C_e + C_l/\tau], \tag{2}$$

where τ is the expected length of a cycle, which is defined as the duration from the epoch the system enters State 2, until it returns to State 2 after visiting one of States 3, 5, 6, 7, 8 and 9 at least once. Thus, within each cycle, the expert comes and returns exactly once, and she is paid the trip charge C_l exactly once. By Wald's First Identity [1], the expected number of visits by the expert per unit time is the reciprocal of τ. Therefore, C_l/τ is the trip charge paid to the expert per unit time.

4. Limiting Availability and Limiting Profit Analysis

In this section, we derive the analytic expressions for the limiting availability A_∞ and the limiting profit per unit time ω for two models: (1) MRE-RPT, (2) SRE-RPT. In view of Assumption 9, let us denote the patience time, the lifetime, the repair times by the regular repairer and the expert respectively as

$$T \sim exp(\alpha), \quad X \sim exp(\lambda), \quad Y \sim exp(\beta), \quad Z \sim exp(\gamma).$$

Here, the parameter of an exponential distribution denotes the rate; and its reciprocal denotes the mean. By the memoryless property of an exponential random variable, the future trajectory of the stochastic process depends only on the present state, while the history of the process can be disregarded. Hence, the process, describing each repair model is a semi-Markov processes (SMP); that is, the system changes states in accordance with a Markov chain, but takes a random amount of time between changes. See [19] for more details on SMP. More specifically, in our models, the embedded discrete time stochastic process (DTSP) is a Markov chain with a finite state space $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and a transition probability matrix $P = ((P_{ij}));\ i, j = 1, \ldots, 9$. The exact expressions for P_{ij} varies across the four models, and will be presented in the respective subsections.

The stationary distribution of a Markov chain gives the limiting probability π_j of transitions entering (also departing) State j. It is unique, and is obtained by solving the following system of equations (for more details see [19], pp. 215-216),

$$\pi_j = \sum_i \pi_i P_{ij}, \ \ \sum_j \pi_j = 1. \tag{3}$$

Moreover, the expected sojourn times in different states are

$$
\begin{aligned}
\mu_1 &= E[X] = \frac{1}{\lambda} \\
\mu_2 &= E[min(X,Y,T)] = \frac{1}{\alpha+\lambda+\beta} \\
\mu_3 &= E[min(X,Z)] = \frac{1}{\lambda+\gamma} \\
\mu_4 &= E[min(X',Y,T')] = \frac{1}{\alpha+\lambda+\beta} \\
\mu_5 &= E[min(X,Y,Z,T)] = \frac{1}{\alpha+\lambda+\beta+\gamma} \\
\mu_6 &= E[min(X,Y,Z)] = \frac{1}{\lambda+\beta+\gamma} \\
\mu_7 &= E[min(Y,Z,T'')] = \frac{1}{\alpha+\beta+\gamma} \\
\mu_8 &= E[min(Y,Z,T')] = \frac{1}{\alpha+\beta+\gamma} \\
\mu_8 &= E[min(Y,Z)] = \frac{1}{\beta+\gamma}
\end{aligned}
\tag{4}
$$

The following theorem, also quoted from [19], pp. 215-216, gives the proportions of time the SMP spends in the different states.

Theorem 1 *For an SMP, if the embedded DTSP is irreducible with stationary probabilities* π*, and if the times between successive visits to any State* k *has a non-lattice distribution with a finite mean, and* μ_k *is the expected sojourn time in State* k *before transition, then the limiting probability that the process will be found in State* k *exists, is independent of the initial state, and is given by*

$$\theta_k = \frac{\pi_k \mu_k}{\sum_{j=1}^{9} \pi_j \mu_j}. \tag{5}$$

In the following subsections, for each of the two models, starting from the transition matrix P, we derive θ_k ($k = 1, \ldots 9$) using (5), (3) and (4). Then we obtain A_∞ using (1). Next, we obtain the analytic expression of τ in each model by solving a suitable system of recursive relations. Subsequently, we obtain ω using (2).

4.1 Model 1: MRE-RPT

For the MRE-RPT repair model, the embedded DTMC has transition matrix

$$
P = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\beta}{\alpha+\lambda+\beta} & 0 & \frac{\alpha}{\alpha+\lambda+\beta} & \frac{\lambda}{\alpha+\lambda+\beta} & 0 & 0 & 0 & 0 & 0 \\
\frac{\gamma}{\lambda+\gamma} & 0 & 0 & 0 & \frac{\lambda}{\lambda+\gamma} & 0 & 0 & 0 & 0 \\
0 & \frac{\beta}{\alpha+\lambda+\beta} & 0 & 0 & \frac{\alpha}{\alpha+\lambda+\beta} & 0 & \frac{\lambda}{\alpha+\lambda+\beta} & 0 & 0 \\
0 & \frac{\gamma}{\alpha+\lambda+\beta+\gamma} & \frac{\beta}{\alpha+\lambda+\beta+\gamma} & 0 & 0 & \frac{\alpha}{\alpha+\lambda+\beta+\gamma} & 0 & \frac{\lambda}{\alpha+\lambda+\beta+\gamma} & 0 \\
0 & 0 & \frac{\beta+\gamma}{\lambda+\beta+\gamma} & 0 & 0 & 0 & 0 & 0 & \frac{\lambda}{\lambda+\beta+\gamma} \\
0 & 0 & 0 & 0 & \frac{\beta+\gamma}{\alpha+\beta+\gamma} & 0 & 0 & 0 & \frac{\alpha}{\alpha+\beta+\gamma} \\
0 & 0 & 0 & 0 & \frac{\beta+\gamma}{\alpha+\beta+\gamma} & 0 & 0 & 0 & \frac{\alpha}{\alpha+\beta+\gamma} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{6}
$$

Solving the system of equations (3), we obtain the stationary distribution as

$$\pi \propto \Big(\frac{\beta\xi_1}{\alpha+\lambda+\beta}+\frac{\gamma\xi_2}{\lambda+\gamma}, \xi_3, \xi_2, \frac{\lambda\xi_3}{\alpha+\lambda+\beta}, 1, \frac{\alpha}{\alpha+\lambda+\beta+\gamma}, \\ (\frac{\lambda}{\alpha+\lambda+\beta})^2\xi_3, \frac{\lambda}{\alpha+\lambda+\beta+\gamma}, \xi_4\Big) \quad (7)$$

where,

$$\xi_1 = 1-\frac{\beta}{\alpha+\lambda+\beta}-\frac{\lambda\beta}{(\alpha+\lambda+\beta)^2}$$
$$\xi_2 = \frac{\alpha\gamma(\lambda+\beta+\gamma)+\beta\xi_1(\lambda+\beta+\alpha)(\lambda+\beta+\gamma)+\alpha\xi_1(\beta+\gamma)(\lambda+\beta+\alpha)}{(\lambda+\beta+\gamma)\big(\xi_1(\lambda+\beta+\alpha)(\lambda+\beta+\gamma+\alpha)-\alpha\gamma\big)}$$
$$\xi_3 = \frac{\alpha+\lambda+\beta}{\alpha}(\xi_2-\frac{\beta}{\alpha+\lambda+\beta+\gamma}-\frac{\alpha(\beta+\gamma)}{(\lambda+\beta+\gamma)(\alpha+\lambda+\beta+\gamma)})$$
$$\xi_4 = \frac{\alpha\lambda}{(\lambda+\beta+\gamma)(\alpha+\lambda+\beta+\gamma)}+\frac{\xi_3\alpha\lambda^2}{(\lambda+\beta+\gamma)(\alpha+\beta+\gamma)^2}+\frac{\alpha\lambda}{(\alpha+\beta+\gamma)(\alpha+\lambda+\beta+\gamma)}.$$

Substituting the mean sojourn times (4) and the stationary distribution (7) into (5), we can obtain expressions for θ_k's. Thereafter, from (1), we get

$$A_\infty = 1-\theta_7-\theta_8-\theta_9$$

where

$$\theta_7 \propto \mu_7\pi_7 = \frac{\xi_3\lambda^2}{(\alpha+\lambda+\beta)^2(\alpha+\beta+\gamma)}$$
$$\theta_8 \propto \mu_8\pi_8 = \frac{\lambda}{(\alpha+\beta+\gamma)(\alpha+\lambda+\beta+\gamma)} \quad (8)$$
$$\theta_9 \propto \mu_9\pi_9 = \frac{\xi_4}{\beta+\gamma}.$$

Next, the expected length of a cycle satisfies the recursive relation

$$\tau = \mu_2+P_{21}(\mu_1+\tau)+P_{23}\sigma_{32}^M+P_{24}\sigma_{42}^M \quad (9)$$

where σ_{32}^M denotes the expected time for the system to go from State 3 to State 2 (via State 1 or State 5) under the MRE policy. The other parameters σ_{42}^M, σ_{52}^M, σ_{62}^M, σ_{72}^M and σ_{82}^M (to be introduced shortly) denote similar quantities. These parameters satisfy

$$\sigma_{32}^M = \mu_3+P_{31}\mu_1+P_{35}\sigma_{52}^M$$
$$\sigma_{52}^M = \mu_5+P_{53}\sigma_{32}^M+P_{56}\sigma_{62}^M+P_{58}\sigma_{82}^M \quad (10)$$
$$\sigma_{62}^M = \mu_6+P_{63}\sigma_{32}^M+P_{69}(\mu_9+\sigma_{52}^M)$$
$$\sigma_{82}^M = \mu_8+P_{85}\sigma_{52}^M+P_{89}(\mu_9+\sigma_{52}^M)$$

Solving the system of equations (10), we obtain

$$\sigma_{52}^M = \frac{\mu_5+P_{53}(\mu_3+P_{31}\mu_1)+P_{56}(\mu_6+P_{63}(\mu_3+P_{31}\mu_1)+P_{69}\mu_9)+P_{58}(\mu_8+P_{89}\mu_9)}{1-P_{53}P_{35}-P_{56}P_{63}P_{35}-P_{56}P_{69}-P_{58}P_{85}-P_{58}P_{89}}. \quad (11)$$

Thereafter, using (11) we obtain an explicit expression for σ_{32}^M and σ_{72}^M which the latter satisfies

$$\sigma_{72}^M = \mu_7+P_{75}\sigma_{52}^M+P_{79}(\mu_9+\sigma_{52}^M). \quad (12)$$

Finally. we have one more relationship

$$\sigma_{42}^M = \mu_4 + P_{45}\sigma_{52}^M + P_{47}\sigma_{72}^M + P_{42}\tau. \tag{13}$$

Substituting the expressions for σ_{32}^M and σ_{42}^M into (9) and solving, we obtain

$$\tau = \frac{\mu_2 + P_{21}\mu_1 + P_{23}\sigma_{32}^M + P_{24}(\mu_4 + P_{45}\sigma_{52}^M + P_{47}\sigma_{72}^M)}{1 - P_{21} - P_{24}P_{42}}. \tag{14}$$

Using expression (14) for τ, we obtain ω from (2).

4.2 Model 2: SRE-RPT

For the SRE-RPT repair model, the embedded DTMC has transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta}{\alpha+\lambda+\beta} & 0 & \frac{\alpha}{\alpha+\lambda+\beta} & \frac{\lambda}{\alpha+\lambda+\beta} & 0 & 0 & 0 & 0 & 0 \\ \frac{\gamma}{\lambda+\gamma} & 0 & 0 & 0 & \frac{\lambda}{\lambda+\gamma} & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta}{\alpha+\lambda+\beta} & 0 & 0 & \frac{\alpha}{\alpha+\lambda+\beta} & 0 & \frac{\lambda}{\alpha+\lambda+\beta} & 0 & 0 \\ 0 & \frac{\gamma}{\alpha+\lambda+\beta+\gamma} & \frac{\beta}{\alpha+\lambda+\beta+\gamma} & 0 & 0 & \frac{\alpha}{\alpha+\lambda+\beta+\gamma} & 0 & \frac{\lambda}{\alpha+\lambda+\beta+\gamma} & 0 \\ 0 & 0 & \frac{\beta+\gamma}{\lambda+\beta+\gamma} & 0 & 0 & 0 & 0 & 0 & \frac{\lambda}{\lambda+\beta+\gamma} \\ 0 & 0 & 0 & \frac{\gamma}{\alpha+\beta+\gamma} & \frac{\beta}{\alpha+\beta+\gamma} & 0 & 0 & 0 & \frac{\alpha}{\alpha+\beta+\gamma} \\ 0 & 0 & 0 & \frac{\gamma}{\alpha+\beta+\gamma} & \frac{\beta}{\alpha+\beta+\gamma} & 0 & 0 & 0 & \frac{\alpha}{\alpha+\beta+\gamma} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

Using linear algebra we are able to solve the system of equations (3) to obtain stationary distribution π for this model. However, due to the complexity of the solution, we apply numerical technique to obtain the stationary distribution for some known values of the parameters. It can be seen that both analytical and numerical solutions are consistent. Having obtained the values of π_j's and the mean sojourn times (4), and substituting them into (5), we can obtain the values of θ_k's. Thereafter, from (1), we get

$$A_\infty = 1 - \theta_7 - \theta_8 - \theta_9.$$

To obtain ω we need to find the expected cycle time τ. Let σ_{32}^S denote the expected time for the system to go from State 3 to State 2 (via State 1 or State 5) under the SRE policy. The other parameters σ_{42}^S, σ_{52}^S, σ_{62}^S, σ_{72}^S and σ_{82}^S (to be introduced shortly) denote similar quantities. They satisfy the recursive relations

$$\begin{aligned} \tau &= \mu_2 + P_{21}(\mu_1 + \tau) + P_{23}\sigma_{32}^S + P_{24}\sigma_{42}^S \\ \sigma_{32}^S &= \mu_3 + P_{31}\mu_1 + P_{35}\sigma_{52}^S \\ \sigma_{42}^S &= \mu_4 + P_{45}\sigma_{52}^S + P_{47}\sigma_{72}^S + P_{42}\tau \\ \sigma_{52}^S &= \mu_5 + P_{53}\sigma_{32}^S + P_{56}\sigma_{62}^S + P_{58}\sigma_{82}^S \\ \sigma_{62}^S &= \mu_6 + P_{63}\sigma_{32}^S + P_{69}(\mu_9 + \sigma_{52}^S) \\ \sigma_{72}^S &= \mu_7 + P_{74}\sigma_{42}^S + P_{75}\sigma_{52}^S + P_{79}(\mu_9 + \sigma_{52}^S) \\ \sigma_{82}^M &= \mu_8 + P_{84}\sigma_{42}^S + P_{85}\sigma_{52}^M + P_{89}(\mu_9 + \sigma_{52}^M). \end{aligned} \tag{16}$$

Rewriting σ_{42}^S as a function of σ_{52}^S and τ, we obtain σ_{52}^S from the fourth equation in (16). Then, we obtain σ_{32}^S and σ_{42}^S. Having obtained all the σ^S's, from the first equation in (16), we get

$$\tau = \frac{\mu_2 + P_{21}\mu_1 + P_{23}\sigma_{32}^S + P_{24}(\mu_4 + P_{45}\sigma_{52}^S + P_{47}\sigma_{72}^S)}{1 - P_{21} - P_{24}P_{42}}. \tag{17}$$

Using expression (17) for τ, we obtain ω from (2).

5. Comparison of Models

In this section, for some choices of values of the parameters, we compare the two repair models discussed in Section 3 in terms of the limiting availability A_∞ and the limiting profit per unit time ω under RPT policy. For a given choice of parameter values, we determine the best model under which both criteria are maximized. We also demonstrate that a system with two spare units has a higher A_∞ and a higher ω than a system supported by only one spare unit when there is only one repair facility; thereafter utilizing second repair facility which allows both repairers work on the failed units at a time, improves both criteria.

We compare the two models MRE-RPT and SRE-RPT in terms of A_∞, ω, Θ_r and Θ_e for systems with either one spare unit ($S = 1$) or two spare units ($S = 2$) when either one repair facility ($R.F = 1$) or two repair facilities ($R.F = 2$) is available. We assume that the expert repairer completes repair quicker than the regular repairer, but she charges a higher rate; that is, $\beta < \gamma$ and $C_r < C_e$. The comparisons are made given the parameter values: $\lambda = 0.5$, $\alpha = 0.3$, $\beta = 0.35$ and $\gamma = 0.75$ and additionally: $R = 20$, $C_r = 1$, $C_e = 5$ and $C_l = 3$. Table 1 shows the results obtained for two repair models MRE and SRE under RPT policy and different number of spares and repair facilities.

Table 1: Calculated results under RPT policy.

Criteria	**SRE**			**MRE**		
	$S=1$ $R.F=1$	$S=2$ $R.F=1$	$S=2$ $R.F=2$	$S=1$ $R.F=1$	$S=2$ $R.F=1$	$S=2$ $R.F=2$
A_∞	0.736	0.801	0.884	0.788	0.844	0.896
ω	11.919	13.640	14.978	12.484	14.068	15.126
Θ_r	0.320	0.442	0.605	0.174	0.227	0.572
Θ_e	0.342	0.327	0.307	0.426	0.457	0.331

We observe the following results:

1. The limiting availability A_∞ is strictly higher under MRE policy than under SRE policy for systems with either one or two spare units, irrespective of the number of repair facilities.

2. The limiting profit per unit time ω is strictly higher under MRE policy than under SRE policy for systems with either one or two spare units, irrespective of the number of repair facilities.

3. Adding one more spare unit to a system backed by only one spare unit increases both A_∞ and ω. For example, A_∞ is below 80% when $S = 1$; but it is more than 80% when $S = 2$. See [3] for further details.

4. Utilizing one more spare unit yields $\Theta_r > \Theta_e$. This implies that we utilize the regular repairer most. Furthermore, adding second repair facility makes the regular repairer busier than the expert which results in even less costs and higher ω.

5. Adding second repair facility to the system with two spare units raises both A_∞ and ω. For example, the limiting availability is increased to 90% under MRE policy.

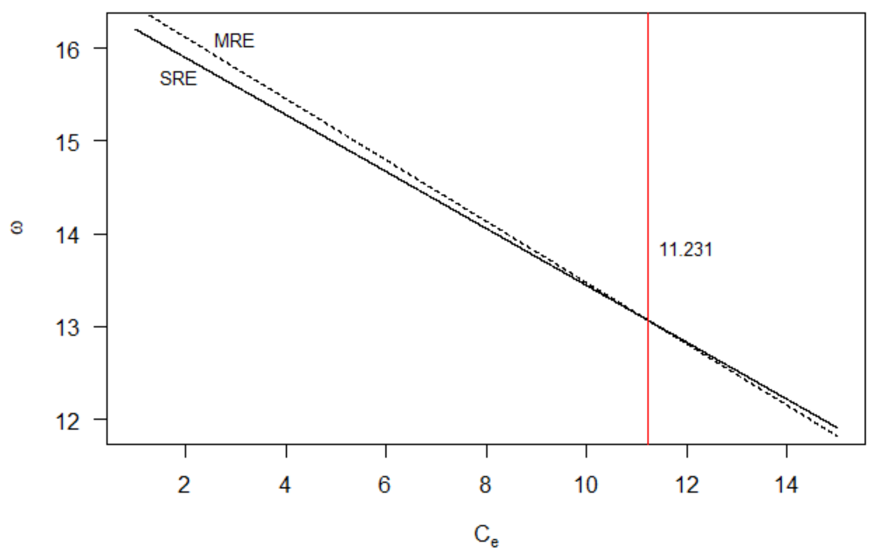

Figure 2: Limiting profit per unit time as a function of C_e for system with $S = 2$ and $R.F = 2$.

Although, for our choice of parameter values, it was seen that ω is larger under MRE policy than under SRE policy, if the expert charges too much, then MRE model may not dominate SRE model in terms of ω. Figure 2 depicts ω for MRE and SRE models as the cost per unit time paid to the expert repairer C_e varies with $R = 20, C_r = 1, C_l = 3$. If the expert charges at a rate less than a threshold, then MRE model yields a higher limiting profit per unit time than SRE model under RPT policy; and the opposite holds if the expert charges above the threshold. In our example, the threshold for C_e is 11.231.

6. Concluding Remarks

In this paper, we extend the results obtained in [3] under random patience time by introducing another repair facility to a cold standby repairable system consisting of two identical units and one repair facility, and serviced by two types of repairers. In a situation where component lifetime is short and repair time is long, multiple spare units are necessary to improve the reliability characteristics of the system. In addition, utilizing multiple repair facilities enable both repairers to work on the failed units simultaneously which results in higher available and more profitable system. In this extended set up, we study the limiting availability and the limiting profit per unit time when lifetime and repair times are exponentially distributed and patience time for the regular repairer is random. Two possible models arise depending on the number of failed units the expert repairer is allowed to repair during each visit. We derive the limiting availability and limiting profit per unit time for each of the two possible models using SMP, which is much simpler than the Laplace transform technique widely used in the literature. As anticipated, we verify that the system supported by two repair facilities results in higher A_∞ and higher ω compared to the system having only one repair facility.

Since the expert repairs faster than the regular repairer, MRE yields a higher A_∞ than SRE. However, in order to maximize ω, the maintenance administrator may adopt either MRE or SRE policy depending on the relative costs payable to the expert (compared to the regular repairer). Thus, given all cost parameters, the maintenance engineer can determine whether MRE or SRE is the preferred policy in terms of ω.

We identify several directions of future research:

- For the purpose of building the repairable models, we have assumed life- and repair times to be exponential. Relaxing these assumptions, though desirable, may prove to be challenging since the stochastic process will no longer be an SMP.
- We assumed that the units are identical. It is desirable to study a more realistic system involving non-identical units with different life- and repair rates. In particular, we must determine which unit should be put on operation and which on repair whenever there are multiple such units.
- We studied the system when patience time is random. It is highly interested to consider the case where patience time is predetermined which is logistically more desirable. However, we fail to use SMP under deterministic patience time policy since Markovian property fails in some states.

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