Exact Solutions To Linear Systems Using Rational Arithmetic And Conversions

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Abstract
This paper presents a new algorithm using rational arithmetic and conversions to calculate exact solutions to any linear system, including inconsistent systems (a null solution) and consistent yet under- and over-specified systems. The algorithm is presented as a MAPLE literate program for ease of implementation in an arbitrary analytical processing environment.

Key Words: Rational Arithmetic, Exact Numerical Methods, Literate Programming, Arbitrary Precision

1. Introduction

The accuracy and precision of statistical calculations is greatly enhanced by the strict use of rational arithmetic and conversions, i.e., expressing all data as rational numbers and expressing all calculations in terms of the addition, subtraction, multiplication, and division of the data. These conventions eliminate all round-off, truncation, and estimation error from statistical calculations, which is a significant improvement over the use of floating point representations. In fact, while only a (relatively) few real numbers may be expressed exactly in any floating point representation (those of the form $\sum_{k=-N}^{M} a_k b^k$ for finite positive integers N and M, with $a_k \in \{0, 1, 2, \ldots, b-1\}$, and $b \in \mathbb{Z}^+ \geq 2$), rational real numbers may be exactly expressed as the ratio of two integers, and the sum, difference, product, and quotient of any such rational number is again a rational number with no calculation error whatsoever. Furthermore, rational numbers include those with finite and infinite yet repeating decimal representations (see Appendix 2). The same cannot be said about floating point numbers nor their arithmetic results, whose error dynamics generally become highly complicated very quickly.

If statistical calculations were limited to arithmetic operations of rational data, the use of rational arithmetic and conversions would be the end of the story – and many algorithms may be expressed this way, including the calculation of a matrix inverse (consisting of the repeated use of row/column reductions without the need for compensating for inconveniences such as ill-conditioned cases, pivoting, dimension exchanges, and the like to compensate for rampant calculation error). However, statistical calculations also involve non-arithmetic operations, such as the square root, e.g., when calculating a sample standard deviation, and such as the exponential, e.g., when calculating a standard normal percentile, and such as dealing with irrational numbers, e.g., when π is involved. Therefore, one might ask: How might rational arithmetic and conversions be extended to the use of irrational data, yet retain the benefits of exact results under common statistical operations, including arithmetic, square root, and exponential operations?

This paper presents the data structures, operations, and analytical methods required to make exact statistical calculations with any kind of numerical data through the foundation of rational arithmetic and conversions. It is the responsibility of the implementing analyst to express the data and the calculation algorithm steps in as rudimentary form as possible,

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i.e., using rational-only data and using only rational arithmetic and conversion operations, to minimize the complexity of the statistical calculation scheme documented herein.

2. Trirational Numbers

For $n \geq 1$, let $\frac{a_0}{b_0}$ be a rational number, I_n be the *Index Set*, a vector of n-many unequal irrational numbers $\{c_i\}$, and V_n be the *Coefficient Set*, a vector of n-many rational numbers $\left\{\frac{a_i}{b_i}\right\}$. Then

$$x = \frac{a_0}{b_0} + \frac{a_1}{b_1}c_1 + \cdots + \frac{a_n}{b_n}c_n$$

is called a *Trirational Number Of Order* n in that it has three parts: (1) A *Purely Rational Part* $\mathcal{R}(x) = \frac{a_0}{b_0}$, (2) a *Purely Irrational Part* $\mathcal{I}(x) = \frac{a_1}{b_1}c_1 + \cdots + \frac{a_n}{b_n}c_n$, and (3) a *Marker Matrix* $\mathcal{M}$, which describe the multiplicative relationships between the elements of the index set (see also Sections 2.2 and 3). Collectively these sets, parts, and markers are called the *Rational Calculation Framework* (or simply "The Framework") for a particular statistical calculation.

Two trirational numbers x and y are said to have the same *Order* if the size of the index set of x is the same as the size of the index set for y, except possibly for the ordering of the elements.

2.1 Addition/Subtraction

Two trirational numbers of the same order are added/subtracted as follows. If $x = \frac{a_0}{b_0} + \frac{a_1}{b_1}c_1 + \cdots + \frac{a_n}{b_n}c_n$ and $y = \frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_1 + \cdots + \frac{a\prime_n}{b'_n}c_n$, then

$$x \pm y = r\left(\frac{a_0}{b_0} \pm \frac{a'_0}{b'_0}\right) + \left(r\left(\frac{a_1}{b_1} \pm \frac{a'_1}{b'_1}\right)c_1 + \cdots + r\left(\frac{a_n}{b_n} \pm \frac{a'_n}{b'_n}\right)c_n\right)$$

where $r(\cdots)$ denotes the reduced form of the rational number, i.e., all common factors have been divided out of the numerator and denominator.

Note the order of $x + y$ is always the common order of x and y.

Two trirational number of different orders are added/subtracted as follows. If $x = \frac{a_0}{b_0} + \frac{a_1}{b_1}c_1 + \cdots + \frac{a_n}{b_n}c_n$ and $y = \frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_1 + \cdots + \frac{a'_n}{b'_n}c_n + \frac{a'_{n+1}}{b'_{n+1}}c_{n+1} + \cdots + \frac{a'_m}{b'_m}c_m$, where $m > n$ and $\frac{a'_i}{b'_i} = \frac{0}{1}$ when c_i is not in the index set I of y, then

$$x \pm y = r\left(\frac{a_0}{b_0} \pm \frac{a'_0}{b'_0}\right) + \left(r\left(\frac{a_1}{b_1} \pm \frac{a'_1}{b'_1}\right)c_1 + \cdots + r\left(\frac{a_n}{b_n} \pm \frac{a'_n}{b'_n}\right)c_n \pm \frac{a'_{n+1}}{b'_{n+1}}c_{n+1} \pm \cdots \pm \frac{a'_m}{b'_m}c_m\right)$$

Note the order of $x \pm y$ is always less than or equal to the maximum of the orders of x and y, depending on whether any of the $\frac{a_i}{b_i} \pm \frac{a'_i}{b'_i}$ is zero.

2.2 Multiplication

Multiplication is the most complicated arithmetic operation for trirational numbers since the product of two irrational numbers may or may not be irrational. The marker matrix expresses these product results; if c_1 is the raw entry (the element in the first column) and c_2 is the column entry (the element in the first row), then the matrix element at their intersection signals the rationality/irrationality of the product.

2.2.1 *Order* 1 *With The Same Irrational Part*

Two trirational numbers of order 1 with the same irrational part are multiplied as follows. If $x = \frac{a_0}{b_0} + \frac{a_1}{b_1} c_1$ and $y = \frac{a'_0}{b'_0} + \frac{a'_1}{b'_1} c_1$, and c_1^2 is rational,[1] then

$$xy = r\left(\frac{a_0}{b_0}\frac{a'_0}{b'_0} + \frac{a_1}{b_1}\frac{a'_1}{b'_1}c_1^2\right) + \left(r\left(\frac{a_1}{b_1}\frac{a'_0}{b'_0} + \frac{a_0}{b_0}\frac{a'_1}{b'_1}\right)c_1\right)$$

and if c_1^2 is irrational, then for $c_2 = c_1^2$, we have

$$xy = r\left(\frac{a_0}{b_0}\frac{a'_0}{b'_0}\right) + \left(r\left(\frac{a_1}{b_1}\frac{a'_0}{b'_0} + \frac{a_0}{b_0}\frac{a'_1}{b'_1}\right)c_1 + r\left(\frac{a_1}{b_1}\frac{a'_1}{b'_1}\right)c_2\right)$$

2.2.2 *Order* 1 *With Different Irrational Parts*

Two trirational numbers of order 1 with different irrational parts are multiplied as follows. If $x = \frac{a_0}{b_0} + \frac{a_1}{b_1} c_1$ and $y = \frac{a'_0}{b'_0} + \frac{a'_1}{b'_1} c_2$, and $c_1 c_2$ is rational,[2] then

$$xy = r\left(\frac{a_0}{b_0}\frac{a'_0}{b'_0} + r\left(\frac{a_1}{b_1}\frac{a'_1}{b'_1}\right)c_1c_2\right) + \left(r\left(\frac{a'_0}{b'_0}\frac{a_1}{b_1}\right)c_1 + r\left(\frac{a_0}{b_0}\frac{a'_1}{b'_1}\right)c_2\right)$$

and if $c_1 c_2$ is irrational, then we have

$$xy = r\left(\frac{a_0}{b_0}\frac{a'_0}{b'_0}\right) + \left(r\left(\frac{a'_0}{b'_0}\frac{a_1}{b_1}\right)c_1 + r\left(\frac{a_0}{b_0}\frac{a'_1}{b'_1}\right)c_2 + r\left(\frac{a_1}{b_1}\frac{a'_1}{b'_1}\right)c_1c_2\right)$$

2.2.3 *Higher Orders*

Two trirational numbers of order $n_x \geq 2$ and $n_y \geq 2$ are multiplied as follows. If

$$x = \frac{a_{x,0}}{b_{x,0}} + \sum_{i=1}^{n_x} \frac{a_{x,i}}{b_{x,i}} c_i$$

and

$$y = \frac{a_{y,0}}{b_{y,0}} + \sum_{j=1}^{n_y} \frac{a_{y,j}}{b_{y,j}} c_j$$

where $\{c_i\}$ and $\{c_j\}$ come from the collection of all such irrational numbers $\{c_k\}$, then for each i from x let

$$\mathcal{M}_x(i) = \{j \geq 1 : \mathcal{M}(i,j) = 1\}$$

for the marker matrix $\mathcal{M}$, where i is the index for c_i. Then $\mathcal{M}_x(i)$ is the set of $\{c_j\}$ from y where $c_i c_j$ is rational. We have[3]

$$\mathcal{M}'_x(i) = \{j \geq 1 : \mathcal{M}(i,j) \leq 0\}$$

Then

$$xy = \left(\mathcal{R}(x)\mathcal{R}(y) + \sum_{i=1}^{n_x} \sum_{j \in \mathcal{M}_x(i)}^{n_y} \frac{a_{x,i}}{b_{x,i}}\frac{a_{y,j}}{b_{y,j}} c_i c_j\right) + \left(\frac{a_{y,0}}{b_{y,0}}\mathcal{I}(x) + \frac{a_{x,0}}{b_{x,0}}\mathcal{I}(y) + \sum_{i=1}^{n_x} \sum_{j \in \mathcal{M}'_x(i)}^{n_y} \frac{a_{x,i}}{b_{x,i}}\frac{a_{y,j}}{b_{y,j}} c_i c_j\right)$$

[1] The rationality of c_1^2 is signaled in the marker matrix entry at the intersection of where c_1 is in the row first element and in the column first element.

[2] See the marker matrix entry at the intersection where c_1 and c_2 are the row/column first elements, respectively.

[3] An entry of -1 in the marker matrix to be assumed irrational until indicated otherwise in the marker matrix, i.e., these terms may be continued in marker matrix manipulations until a particular marker matrix shows it should be treated as rational.

2.3 Reciprocal

The reciprocal is used to implement division as the multiplication of a trirational numerator number by the reciprocal of the trirational denominator number.

2.3.1 Order 1

The reciprocal of a non-zero trirational number of order 1 is calculated as follows. If $x = \frac{a_0}{b_0} + \frac{a_1}{b_1}c_1$, where $a_0 \neq 0$, then

$$\frac{1}{\frac{a_0}{b_0} + \frac{a_1}{b_1}c_1} = \frac{b_0}{a_0} - \frac{b_0}{a_0}\frac{a_1}{b_1}\frac{c_1}{\frac{a_0}{b_0} + \frac{a_1}{b_1}c_1}$$

since

$$\left(\frac{b_0}{a_0} - \frac{b_0}{a_0}\frac{a_1}{b_1}\frac{c_1}{\frac{a_0}{b_0} + \frac{a_1}{b_1}c_1}\right)\left(\frac{a_0}{b_0} + \frac{a_1}{b_1}c_1\right) = \left(\frac{a_0}{b_0}\frac{b_0}{a_0}\right) + \left(\frac{b_0}{a_0}\frac{a_1}{b_1} - \frac{b_0}{a_0}\frac{a_1}{b_1}\right)c_1 = 1$$

where $\frac{c_1}{\frac{a_0}{b_0}+\frac{a_1}{b_1}c_1}$ is irrational.[4] Note that the reciprocal is still of order 1, however, the purely irrational part is based on a different irrational number.

Note also that $\mathcal{R}\left(\frac{b_0}{a_0} - \frac{b_0}{a_0}\frac{a_1}{b_1}\frac{c_1}{\frac{a_0}{b_0}+\frac{a_1}{b_1}c_1}\right) = \frac{b_0}{a_0}$ and $\mathcal{I}\left(\frac{b_0}{a_0} - \frac{b_0}{a_0}\frac{a_1}{b_1}\frac{c_1}{\frac{a_0}{b_0}+\frac{a_1}{b_1}c_1}\right) = -\frac{b_0}{a_0}\frac{a_1}{b_1}$, so that

$$\frac{1}{\frac{b_0}{a_0} - \frac{b_0}{a_0}\frac{a_1}{b_1}\frac{c_1}{\frac{a_0}{b_0}+\frac{a_1}{b_1}c_1}} = \frac{1}{\frac{b_0}{a_0}} - \left(\frac{1}{\frac{b_0}{a_0}}\right)\left(-\frac{b_0}{a_0}\frac{a_1}{b_1}\right)\frac{\frac{c_1}{\frac{a_0}{b_0}+\frac{a_1}{b_1}c_1}}{\frac{b_0}{a_0} - \frac{b_0}{a_0}\frac{a_1}{b_1}\frac{c_1}{\frac{a_0}{b_0}+\frac{a_1}{b_1}c_1}} = \frac{a_0}{b_0} + \frac{a_1}{b_1}c_1$$

This means the reciprocal of the reciprocal of a trirational number of order 1 is that trirational number of order 1.

For the case where $a_0 = 0$, so that $x = \frac{a_1}{b_1}c_1$, where $a_1 \neq 0$, then

$$\frac{1}{\frac{a_1}{b_1}c_1} = \frac{b_1}{a_1}\frac{1}{c_1}$$

where $\frac{1}{c_1}$ is irrational since c_1 is irrational.

2.3.2 Higher Orders

The reciprocal of a non-zero trirational number of order $n_y \geq 2$ is given by

$$\frac{1}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i} = \frac{b_{y,0}}{a_{y,0}} - \frac{b_{y,0}}{a_{y,0}}\sum_{j=1}^{n_y}\frac{a_{y,j}}{b_{y,j}}\frac{c_j}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i}$$

where $a_{y,0} \neq 0$.

[4] This value must be irrational when c_1 is irrational, since otherwise

$$\frac{c_1}{\frac{a_0}{b_0} + \frac{a_1}{b_1}c_1} = \frac{n}{m}$$

would mean

$$c_1 = \frac{\frac{n}{m}\frac{a_0}{b_0}}{1 - \frac{n}{m}\frac{a_1}{b_1}}$$

which is rational.

Note that each $\frac{c_j}{\frac{a_{y,0}}{b_{y,0}}+\sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i}$ is irrational[5] and that

$$\frac{1}{\frac{b_{y,0}}{a_{y,0}} - \frac{b_{y,0}}{a_{y,0}}\sum\limits_{j=1}^{n_y}\frac{a_{y,j}}{b_{y,j}}\frac{c_j}{\frac{a_{y,0}}{b_{y,0}}+\sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i}} = \frac{1}{\frac{b_{y,0}}{a_{y,0}}} - \frac{1}{\frac{b_{y,0}}{a_{y,0}}}\sum_{j=1}^{n_y}\left(-\frac{b_{y,0}}{a_{y,0}}\frac{a_{y,j}}{b_{y,j}}\right)\frac{\frac{c_j}{\frac{a_{y,0}}{b_{y,0}}+\sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i}}{\frac{b_{y,0}}{a_{y,0}} - \frac{b_{y,0}}{a_{y,0}}\sum\limits_{j=1}^{n_y}\frac{a_{y,j}}{b_{y,j}}\frac{c_j}{\frac{a_{y,0}}{b_{y,0}}+\sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i}} = \frac{a_{y,0}}{b_{y,0}}+\sum_{j=1}^{n_y}\frac{a_{y,j}}{b_{y,j}}c_j$$

This means the reciprocal of the reciprocal of a trirational number of order $n_y \geq 2$ is that trirational number of order $n_y \geq 2$.

For the case where $a_{y,0} = 0$, so that $x = \sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i \neq 0$, where none of the $a_{y,i}$ are zero, then $\frac{1}{\sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i}$ is the reciprocal of x, where $\frac{1}{\sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i}$ is (pure) irrational[6] since each c_i is irrational and all $\frac{a_{y,i}}{b_{y,i}}$ are (non-zero) rational.

2.4 Division

Division is implemented as the multiplication of the trirational numerator number by the reciprocal of the trirational denominator number.

2.4.1 Order 1 *With The Same Irrational Part*

Two trirational numbers of order 1 with the same irrational part are divided as follows. If $x = \frac{a_0}{b_0} + \frac{a_1}{b_1}c_1$ and $y = \frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_1 \neq \frac{0}{1}$, and c_1^2 is rational,[7] then

$$\frac{x}{y} = r\left(\frac{\frac{a_1}{b_1}\frac{a'_1}{b'_1}c_1^2 - \frac{a_0}{b_0}\frac{a'_0}{b'_0}}{\left(\frac{a'_1}{b'_1}\right)^2 c_1^2 - \left(\frac{a'_0}{b'_0}\right)^2}\right) + r\left(\frac{\frac{a_0}{b_0}\frac{a'_1}{b'_1} - \frac{a_1}{b_1}\frac{a'_0}{b'_0}}{\left(\frac{a'_1}{b'_1}\right)^2 c_1^2 - \left(\frac{a'_0}{b'_0}\right)^2}\right)c_1$$

since

$$\mathcal{R}\left(\frac{x}{y}y\right) = \frac{\frac{a_1}{b_1}\frac{a'_1}{b'_1}c_1^2 - \frac{a_0}{b_0}\frac{a'_0}{b'_0}}{\left(\frac{a'_1}{b'_1}\right)^2 c_1^2 - \left(\frac{a'_0}{b'_0}\right)^2}\frac{a'_0}{b'_0} + \frac{\frac{a_0}{b_0}\frac{a'_1}{b'_1} - \frac{a_1}{b_1}\frac{a'_0}{b'_0}}{\left(\frac{a'_1}{b'_1}\right)^2 c_1^2 - \left(\frac{a'_0}{b'_0}\right)^2}\frac{a'_1}{b'_1}c_1^2 = \frac{a_0}{b_0} = \mathcal{R}\left(x\right)$$

and

$$\mathcal{I}\left(\frac{x}{y}y\right) = \frac{\frac{a_1}{b_1}\frac{a'_1}{b'_1}c_1^2 - \frac{a_0}{b_0}\frac{a'_0}{b'_0}}{\left(\frac{a'_1}{b'_1}\right)^2 c_1^2 - \left(\frac{a'_0}{b'_0}\right)^2}\frac{a'_1}{b'_1} + \frac{\frac{a_0}{b_0}\frac{a'_1}{b'_1} - \frac{a_1}{b_1}\frac{a'_0}{b'_0}}{\left(\frac{a'_1}{b'_1}\right)^2 c_1^2 - \left(\frac{a'_0}{b'_0}\right)^2}\frac{a'_0}{b'_0} = \frac{a_1}{b_1} = \mathcal{I}\left(x\right)$$

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[5]If $\frac{c_j}{\frac{a_{y,0}}{b_{y,0}}+\sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i}$ were rational, say r, for any j, then

$$c_j - \sum_{i=1}^{n_y}\left(r\frac{a_{y,i}}{b_{y,i}}\right)c_i = r\frac{a_{y,0}}{b_{y,0}}$$

which is a linear rational combination of reduced irrational numbers that equals a rational number; this is a contradiction.

[6]If $\frac{1}{\sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i}$ were rational, then $\sum\limits_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}}c_i$ would be rational, and the rational linear combination of irrationals is irrational – a contradiction.

[7]The same rules for determining the rationality of c_1^2 found in Section 2.2.1 apply here as well.

If c_1^2 is irrational,[8] then

$$\begin{aligned}\frac{x}{y} &= x\left(\frac{1}{y}\right)\\ &= \left(\frac{a_0}{b_0}+\frac{a_1}{b_1}c_1\right)\left(\frac{b_0'}{a_0'}-\frac{b_0'}{a_0'}\frac{a_1'}{b_1'}\frac{c_1}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}\right)\\ &= r\left(\frac{b_0'}{a_0'}\frac{a_0}{b_0}\right)+\left(r\left(\frac{b_0'}{a_0'}\frac{a_1}{b_1}\right)c_1-r\left(\frac{a_0}{b_0}\frac{b_0'}{a_0'}\frac{a_1'}{b_1'}\right)\frac{c_1}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}-r\left(\frac{a_1}{b_1}\frac{b_0'}{a_0'}\frac{a_1'}{b_1'}\right)\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}\right)\end{aligned} \tag{1}$$

where the purely irrational part is based on two or three irrational numbers,[9] namely

$$\left\{c_1, \frac{c_1}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}\right\} \quad \text{or} \quad \left\{c_1, \frac{c_1}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}, \frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}\right\}$$

depending whether $\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$ is rational or irrational. This determination is found in the $\mathcal{M}$ matrix (see Section 2.2). However, see also Appendix 1 for some simplifying conditions for determining the rationality of $\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$.

2.4.2 Order 1 With Different Irrational Parts

Two trirational numbers of order 1 with different irrational parts are divided as follows. If $x = \frac{a_0}{b_0}+\frac{a_1}{b_1}c_1$ and $y = \frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_2 \neq \frac{0}{1}$, and c_1c_2 is rational,[10] then

$$\frac{x}{y} = r\left(\frac{\frac{a_1}{b_1}\frac{a_1'}{b_1'}c_1c_2-\frac{a_0}{b_0}\frac{a_0'}{b_0'}}{\left(\frac{a_1'}{b_1'}\right)^2 c_1c_2-\left(\frac{a_0'}{b_0'}\right)^2}\right)+r\left(\frac{\frac{a_0}{b_0}\frac{a_1'}{b_1'}-\frac{a_1}{b_1}\frac{a_0'}{b_0'}}{\left(\frac{a_1'}{b_1'}\right)^2 c_1c_2-\left(\frac{a_0'}{b_0'}\right)^2}\right)c_1$$

since

$$\mathcal{R}\left(\frac{x}{y}y\right) = \frac{\frac{a_1}{b_1}\frac{a_1'}{b_1'}c_1c_2-\frac{a_0}{b_0}\frac{a_0'}{b_0'}}{\left(\frac{a_1'}{b_1'}\right)^2 c_1c_2-\left(\frac{a_0'}{b_0'}\right)^2}\frac{a_0'}{b_0'}+\frac{\frac{a_0}{b_0}\frac{a_1'}{b_1'}-\frac{a_1}{b_1}\frac{a_0'}{b_0'}}{\left(\frac{a_1'}{b_1'}\right)^2 c_1c_2-\left(\frac{a_0'}{b_0'}\right)^2}\frac{a_1'}{b_1'}c_1c_2 = \frac{a_0}{b_0} = \mathcal{R}(x)$$

and

$$\mathcal{I}\left(\frac{x}{y}y\right) = \frac{\frac{a_1}{b_1}\frac{a_1'}{b_1'}c_1c_2-\frac{a_0}{b_0}\frac{a_0'}{b_0'}}{\left(\frac{a_1'}{b_1'}\right)^2 c_1c_2-\left(\frac{a_0'}{b_0'}\right)^2}\frac{a_1'}{b_1'}+\frac{\frac{a_0}{b_0}\frac{a_1'}{b_1'}-\frac{a_1}{b_1}\frac{a_0'}{b_0'}}{\left(\frac{a_1'}{b_1'}\right)^2 c_1c_2-\left(\frac{a_0'}{b_0'}\right)^2}\frac{a_0'}{b_0'} = \frac{a_1}{b_1} = \mathcal{I}(x)$$

If c_1c_2 is irrational, then

$$\frac{x}{y} = x\left(\frac{1}{y}\right)$$

[8]The same rules for determining the rationality of c_1^2 found in Section 2.2.2 apply here as well.

[9]See Appendix 1 for the proof that $\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$ is rational if and only if c_1 has the form $\pm\frac{2\sqrt{\frac{n'}{m'}}\frac{a_0'}{b_0'}}{1\mp\sqrt{\frac{n'}{m'}}\frac{a_1'}{b_1'}}$ for rational $\frac{a_0'}{b_0'}$ and $\frac{a_1'}{b_1'}$, and non-perfect squares n' and m'.

[10]The same rules for determining the rationality of c_1^2 found in Section 2.2 apply here as well.

$$= \left(\frac{a_0}{b_0} + \frac{a_1}{b_1}c_1\right)\left(\frac{b'_0}{a'_0} - \frac{b'_0}{a'_0}\frac{a'_1}{b'_1}\frac{c_2}{\frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_2}\right)$$

$$= r\left(\frac{b'_0}{a'_0}\frac{a_0}{b_0}\right) + \left(r\left(\frac{b'_0}{a'_0}\frac{a_1}{b_1}\right)c_1 - r\left(\frac{a_0}{b_0}\frac{b'_0}{a'_0}\frac{a'_1}{b'_1}\right)\frac{c_2}{\frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_2} - r\left(\frac{a_1}{b_1}\frac{b'_0}{a'_0}\frac{a'_1}{b'_1}\right)\frac{c_1 c_2}{\frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_2}\right) \tag{2}$$

where the purely irrational part is based on two or three irrational numbers, namely

$$\left\{c_1, \frac{c_2}{\frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_2}\right\} \quad \text{or} \quad \left\{c_1, \frac{c_2}{\frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_2}, \frac{c_1 c_2}{\frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_2}\right\}$$

depending whether $\frac{c_1 c_2}{\frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}c_2}$ is rational or irrational. This determination is found in the $\mathcal{M}$matrix (see Section 2.2).

2.4.3 Higher Orders

Two trirational numbers of order $n_x \geq 2$ and $n_y \geq 2$ are divided as follows. If

$$x = \frac{a_{x,0}}{b_{x,0}} + \sum_{i=1}^{n_x} \frac{a_{x,i}}{b_{x,i}} c_i$$

and

$$\frac{1}{y} = \frac{b_{y,0}}{a_{y,0}} - \frac{b_{y,0}}{a_{y,0}} \sum_{j=1}^{n_y} \frac{a_{y,j}}{b_{y,j}} \frac{c_j}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y} \frac{a_{y,i}}{b_{y,i}} c_i}$$

where $\{c_i\}$ and $\left\{\frac{c_j}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y} \frac{a_{y,i}}{b_{y,i}} c_i}\right\}$ come from the collection of all such irrational numbers $\{c_k\}$, then for each i from x let

$$\mathcal{M}_x(i) = \{j \geq 1 : \mathcal{M}(i,j) = 1\}$$

that is the set of $\left\{\frac{c_j}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y} \frac{a_{y,i}}{b_{y,i}} c_i}\right\}$ from $\frac{1}{y}$ where $\frac{c_i c_j}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y} \frac{a_{y,i}}{b_{y,i}} c_i}$ is rational. We have

$$\mathcal{M}'_x(i) = \mathcal{M}_x(i) = \{j \geq 1 : \mathcal{M}(i,j) \leq 0\}$$

Then

$$\frac{x}{y} = x\left(\frac{1}{y}\right) = \left(\frac{a_{x,0}}{b_{x,0}} + \sum_{i=1}^{n_x} \frac{a_{x,i}}{b_{x,i}} c_i\right)\left(\frac{b_{y,0}}{a_{y,0}} - \frac{b_{y,0}}{a_{y,0}} \sum_{j=1}^{n_y} \frac{a_{y,j}}{b_{y,j}} \frac{c_j}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y} \frac{a_{y,i}}{b_{y,i}} c_i}\right)$$

$$= \left(\begin{array}{c} \left(\frac{a_{x,0}}{b_{x,0}}\frac{b_{y,0}}{a_{y,0}} - \sum_{i=1}^{n_x}\sum_{j \in \mathcal{D}_x(i)}^{n_y} \left(\frac{b_{y,0}}{a_{y,0}}\frac{a_{x,i}}{b_{x,i}}\frac{a_{y,j}}{b_{y,j}}\right)\frac{c_i c_j}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}} c_i}\right) \\ + \left(\sum_{i=1}^{n_x}\left(\frac{a_{x,i}}{b_{x,i}}\frac{b_{y,0}}{a_{y,0}}\right)c_i - \sum_{j=1}^{n_y}\left(\frac{a_{x,0}}{b_{x,0}}\frac{b_{y,0}}{a_{y,0}}\frac{a_{y,j}}{b_{y,j}}\right)\frac{c_j}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}} c_i} - \sum_{i=1}^{n_x}\sum_{j \in \mathcal{D}'_x(i)}^{n_y}\left(\frac{b_{y,0}}{a_{y,0}}\frac{a_{x,i}}{b_{x,i}}\frac{a_{y,j}}{b_{y,j}}\right)\frac{c_i c_j}{\frac{a_{y,0}}{b_{y,0}} + \sum_{i=1}^{n_y}\frac{a_{y,i}}{b_{y,i}} c_i}\right) \end{array}\right)$$

2.5 Square Root

Definition 1 *The square root of a trirational number* $\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i$ *of order* $N \geq 1$ *is a pure irrational trirational number of order* 1 *given by* $\sqrt{\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i}$.

The following sequence of claims shows that the defined form for the square root of a trirational number, up to a constant factor, is the only one generally available.

Claim 2 *(A) The square root of a trirational number* $\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i$ *of order* $N \geq 1$*, where not all* $\left\{\frac{a_i}{b_i}\right\}_{i=1}^{N}$ *are zero, is irrational; (B)* $x = \sqrt{\frac{a_0}{b_0} + \frac{a_1}{b_1} c_1}$ *cannot be expressed as* $x = \frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}\sqrt{c_1}$ *for any rational numbers* $\left\{\frac{a'_0}{b'_0}, \frac{a'_1}{b'_1}\right\}$*; (C) If* $\frac{a_0}{b_0} + \sum_{i=1}^{N-1} \frac{a_i}{b_i} c_i \equiv \frac{1}{4}\left(\frac{m}{n}\frac{a_N}{b_N} - \frac{n}{m}\right)^2 c_N$ *for some integers* n *and* m*, and for* $N \geq 2$*, then* $\sqrt{\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i}$ *cannot be expressed as* $x = \frac{a'_0}{b'_0} + \sum_{i=1}^{N} \frac{a'_i}{b'_i}\sqrt{c_i}$*, for any rational numbers* $\left\{\frac{a'_i}{b'_i}\right\}_{i=0}^{N}$*; (D) If* $\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i \equiv \frac{1}{4}\left(\frac{m}{n}\frac{a_{N+1}}{b_{N+1}} - \frac{n}{m}\right)^2 c_{N+1}$ *for some integers* n *and* m*, and for some choice of* $\frac{a_{N+1}}{b_{N+1}}$ *and* c_{N+1}*, then* $\sqrt{\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i} \equiv \frac{1}{2}\left|\frac{m}{n}\frac{a_{N+1}}{b_{N+1}} - \frac{n}{m}\right|\sqrt{c_{N+1}}$.

Proof. (A) If $\sqrt{\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i}$, where not all $\left\{\frac{a_i}{b_i}\right\}_{i=1}^{N}$ are zero, were rational, then $\sqrt{\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i} = \frac{n}{m}$ for some integers n and m. Then $\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i = \frac{n^2}{m^2}$, which is a contradiction.

(B) Suppose

$$\sqrt{\frac{a_0}{b_0} + \frac{a_1}{b_1} c_1} = \frac{a'_0}{b'_0} + \frac{a'_1}{b'_1}\sqrt{c_1}$$

for some rational numbers $\frac{a'_0}{b'_0}$ and $\frac{a'_1}{b'_1}$. Then

$$\frac{a_0}{b_0} + \frac{a_1}{b_1} c_1 = \left(\frac{a'_0}{b'_0}\right)^2 + 2\frac{a'_0}{b'_0}\frac{a'_1}{b'_1}\sqrt{c_1} + \left(\frac{a'_1}{b'_1}\right)^2 c_1$$

and since c_1 and $\sqrt{c_1}$ are irrational, we have

$$\frac{a'_0}{b'_0} = \sqrt{\frac{a_0}{b_0}}$$

Therefore, assuming $\sqrt{\frac{a_0}{b_0}}$ were rational, we have

$$\left(\frac{a'_1}{b'_1}\right)^2 c_1 + 2\sqrt{\frac{a_0}{b_0} c_1}\left(\frac{a'_1}{b'_1}\right) - \frac{a_1}{b_1} c_1 = 0$$

This means

$$\frac{a_1'}{b_1'} = \frac{\pm\sqrt{\frac{a_0}{b_0} + \frac{a_1}{b_1}c_1} - \sqrt{\frac{a_0}{b_0}}}{\sqrt{c_1}}$$

must be rational, say, $\frac{n}{m}$ for some integers n and m. Hence, we have

$$\frac{a_0}{b_0} + \frac{a_1}{b_1}c_1 = \left(\sqrt{\frac{a_0}{b_0}} + \frac{n}{m}\sqrt{c_1}\right)^2$$

or

$$c_1 = 4\frac{\frac{n^2}{m^2}\frac{a_0}{b_0}}{\left(\frac{n^2}{m^2} - \frac{a_1}{b_1}\right)^2}$$

which is a contradiction.

(C) Part B establishes the basis for induction on N. Now suppose $\sqrt{\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i}c_i}$ cannot be expressed as $\frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'}\sqrt{c_i}$ for some value of N, yet

$$\sqrt{\frac{a_0}{b_0} + \sum_{i=1}^{N+1} \frac{a_i}{b_i}c_i} = \frac{a_0'}{b_0'} + \sum_{i=1}^{N+1} \frac{a_i'}{b_i'}\sqrt{c_i}$$

for some rational numbers $\left\{\frac{a_i'}{b_i'}\right\}_{i=0}^{N+1}$, which means

$$\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i}c_i = \left(\begin{array}{l} \left(\frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'}\sqrt{c_i}\right)^2 \\ +2\left(\frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'}\sqrt{c_i}\right)\left(\frac{a_{N+1}'}{b_{N+1}'}\sqrt{c_{N+1}}\right) \\ +\left(\left(\frac{a_{N+1}'}{b_{N+1}'}\right)^2 - \frac{a_{N+1}}{b_{N+1}}\right)c_{N+1} \end{array}\right) \tag{3}$$

Since

$$\left(\frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'}\sqrt{c_i}\right)^2 = \frac{1}{4}\left(\frac{m}{n}\frac{a_{N+1}}{b_{N+1}} - \frac{n}{m}\right)^2 c_{N+1}$$

for some integers n and m, we have

$$c_{N+1} = 4\frac{\left(\frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'}\sqrt{c_i}\right)^2}{\left(\frac{a_{N+1}}{b_{N+1}} - \frac{n^2}{m^2}\right)^2}\frac{n^2}{m^2}$$

so that

$$\left(\frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'}\sqrt{c_i}\right)^2 + \frac{a_{N+1}}{b_{N+1}}c_{N+1} = \left(\left(\frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'}\sqrt{c_i}\right) + \frac{n}{m}\sqrt{c_{N+1}}\right)^2$$

which means

$$\frac{\pm\sqrt{\left(\frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'}\sqrt{c_i}\right)^2 + \frac{a_{N+1}}{b_{N+1}}c_{N+1}} - \left(\frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'}\sqrt{c_i}\right)}{\sqrt{c_{N+1}}} = \frac{n}{m}$$

is rational. Set

$$\frac{a'_{N+1}}{b'_{N+1}} = \frac{\pm\sqrt{\left(\frac{a'_0}{b'_0} + \sum_{i=1}^{N} \frac{a'_i}{b'_i}\sqrt{c_i}\right)^2 + \frac{a_{N+1}}{b_{N+1}} c_{N+1}} - \left(\frac{a'_0}{b'_0} + \sum_{i=1}^{N} \frac{a'_i}{b'_i}\sqrt{c_i}\right)}{\sqrt{c_{N+1}}}$$

so that

$$\left(\frac{a'_{N+1}}{b'_{N+1}}\right)^2 c_{N+1} + 2\left(\frac{a'_0}{b'_0} + \sum_{i=1}^{N} \frac{a'_i}{b'_i}\sqrt{c_i}\right)\sqrt{c_{N+1}}\frac{a'_{N+1}}{b'_{N+1}} - \frac{a_{N+1}}{b_{N+1}} c_{N+1} = 0$$

and from (3) we have

$$\frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i = \left(\frac{a'_0}{b'_0} + \sum_{i=1}^{N} \frac{a'_i}{b'_i}\sqrt{c_i}\right)^2$$

which is a contradiction.

(D) If $\frac{a_0}{b_0} + \sum_{i=1}^{N-1} \frac{a_i}{b_i} c_i = \frac{1}{4}\left(\frac{m}{n}\frac{a_N}{b_N} - \frac{n}{m}\right)^2 c_N$ for some integers n and m, and for some choice of $\frac{a_{N+1}}{b_{N+1}}$ and c_{N+1}, then

$$\sqrt{\frac{a_0}{b_0} + \sum_{i=1}^{N-1} \frac{a_i}{b_i} c_i} = \sqrt{\frac{1}{4}\left(\frac{m}{n}\frac{a_N}{b_N} - \frac{n}{m}\right)^2 c_N} = \frac{1}{2}\left|\frac{m}{n}\frac{a_N}{b_N} - \frac{n}{m}\right|\sqrt{c_N}$$

■

The following miscellaneous relationships between trirational numbers and their square roots demonstrate one possible system for restricting the complexity of algorithm results that use the square root.

Definition 3 *The irrational part of a trirational number consisting of the square roots of non-perfect square reduced rational numbers is said to be* Closed *if the reduced form of any product or quotient of two distinct members is either rational or a rational multiple of a member of the set.*

For example, the set of the square roots of non-perfect square reduced rational numbers

$$A = \left\{\sqrt{2}, \sqrt{3}, \sqrt{6}, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{2}}\right\}$$

is closed, since

$$
\begin{array}{ll||ll}
\sqrt{2} * \sqrt{3} = \sqrt{6}, & \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}} & \sqrt{6} * \sqrt{\frac{1}{2}} = \sqrt{3} & \frac{\sqrt{6}}{\sqrt{\frac{1}{2}}} = 2\sqrt{3} \\
\sqrt{2} * \sqrt{6} = 2\sqrt{3}, & \frac{\sqrt{2}}{\sqrt{6}} = \sqrt{\frac{1}{3}} & \sqrt{6} * \sqrt{\frac{1}{3}} = \sqrt{2} & \frac{\sqrt{6}}{\sqrt{\frac{1}{3}}} = 3\sqrt{2} \\
\sqrt{2} * \sqrt{\frac{1}{2}} = 1, & \frac{\sqrt{2}}{\sqrt{\frac{1}{2}}} = 2 & \sqrt{6} * \sqrt{\frac{2}{3}} = 2, & \frac{\sqrt{6}}{\sqrt{\frac{2}{3}}} = 3 \\
\sqrt{2} * \sqrt{\frac{1}{3}} = \sqrt{\frac{2}{3}}, & \frac{\sqrt{2}}{\sqrt{\frac{1}{3}}} = \sqrt{6} & \sqrt{6} * \sqrt{\frac{3}{2}} = 3, & \frac{\sqrt{6}}{\sqrt{\frac{3}{2}}} = 2 \\
\sqrt{2} * \sqrt{\frac{2}{3}} = 2\sqrt{\frac{1}{3}}, & \frac{\sqrt{2}}{\sqrt{\frac{2}{3}}} = \sqrt{3} & \sqrt{\frac{1}{2}} * \sqrt{\frac{1}{3}} = 3, & \frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{3}}} = \sqrt{\frac{3}{2}} \\
\sqrt{2} * \sqrt{\frac{3}{2}} = \sqrt{3}, & \frac{\sqrt{2}}{\sqrt{\frac{3}{2}}} = 2\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} * \sqrt{\frac{2}{3}} = \sqrt{\frac{1}{3}}, & \frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{2}{3}}} = \frac{1}{2}\sqrt{3} \\
\sqrt{3} * \sqrt{6} = 2\sqrt{3}, & \frac{\sqrt{3}}{\sqrt{6}} = \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} * \sqrt{\frac{3}{2}} = \frac{1}{2}\sqrt{3}, & \frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{2}}} = \sqrt{\frac{1}{3}} \\
\sqrt{3} * \sqrt{\frac{1}{2}} = \sqrt{\frac{3}{2}}, & \frac{\sqrt{3}}{\sqrt{\frac{1}{2}}} = \sqrt{6} & \sqrt{\frac{1}{3}} * \sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{2}, & \frac{\sqrt{\frac{1}{3}}}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{1}{2}} \\
\sqrt{3} * \sqrt{\frac{1}{3}} = 1, & \frac{\sqrt{3}}{\sqrt{\frac{1}{3}}} = 3 & \sqrt{\frac{1}{3}} * \sqrt{\frac{3}{2}} = \sqrt{\frac{1}{2}}, & \frac{\sqrt{\frac{1}{3}}}{\sqrt{\frac{3}{2}}} = \frac{1}{3}\sqrt{2} \\
\sqrt{3} * \sqrt{\frac{2}{3}} = \sqrt{2}, & \frac{\sqrt{3}}{\sqrt{\frac{2}{3}}} = 3\sqrt{\frac{1}{2}} & \sqrt{\frac{2}{3}} * \sqrt{\frac{3}{2}} = 1, & \frac{\sqrt{\frac{2}{3}}}{\sqrt{\frac{3}{2}}} = \frac{2}{3} \\
\sqrt{3} * \sqrt{\frac{3}{2}} = 3\sqrt{\frac{1}{2}}, & \frac{\sqrt{3}}{\sqrt{\frac{3}{2}}} = \sqrt{2} & \sqrt{x} * \sqrt{x} = x, & x \in \left\{2, 3, 6, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}\right\}
\end{array}
$$

Claim 4 *If p and q are positive prime numbers, then* $A = \left\{\sqrt{p}, \sqrt{q}, \sqrt{pq}, \sqrt{\frac{1}{p}}, \sqrt{\frac{1}{q}}, \sqrt{\frac{p}{q}}, \sqrt{\frac{q}{p}}\right\}$ *is closed.*

Proof. Every element of A is of the form $\sqrt{\frac{p^a q^b}{p^c q^d}}$ for $\{0,1\}$ integers a, b, c, and d. For example, $\sqrt{\frac{q}{p}}$ has (a,b,c,d) as $(0,1,1,0)$. Then the product of any such $\sqrt{\frac{p^a q^b}{p^c q^d}}$ with another distinct one is additive in the (a,b,c,d). Suppose (a',b',c',d') is the sum of such a product. Then the values of a', b', c', and d' may be $\{0,1,2\}$; if any is 2, say, a', then $\sqrt{p^{a'}} = \sqrt{p^2} = p$, which is rational. All other terms in (a',b',c',d') are in $\{0,1\}$, which makes $\sqrt{\frac{p^{a'} q^{b'}}{p^{c'} q^{d'}}}$ part of A.

The quotient of any such $\sqrt{\frac{p^a q^b}{p^c q^d}}$ with another distinct one is subtracting in the (a,b,c,d). Suppose (a'',b'',c'',d'') is the difference of such a quotient. Then the values of a'', b'', c'', and d'' may be $\{-1,0,1\}$; if $a'' = -1$, then this is the same as $(0, b'', c''+1, d'')$, and if $b'', c'', d'' = -1$, then this is the same as $(a'', 0, c'', d''+1)$, $(a''+1, b'', 0, d'')$, and $(a'', b''+1, c'', 0)$, respectively. This makes all terms (a'',b'',c'',d'') be in $\{0,1,2\}$, which, as previously shown, makes all such combinations $\sqrt{\frac{p^{a''} q^{b''}}{p^{c''} q^{d''}}}$ part of A. ■

Corollary 5 *If* $\{p_j\}_{j=1}^m$ *is a set of positive prime numbers, then* $A = \left\{\sqrt{\frac{\prod_{j=1}^m p_j^{a_j}}{\prod_{j=1}^m p_j^{b_j}}}\right\}$ *for all reduced combinations of* $\{0,1\}$ *integers* $\{a_j, b_j\}$ *is closed.*

Claim 6 *If a set of trirational numbers have a closed irrational set, then the sum, difference, and product of those numbers have the same irrational set.*

Proof. Let a set of trirational numbers have a closed irrational set $I = \{c_k\}_{k=1}^{N}$. If $x = \frac{a_0}{b_0} + \sum_{i=1}^{N} \frac{a_i}{b_i} c_i$ and $y = \frac{a_0'}{b_0'} + \sum_{i=1}^{N} \frac{a_i'}{b_i'} c_i$, then

(Sum/Difference)

$$x \pm y = r\left(\frac{a_0}{b_0} \pm \frac{a_0'}{b_0'}\right) + \sum_{i=1}^{N} r\left(\frac{a_1}{b_1} \pm \frac{a_1'}{b_1'}\right) c_i$$

which has irrational set I.

(Product)

$$xy = \left(\mathcal{R}(x)\mathcal{R}(y) + \sum_{i=1}^{n_x} \sum_{j \in \mathcal{M}_x(i)}^{n_y} \frac{a_{x,i}}{b_{x,i}} \frac{a_{y,j}}{b_{y,j}} c_i c_j\right) + \left(\frac{a_{y,0}}{b_{y,0}} \mathcal{I}(x) + \frac{a_{x,0}}{b_{x,0}} \mathcal{I}(y) + \sum_{i=1}^{n_x} \sum_{j \in \mathcal{M}_x'(i)}^{n_y} \frac{a_{x,i}}{b_{x,i}} \frac{a_{y,j}}{b_{y,j}} c_i c_j\right)$$

where $\mathcal{I}(x)$, $\mathcal{I}(y)$, and $\{c_i c_j\}$ are all part of I, since it is closed. ∎

Note that the quotient of such numbers involves irrational numbers not in the closed set.

2.6 Exponential/Logarithm

The exponential and logarithmic functions act like the square root in that an entirely irrational number is produced by the operation when the argument is rational. For irrational arguments, it is reasonable to assume the exponential and logarithmic functions are irrational until indicated otherwise in the marker matrix.

Claim 7 *For a positive integer q, e^q is irrational.*

Proof. (Inspired by Fourier[11]) Suppose e^q were rational. Then $e^q = \frac{n}{m}$ for positive integers n and $m \geq 2$ (since e^q is not an integer). From the Taylor Series expansion of $f(x) = e^x$ at $x = 0$, with the standard Lagrange form of the remainder term, we have

$$e^x - \sum_{k=0}^{v} \frac{1}{k!} x^k < \frac{e^{\xi} |x|^{v+1}}{(v+1)!}$$

for every integer $v \geq 0$ for some value $\xi \in (-q, q)$. Hence, we have

$$e^q - \sum_{k=0}^{v} \frac{1}{k!} q^k < \frac{e^q q^{v+1}}{(v+1)!}$$

Using $\Gamma(x)$ as a continuous interpolation of the discrete factorial, we have

$$\frac{d}{dx} \frac{q^x}{\Gamma(x)} = \frac{q^x}{\Gamma(x)} (\ln q - \Psi(x)) < 0$$

for large enough x, and $\frac{q^x}{\Gamma(x)}$ is bounded below by 0. Therefore $\lim_{x \to \infty} \frac{q^x}{\Gamma(x)}$ exists as a real number.

[11] Jean-Baptiste Joseph Fourier (1768-1830)

We also have

$$\lim_{x\to\infty}\frac{q^x}{\Gamma(x)}=\lim_{x\to\infty}\frac{q^x\ln q}{\Psi(x)\Gamma(x)}=\left(\lim_{x\to\infty}\frac{q^x}{\Gamma(x)}\right)\left(\frac{\ln q}{\lim_{x\to\infty}\Psi(x)}\right)=0$$

for the Digamma Function Ψ, which means for $v>qe^q-1$ we have

$$e^q-\sum_{k=0}^{v}\frac{1}{k!}q^k<\frac{qe^q}{v+1}\left(\frac{q^v}{v!}\right)<\frac{q^v}{v!}\to 0 \text{ as } v\to\infty$$

so that $\left\{e^q-\sum_{k=0}^{v}\frac{1}{k!}q^k\right\}$ is a Cauchy sequence in v that uniformly converges given q.

For fixed q, choose $v_1>v_0\geq m$ large enough so that

$$\frac{q^{v_1}}{v_1!}<\frac{1}{2(v_0!)} \quad \text{and} \quad \sum_{k=v_0+1}^{v_1}\frac{1}{k!}q^k<\frac{1}{2(v_0!)}$$

where v_0 is a multiple of m.

We have

$$\begin{aligned}e^q-\sum_{k=0}^{v_0}\frac{1}{k!}q^k&=\left(e^q-\sum_{k=0}^{v_1}\frac{1}{k!}q^k\right)+\sum_{k=v_0+1}^{v_1}\frac{1}{k!}q^k\\&<\frac{q^{v_1}}{v_1!}+\frac{1}{2(v_0!)}\\&<\frac{1}{2(v_0!)}+\frac{1}{2(v_0!)}\\&=\frac{1}{v_0!}\end{aligned}$$

which means

$$0<v_0!\left(e^q-\sum_{k=0}^{v_0}\frac{1}{k!}q^k\right)<1$$

However,

$$v_0!\left(e^q-\sum_{k=0}^{v_0}\frac{1}{k!}q^k\right)=v_0!\left(\frac{n}{m}-\sum_{k=0}^{v_0}\frac{1}{k!}q^k\right)=n\left(\frac{v_0}{m}\right)(v_0-1)!-\sum_{k=0}^{v_0}\frac{v_0!}{k!}q^k$$

is an integer, since v_0, q, and k are integers, v_0 is a multiple of m, and $k\leq v_0$. This is a contradiction, since there are no positive integers less than 1. ■

Claim 8 *For positive rational $\frac{q}{p}$, $e^{\frac{q}{p}}$ is irrational.*

Proof. We may take $p,q>0$. If $e^{\frac{q}{p}}$ were rational, then $e^{\frac{q}{p}}=\frac{n}{m}$ for non-zero integers n and m, which would mean

$$e^q=\frac{n^p}{m^p}$$

which is rational, since p is a positive integer; this is a contradiction of the previous claim, since q is a positive integer. ■

Corollary 9 *For rational $\frac{q}{p}$, $e^{\frac{q}{p}}$ is irrational.*

Proof. For positive rational $\frac{q}{p}$, the previous claim shows that $e^{\frac{q}{p}}$ is irrational. For negative $\frac{q}{p}$, suppose $e^{\frac{q}{p}}$ is rational. Then $e^{-\frac{q}{p}}$ is rational. However, $-\frac{q}{p}>0$, which is a contradiction of the previous claim. ■

3. Updating The Marker Matrices

After several arithmetic and conversion operation on trirational numbers a new (revised) marker matrix is produced. The new marker matrix contains additional entries produced by arithmetic and conversion operations that were not covered by the marker matrices used as input to the operation.

However, the values required in the new marker matrix are (usually) unknown to the algorithm code, so that -1 is used generically in those cases. It then becomes the responsibility of the implementing analyst to update the new marker matrix (replacing the -1 values) with the required values (either 0 for "irrational" or 1 for "rational") before using the new marker matrix in subsequent calculation steps. This may be accomplished by interrupting the algorithm processing so that the required values may be entered as needed, after which the algorithm processing continues, or by setting an argument in the implementation code that assumes all zero entries in all marker positions, thereby eliminating all rational part consolidation during algorithm processing. In this latter case, no marker updating occurs and the final calculated value is used to produce a decimal floating point value if desired.

For the purposes of this paper, a new marker matrix may be produced with unknown values signaled as -1, whenever applicable, without further attempts to automatically update the new marker matrix nor to provide user intervention opportunities between algorithm processing steps. The particular method by which a user would intervene in updating the marker matrix, when needed, between algorithm processing steps, is left to the implementing analyst.

4. Example Use: The Matrix Inverse

Given a square matrix $M = (m_{ij})$ of dimension n, the following algorithm either produces the matrix inverse M^{-1} of dimension n, or determines that no such inverse exists. The matrix $A = (a_{ij})$ is $I_{n\times n}$ (the $n \times n$ identity matrix) at the beginning of the algorithm. Every algebraic operation should be applied only to matrix A.

1. Set $i = 1$, $j = 1$, and $u = 0$.

2. If $m_{ii} \neq 0$, divide row i by m_{ii} and set $u = 0$; otherwise add 1 to u, and if $i+u \leq n$, then add row $i + u$ to row i (by column), and repeat this step; otherwise return "M does not have an inverse."

3. Multiply row i by $m_{i+j,i}$ and subtract these values from row $i + j$.

4. Add 1 to j; if $(i + j + 1) \bmod n \neq i \bmod n$, then skip to Step 3; otherwise continue.

5. Add 1 to i; if $i \leq n$, then set $j = 1$ and skip to Step 2; otherwise return A.

Note that the following MAPLE implementation uses Rational Arithmetic And Conversion (RAC) routines in place of the regular set of arithmetic and conversion operations that MAPLE normally provides.

```
MI:=proc(MT,MX)
   local n,ii,jj,kk,uu,idx,tmp,A::Matrix,AX::Matrix,
         MV::Matrix,MVX::Matrix,MW::Matrix,MWX::Matrix,val;
   description "Algorithm For Calculating Matrix Inverse
                Using Rational Arithmetic And Conversions";
```

```
options ‘Copyright 2018 PQI Consulting All Rights Reserved‘;
n:=RowDimension(MT);
if(ColumnDimension(MT)<>n) then
   return "Matrix Not Square";
end if;
A:=Matrix(n,n);
for kk from 1 to n do:
   for uu from 1 to n do:
      if(kk=uu) then A[kk,kk]:=[1,1];
      else A[kk,uu]:=[0,1];
      end if;
   end do;
end do;
AX:=Matrix(1,1,[M]);
MV:=Matrix(n,n,MT); MVX:=copy(MX);
MW:=Matrix(n,n,MT); MWX:=copy(MW);
ii:=1; uu:=0;
while(ii<=n) do:
   if(MW[ii,ii]<>[0,1]) then
      uu:=0;
      for kk from 1 to n do:
         tmp:=RAC_Div(A[ii,kk],
                      RAC_MRGenerate(A[ii,kk],
                                     Matrix(1,1,[M])),
                      MV[ii,ii],
                      RAC_MRGenerate(MV[ii,ii],
                                     Matrix(1,1,[M])));
         tmp:=RAC_Sweep(tmp);
         A[ii,kk]:=tmp[1]; AX:=RAC_MRMeld(AX,tmp[2]);
         tmp:=RAC_Div(MW[ii,kk],
                      RAC_MRGenerate(MW[ii,kk],
                                     Matrix(1,1,[M])),
                          MV[ii,ii],
                          RAC_MRGenerate(MV[ii,ii],
                                         Matrix(1,1,[M])));
         tmp:=RAC_Sweep(tmp);
         MW[ii,kk]:=tmp[1]; MWX:=RAC_MRMeld(MWX,tmp[2]);
      end do;
      jj:=0;
      while((ii+jj+1) mod n <> ii mod n) do:
         idx:=1+((ii+jj) mod n);
         for kk from 1 to n do:
            tmp:=RAC_Mul(A[ii,kk],
                         RAC_MRGenerate(A[ii,kk],
                                        Matrix(1,1,[M])),
                         MV[idx,ii],
                         RAC_MRGenerate(MV[idx,ii],
                                        Matrix(1,1,[M])));
            tmp:=RAC_Sub(A[idx,kk],AX,tmp);
            tmp:=RAC_Sweep(tmp);
            A[idx,kk]:=tmp[1]; AX:=RAC_MRMeld(AX,tmp[2]);
            tmp:=RAC_Mul(MW[ii,kk],
                         RAC_MRGenerate(MW[ii,kk],
                                        Matrix(1,1,[M])),
                             MV[idx,ii],
                             RAC_MRGenerate(MV[idx,ii],
                                            Matrix(1,1,[M])));
```

```
                tmp:=RAC_Sub(MV[idx,kk],MVX,tmp);
                tmp:=RAC_Sweep(tmp);
                MW[idx,kk]:=tmp[1]; MWX:=RAC_MRMeld(MWX,tmp[2]);
             end do:
             jj:=jj+1;
          end do;
          ii:=ii+1;
       else
          if(ii=n or (ii+uu)>n) then
     return "Inverse Does Not Exist";
          else
             uu:=uu+1;
             for kk from 1 to n do:
                tmp:=RAC_Add(MW[ii,kk],
                             RAC_MRGenerate(MW[ii,kk],
                                            Matrix(1,1,[M])),
                                 MW[ii+uu,kk],
                                 RAC_MRGenerate(MW[ii+uu,kk],
                                                Matrix(1,1,[M])));
                tmp:=RAC_Sweep(tmp);
                MW[ii,kk]:=tmp[1]; MWX:=RAC_MRMeld(MWX,tmp[2]);
                tmp:=RAC_Add(A[ii,kk],
                             RAC_MRGenerate(A[ii,kk],
                                            Matrix(1,1,[M])),
                             A[ii+uu,kk],
                             RAC_MRGenerate(A[ii+uu,kk],
                                            Matrix(1,1,[M])));
                tmp:=RAC_Sweep(tmp);
                A[ii,kk]:=tmp[1]; AX:=RAC_MRMeld(AX,tmp[2]);
             end do;
          end if;
       end if;
       MV:=copy(MW);
    end do;
    return A,AX;
end proc;
```

The following example of this implementation demonstrate the use of pure trirational numbers as well as those with an error term E_1.

```
> M1:=Matrix(5,5,[[[0,1],[3,1],[-7,1],[9,1],[-11,1]],
> [[0,1],[0,1],[5,1],[5,1],[-3,1]],
> [[1,1],[-3,1],[1,1],[7,1],[8,1]],
> [[0,1],[0,1],[1,1],[0,1],[1,1]],
> [[1,1],[2,1],[1,1],[2,1],[1,1]]]):
> M2a:=Matrix(5,5,[[[0,1],[3,1],[-7,1],[9,1],[-11,1]],
> [[0,1],[0,1],[5,1],[5,1],[-3,1]],
> [[1,1],[-3,1],[1,1],[7,1,1,1,E_1],[8,1]],
> [[0,1],[0,1],[1,1],[0,1],[1,1]],
> [[1,1],[2,1],[1,1],[2,1],[1,1]]]):
> MI(M1,Matrix(1,1,[M]))[1];
> MI(M2a,Matrix(2,2,[[M,E_1],[E_1,0]]))[1];
```

$$\begin{bmatrix} [-46,97] & [164,485] & [56,485] & [-583,97] & [429,485] \\ [15,97] & [-83,485] & [-52,485] & [188,97] & [52,485] \\ [-5,97] & [12,97] & [-3,97] & [2,97] & [3,97] \\ [8,97] & [1,485] & [24,485] & [55,97] & [-24,485] \\ [5,97] & [-12,97] & [3,97] & [95,97] & [-3,97] \end{bmatrix}$$

$$\begin{bmatrix} [0,1,-2,1,\frac{115+8\,E\_1}{485+24\,E\_1}] & [0,1,4,1,\frac{41+2\,E\_1}{485+24\,E\_1}] & [0,1,56,1,(485+24\,E\_1)^{-1}] & [0,1,-11,1,\frac{265+16\,E\_1}{485+24\,E\_1}] & [0,1,3,1,\frac{143+8\,E\_1}{485+24\,E\_1}] \\ [0,1,1,1,\frac{75+8\,E\_1}{485+24\,E\_1}] & [0,1,-1,1,\frac{83+4\,E\_1}{485+24\,E\_1}] & [0,1,-52,1,(485+24\,E\_1)^{-1}] & [0,1,4,1,\frac{235+19\,E\_1}{485+24\,E\_1}] & [0,1,52,1,(485+24\,E\_1)^{-1}] \\ [0,1,-25,1,(485+24\,E\_1)^{-1}] & [0,1,3,1,\frac{20+E\_1}{485+24\,E\_1}] & [0,1,-15,1,(485+24\,E\_1)^{-1}] & [0,1,1,1,\frac{10+9\,E\_1}{485+24\,E\_1}] & [0,1,15,1,(485+24\,E\_1)^{-1}] \\ [0,1,40,1,(485+24\,E\_1)^{-1}] & [0,1,1,1,(485+24\,E\_1)^{-1}] & [0,1,24,1,(485+24\,E\_1)^{-1}] & [0,1,275,1,(485+24\,E\_1)^{-1}] & [0,1,-24,1,(485+24\,E\_1)^{-1}] \\ [0,1,25,1,(485+24\,E\_1)^{-1}] & [0,1,-3,1,\frac{20+E\_1}{485+24\,E\_1}] & [0,1,15,1,(485+24\,E\_1)^{-1}] & [0,1,5,1,\frac{95+3\,E\_1}{485+24\,E\_1}] & [0,1,-15,1,(485+24\,E\_1)^{-1}] \end{bmatrix}$$

5. Miscellaneous Relationships Between The Rationality Of A Function And Its Arguments

Claim 10 *If c_i and c_j are irrational, then $\frac{c_i}{a+bc_j}$ rational for $a \neq 0$ and b rational means $\frac{c_i}{c_j}$ is irrational.*

Proof. Suppose $\frac{c_j}{c_i} = r$ is rational. Then

$$\frac{a+bc_j}{c_i} = \frac{a+brc_i}{c_i} = br + a\frac{1}{c_i}$$

is irrational for $a \neq 0$ and b rational. ■

Corollary 11 *If c_i and c_j are irrational, then $\frac{c_i}{c_j}$ rational means $\frac{c_i}{a+bc_j}$ is irrational for $a \neq 0$ and b rational.*

Claim 12 *π is irrational.*

Proof. (Due to Niven[12]) We first establish the following lemma.

Lemma 13 *For n a positive integer and f a C^∞ function on a superset of $[0,\pi]$, we have*

$$\int_0^\pi f(x)\sin x\,dx = \sum_{k=0}^{n-1}(-1)^k\left(f^{(2k)}(\pi)+f^{(2k)}(0)\right) + (-1)^n\int_0^\pi f^{(2n)}(x)\sin x\,dx$$

Proof. (By induction on n) For $n = 1$, set $u = f(x)$ and $dv = \sin x\,dx$. Then $du = f^{(1)}(x)\,dx$ and $v = -\cos x$, so that

$$\begin{aligned} \int_0^\pi f(x)\sin x\,dx &= -f(x)\cos x\Big|_{x=0}^{x=\pi} + \int_0^\pi f^{(1)}(x)\cos x\,dx \\ &= f(\pi)+f(0)+\left(f^{(1)}(x)\sin x\Big|_{x=0}^{x=\pi} - \int_0^\pi f^{(2)}(x)\sin x\,dx\right) \\ &= f(\pi)+f(0)-\int_0^\pi f^{(2)}(x)\sin x\,dx \\ &= \sum_{k=0}^{1-1}(-1)^k\left(f^{(2k)}(\pi)+f^{(2k)}(0)\right)+(-1)^1\int_0^\pi f^{(2)}(x)\sin x\,dx \end{aligned}$$

[12]Ivan Morton Niven (1915-1999)

where the second integration by parts uses $u = f'(x)$ and $dv = \cos x\, dx$.

Now suppose we have

$$\int_0^\pi f(x)\sin x\, dx = \sum_{k=0}^{n-1} (-1)^k \left(f^{(2k)}(\pi) + f^{(2k)}(0)\right) + (-1)^n \int_0^\pi f^{(2n)}(x)\sin x\, dx$$

Then with two integration by parts we have

$$\int_0^\pi f^{(2n)}(x)\sin x\, dx = f^{(2n)}(\pi) + f^{(2n)}(0) - \int_0^\pi f^{(2n+2)}(x)\sin x\, dx$$

so that

$$\begin{aligned}\int_0^\pi f(x)\sin x\, dx &= \sum_{k=0}^{n-1} (-1)^k \left(f^{(2k)}(\pi) + f^{(2k)}(0)\right) + (-1)^n \left(f^{(2n)}(\pi) + f^{(2n)}(0) - \int_0^\pi f^{(2n+2)}(x)\sin x\, dx\right)\\ &= \sum_{k=0}^{(n+1)-1} (-1)^k \left(f^{(2k)}(\pi) + f^{(2k)}(0)\right) + (-1)^{n+1} \int_0^\pi f^{(2(n+1))}(x)\sin x\, dx\end{aligned}$$

■

Suppose π is rational, say, $\pi = \frac{n}{m}$ where n and m are integers, and define

$$f(x; w) = \frac{1}{w!} x^w (n - mx)^w$$

for some positive integer w. This is a $2w$-degree polynomial in x. Hence, $f^{(r)}(x; w) \equiv 0$ for all $r > 2w$ regardless of w. This means

$$\int_0^\pi f(x; w)\sin x\, dx = \sum_{k=0}^{w} (-1)^k \left(f^{(2k)}(\pi; w) + f^{(2k)}(0; w)\right)$$

However, we also have

$$f(\pi - x; w) = \frac{1}{w!}\left(\frac{n}{m} - x\right)^w \left(n - m\left(\frac{n}{m} - x\right)\right)^w = \frac{1}{w!} x^w (n - mx)^w = f(x; w)$$

so that

$$(-1)^{2k} f^{(2k)}(\pi - x; w) \equiv f^{(2k)}(x; w)$$

or

$$f^{(2k)}(0; w) = f^{(2k)}(\pi; w)$$

Therefore, we have

$$\int_0^\pi f(x; w)\sin x\, dx = 2\sum_{k=0}^{w} (-1)^k f^{(2k)}(0; w)$$

for any positive integer w.

Furthermore, $f^{(2k)}(x; w)$ consists of a sum of terms of the form $\frac{1}{w!}\frac{d^r}{dx^r}(x^w)\frac{d^s}{dx^s}((n - mx)^w)$, which are non-zero only when $r = w$, and since $\frac{1}{w!}\frac{d^w}{dx^w}(x^w)\big|_{x=0} = 1$ and $\frac{d^s}{dx^s}((n - mx)^n)\big|_{x=0} > 0$ is an integer, then $f^{(2k)}(0; w)$ is a positive integer for any k, which means $\int_0^\pi f(x; w)\sin x\, dx$ is a positive integer.

Finally, we have

$$0 \leq \sin x \leq 1 \quad \text{and} \quad 0 \leq x(n - mx) \leq xn$$

when $0 \le x \le \pi$, so that

$$\int_0^\pi f(x;w)\sin x\,dx = \int_0^\pi \frac{1}{w!}x^w (n-mx)^w \sin x\,dx \le \int_0^\pi \frac{1}{w!}(xn)^w\,dx = \frac{1}{n(w+1)!}(\pi n)^{w+1} < 1$$

for large enough w. This is a contradiction, since $\int_0^\pi f(x;w)\sin x\,dx$ is a positive integer regardless of w. ■

Even though RAC operations need not distinguish rational and irrational numbers by any other classification system, it is useful to know when trirational numbers involve algebraic versus transcendental numbers, as the latter classification necessarily implies irrationality.

Definition 14 *A real number is said to be* algebraic *if it is the zero of a finite degree polynomial with integer coefficients; otherwise it is said to be* transcendental.

Corollary 15 *Every transcendental number is irrational and every rational number is algebraic. However, not every irrational number is transcendental, and not every algebraic number is rational.*

Proof. Suppose x were transcendental and rational. Since it is rational, say $x = \frac{n}{m}$, then for $f(x) = mx - n$ we have $f(x) = 0$, and since n and m are integers, x is algebraic – a contradiction. So if x must be transcendental, then it must also be irrational, or if x must be rational, then it must also be algebraic.

In particular, $\sqrt{2}$ is irrational yet $\left(\sqrt{2}\right)^2 - 2 = 0$, so $\sqrt{2}$ is algebraic. Therefore, not every irrational number is transcendental, and not every algebraic number is rational. ■

Lemma 16 *π is transcendental.*

Proof. Suppose π were algebraic, i.e., there is a finite degree polynomial

$$P(x) = \sum_{n=0}^{N} a_n x^n$$

with integer coefficients a_n such that $P(\pi) = 0$. We have

$$P(ix) = \sum_{n=0}^{N} a_n (ix)^n = \left(\sum_{\substack{n=0 \\ n \bmod 4=0}}^{N} a_n x^n - \sum_{\substack{n=0 \\ n \bmod 4=2}}^{N} a_n x^n \right) + i \left(\sum_{\substack{n=0 \\ n \bmod 4=1}}^{N} a_n x^n - \sum_{\substack{n=0 \\ n \bmod 4=3}}^{N} a_n x^n \right)$$

so that

$$0 = P(\pi) = P(i(-i\pi)) = \left(\sum_{\substack{n=0 \\ n \bmod 4=0}}^{N} a_n \pi^n - \sum_{\substack{n=0 \\ n \bmod 4=2}}^{N} a_n \pi^n \right) - i \left(\sum_{\substack{n=0 \\ n \bmod 4=1}}^{N} a_n \pi^n - \sum_{\substack{n=0 \\ n \bmod 4=3}}^{N} a_n \pi^n \right)$$

which means the complex conjugate is also zero, i.e., we have

$$P(i\pi) = 0$$

which means $i\pi$ is algebraic.

However, $e^{i\pi} = \cos\pi + i\sin\pi = -1$ which is not transcendental; this is a contradiction of the Hermite-Lindemann-Weierstraß-Baker[13] Theorem (which implies that e^u is transcendental when u is algebraic). ■

[13]Charles Hermite (1822-1901), Carl Louis Ferdinand von Lindemann (1852-1939), Karl Theodor Wilhelm Weierstraß (1815-1897), and Alan Baker (1939-).

Claim 17 *For positive integer $q \geq 2$, π^q is irrational.*

Proof. Suppose $\pi^q = \frac{n}{m}$ for integers n and m. Then for $f(x) = mx^q - n$ we have $f(\pi) = 0$. Since n, m, and q were integers, this means π is algebraic – a contradiction of the previous lemma. ■

Claim 18 *For positive rational $\frac{q}{p}$, $\pi^{\frac{q}{p}}$ is irrational.*

Proof. We may take $p, q > 0$. If $\pi^{\frac{q}{p}}$ were rational, then $\pi^{\frac{q}{p}} = \frac{n}{m}$ for non-zero integers n and m, which would mean

$$\pi^q = \frac{n^p}{m^p}$$

which is rational, since p is a positive integer; this is a contradiction of the previous claim, since q is a positive integer. ■

Algebraic numbers are also useful in determining whether a number is irrational or not: If assuming a number is algebraic leads to a contradiction, then the number must be transcendental and therefore irrational. In particular, algebraic numbers are closed under arithmetic operations.

Claim 19 *Algebraic numbers form a field under addition and multiplication.*

Proof. We have 0 is the solution to $x = 0$, so that 0 is algebraic, and $0 + 0 = 0 = 0 * 0$.

Let $x_0 \neq 0$ and $y_0 \neq 0$ be algebraic numbers, where $P(x) = \sum_{k=0}^{n} a_k x^k$ and $Q(y) = \sum_{r=0}^{m} b_r y^r$ are polynomials with integer coefficients such that $P(x_0) = 0 = Q(y_0)$, and let $\{x_0, x_1, \ldots x_{n-1}\}$ be the zeros of P, and let $\{y_0, y_1, \ldots, y_{m-1}\}$ be the zeros of Q.

(a) Clearly 0 is the additive identity.

(b) We have

$$P(x) = \sum_{k \leq n \text{ even}} a_k x^k + \sum_{j \leq n \text{ odd}} a_j x^j$$

so that

$$W_1(x) = P(-x) = \sum_{k \leq n \text{ even}} a_k x^k - \sum_{j \leq n \text{ odd}} a_j x^j$$

is a polynomial in x with integer coefficients such that $W_1(-x_0) = 0$, which means $-x_0$ is algebraic (the additive inverse).

(c) Clearly 1 is the multiplicative identity.

(d) We have

$$W_2(x) = x^n P\left(\frac{1}{x}\right) = \sum_{k=0}^{n} x^{n-k} = \sum_{k=0}^{n} a_{n-k} x^k$$

is a polynomial in x with integer coefficients such that $W_2\left(\frac{1}{x_0}\right) = 0$, which means $\frac{1}{x_0}$ is algebraic (the multiplicative inverse).

(e) Let

$$Q_0(x; z) = Q(z - x) = \sum_{r=0}^{m} b_r (z - x)^r = \sum_{r=0}^{m} c_r(z) x^r$$

where $\{c_r(z)\}$ are polynomials in z with integer coefficients (since the $\{b_r\}$ are integers). Note that for $z = x_0 + y_0$, we have

$$Q_0(x_0; x_0 + y_0) = Q(y_0) = 0$$

which means $Q_0(x; x_0 + y_0)$ and $P(x)$ have a zero in common, namely x_0.

Hence, the resultant $W_1(z)$ of $P(x)$ and $Q_0(x; x_0 + y_0)$ is a polynomial in z with integer coefficients (since the $\{a_k\}$ and $\{c_r(z)\}$ are integers) where

$$W_1(x_0 + y_0) = 0$$

Therefore, $x_0 + y_0$ is algebraic.

(f) Let

$$Q_1(x; z) = x^m Q\left(\frac{z}{x}\right) = \sum_{r=0}^{m} b_r x^m \left(\frac{z}{x}\right)^r = \sum_{r=0}^{m} c_{m-r}(z)\, x^r$$

where $\{c_{m-r}(z)\}$ are polynomials in z with integer coefficients (since the $\{b_r\}$ are integers). Note that for $z = x_0 y_0$, we have

$$Q_1(x_0; x_0 y_0) = x_0^m Q(y_0) = 0$$

which means $Q_1(x; x_0 y_0)$ and $P(x)$ have a zero in common, namely x_0.

Hence, the resultant $W_2(z)$ of $P(x)$ and $Q_1(x; x_0 y_0)$ is a polynomial in z with integer coefficients (since the $\{a_k\}$ and $\{c_{m-r}(z)\}$ are integers) where

$$W_2(x_0 y_0) = 0$$

Therefore, $x_0 y_0$ is algebraic. ■

In comparison, irrational numbers do not even have the most fundamental structure under arithmetic operations.

Claim 20 *Irrational numbers do not form a group under addition.*

Proof. We have $\sqrt{2}$ and $1-\sqrt{2}$ are irrational,[14] yet $\sqrt{2}+\left(1-\sqrt{2}\right) = 1$ is not irrational. ■

Claim 21 *Irrational numbers do not form a group under multiplication.*

Proof. We have $\sqrt{2}$ is irrational, however $\left(\sqrt{2}\right)\left(\sqrt{2}\right) = 2$ is not irrational. ■

Claim 22 *Irrational numbers do not form a group under exponentiation.*

Proof. If x were the exponentiation identity, then $y^x = y$ for every irrational y. This means $y^{x-1} = 1$ (since $y \neq 0$), for every irrational y, which means $x = 1$, which is not irrational. ■

Claim 23 *Irrational numbers do not form a group under non-zero rational exponentiation.*

Proof. We have $\sqrt{2}$ is irrational and $\left(\sqrt{2}\right)^2 = 2$ is not irrational. ■

Claim 24 *Irrational numbers do not form a group under irrational exponentiation.*

Proof. Besides the fact that there is no irrational exponentiation identity, we have $\sqrt{e}$ and $\ln 4$ are both irrational (by Claim X), so that $\left(e^{\frac{1}{2}}\right)^{\ln 4} = e^{\frac{1}{2}(2\ln 2)} = 2$ is not irrational. ■

However, irrational numbers do have some properties that are useful in determining marker matrix entries.

[14] If $1-\sqrt{2} = \frac{n}{m}$ were rational, for integers n and m, then $\sqrt{2} = \frac{m-n}{m}$ would be rational; a contradiction.

Claim 25 *The reciprocal of an irrational number is irrational.*

Proof. Suppose $\frac{1}{x}$ is rational, i.e., $\frac{1}{x} = \frac{n}{m}$ for non-zero integers n and m. Then $x = \frac{m}{n}$ is rational. ■

Claim 26 *The product of non-zero rational and irrational numbers is irrational.*

Proof. Let x be a rational number, say $\frac{n'}{m'}$, for non-zero integers n' and m', and let y be an irrational number. Suppose $xy = \frac{n}{m}$ for non-zero integers n and m. Then $y = \frac{nm'}{mn'}$; this is a contradiction. ■

Corollary 27 *The quotient of non-zero rational and irrational numbers is irrational.*

Proof. Let x be a non-zero rational number and let y be an irrational number. Then $\frac{1}{y}$ is irrational, so that $\frac{x}{y} = x\left(\frac{1}{y}\right)$ is the product of a rational number (x) with an irrational number $\left(\frac{1}{y}\right)$, which is irrational. Therefore $\frac{y}{x} = \frac{1}{\frac{x}{y}}$ is also irrational. ■

Claim 28 *The natural logarithm of a positive rational number is irrational.*

Proof. Suppose $\ln r = \frac{n}{m}$, where r is a positive rational number, and n and m are non-zero integers. Then $e^{\frac{n}{m}} = r$ is rational, which contradicts the previous corollary. ■

Claim 29 *For a positive number a and non-zero rational number r, if $\ln a$ is rational, then a^r is irrational.*

Proof. Suppose a^r is rational. Since $a^r = e^{r \ln a}$, then $r \ln a$ must be irrational, which means $\ln a$ must be irrational. ■

Note that the reverse of this claim is not true: $2^{\frac{1}{2}}$ is irrational, yet $\ln 2$ is not rational. However, e^2 is irrational, and $\ln e = 1$ is rational.

Claim 30 *For a positive number a, if $\ln a$ is rational, then the logarithm base a of a positive rational number is irrational.*

Proof. Suppose $\ln a$ is rational, say, $\ln a = \frac{n}{m}$ for non-zero integers n and m. Then

$$\log_a r = \frac{\ln r}{\ln a} = \frac{m}{n} \ln r$$

for rational r. However, since r is rational, then $\ln r$ is irrational, which means $\frac{m}{n} \ln r$ is irrational. ■

Note that the reverse of this claim is not true: $\log_2 3$ is irrational, yet $\ln 2$ is not rational. However, $\log_{e^2} 3$ is irrational, yet $\ln e^2 = 2$ is rational.

Claim 31 *For condition G on parameter $\alpha \in W$, if $f : U \times W \subset \mathbb{R} \times \mathbb{R} \to V \subset \mathbb{R}$ is a function such that $f(r; \alpha) \in V$ is irrational when $r \in U$ is rational and condition G applies, then every element of $\varnothing \neq f^{-1}(s) \in U \times \{\alpha\}$ is irrational when $s \in V$ is rational and condition G applies.*

Proof. Given condition G on parameter $\alpha \in W$, let $s \in \mathbb{R}$ be rational. Suppose $r \times \{\alpha\} \in f^{-1}(s) \subset U \times \{\alpha\}$ is rational. Then $s = f(r; \alpha) \in V$ is irrational since $r \in U$ is rational and condition G applies; this is a contradiction. ■

In particular, for $U = \mathbb{R} - \{0\}$, $V = W = (0, \infty) - \{1\}$, the function $f(r; \alpha) = \alpha^r$, and where "$\ln \alpha$ is rational" is the condition G on α, then Claim (29) is the statement that f is irrational when r is rational, and Claim (30) is the statement that $f^{-1}(s) = \log_\alpha s$ is irrational when s is rational.

6. Coding Notes

A trirational number may be represented in programming code (regardless of language or context) starting as a set of consecutive signed/unsigned unitbytes[15] (representing the numerator and denominator, respectively, of a reduced rational number), in the following order: First for the rational part, then one each for each element of the index set (representing the coefficients for the irrational part), ending with single unsigned unitbytes for each member of the index set. Since the index set potentially changes as the result of any particular operation for any given framework, a potentially new storage requirement may be needed as the result of any rational arithmetic or conversion operation.

A rational arithmetic or conversion operation may be thought of as a transformation from one (for unitary operations and conversions) or two (for binary operations) trirational number(s) to another trirational number, along with a transformation of one/two marker matrices into a resulting marker matrix. These transformations define how each range signed/unsigned unitbytes in the rational part, the index set coefficients part, and the index set itself are functions of the corresponding elements in the domain value(s) based on the chosen arithmetic/conversion function and the markers in the rational calculation framework. A statistical calculation using rational arithmetic and conversions is therefore simply an ordered list of these transformations with conditional logic for determining when a final exact or precisely error-bounded approximation is available.

7. Appendix 1: Rationality of $\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$

Claim 32 *If* $\frac{a_0'}{b_0'}$ *and* $\frac{a_1'}{b_1'}$ *are rational, then* $\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$ *is irrational if and only if* $\left(\frac{c_1}{2\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}\right)^2$ *is irrational.*

Proof. Suppose $\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$ is rational, which means

$$\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1} = \frac{n}{m}$$

for some integers n and m.

We have

$$\begin{aligned}
\left(\frac{a_1'}{b_1'}\right)^2 + 4\frac{m}{n}\frac{a_0'}{b_0'} &= \left(\frac{a_1'}{b_1'}\right)^2 + 4\left(\frac{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}{c_1^2}\right)\frac{a_0'}{b_0'} \\
&= \frac{\left(\frac{a_1'}{b_1'}c_1\right)^2 + 2\left(2\frac{a_0'}{b_0'}\right)\left(\frac{a_1'}{b_1'}c_1\right) + \left(2\frac{a_0'}{b_0'}\right)^2}{c_1^2} \\
&= \left(\frac{2\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}{c_1}\right)^2
\end{aligned}$$

[15] A "unitbyte" is any memory or register space that is considered the smallest collection of bytes for any particular numerical value. For the purposes of this paper, a unitbyte shall be an octabyte (64 bits) to facilitate the use of MAPLE code in assembly-level implementations.

is rational.

Furthermore, if $\left(\frac{c_1}{2\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}\right)^2$ is rational, then

$$\frac{1}{4\frac{a_0'}{b_0'}}\left(\left(\frac{2\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}{c_1}\right)^2-\left(\frac{a_1'}{b_1'}\right)^2\right)=\frac{1}{4\frac{a_0'}{b_0'}}\left(\frac{4\left(\frac{a_0'}{b_0'}\right)^2+4\frac{a_0'}{b_0'}\frac{a_1'}{b_1'}c_1}{c_1^2}\right)=\frac{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}{c_1^2}$$

is rational. ■

Then $\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$ is rational if and only if $\left(\frac{c_1}{2\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}\right)^2$ is rational, which means there are non-perfect square[16] integers n' and m' such that

$$\frac{c_1}{2\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}=\pm\sqrt{\frac{n'}{m'}}$$

which means

$$c_1=\pm\frac{2\sqrt{\frac{n'}{m'}}\frac{a_0'}{b_0'}}{1\mp\sqrt{\frac{n'}{m'}}\frac{a_1'}{b_1'}} \tag{4}$$

Therefore, if c_1 is of the form in (4) for some non-perfect square integers n' and m', then $\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$ is rational; otherwise, it is irrational.

For example, $c_1=\pm\frac{2\sqrt{2}}{1\mp\sqrt{2}}$ is of the form in (4), with $\frac{n'}{m'}=2$ and $\frac{a_0'}{b_0'}=1=\frac{a_1'}{b_1'}$, so that $\frac{c_1^2}{1+c_1}=\frac{\left(\pm\frac{2\sqrt{2}}{1\mp\sqrt{2}}\right)^2}{1\pm\frac{2\sqrt{2}}{1\mp\sqrt{2}}}=-8$ is rational. Likewise, $c_1=\frac{2\sqrt{3}}{1-\sqrt{2}}$ is not of the form in (4), so we have $\frac{c_1^2}{1+c_1}=\frac{\left(\frac{2\sqrt{3}}{1-\sqrt{2}}\right)^2}{1+\frac{2\sqrt{3}}{1-\sqrt{2}}}=\frac{24}{73}\sqrt{2}-\frac{120}{73}\sqrt{3}-\frac{168}{73}\sqrt{6}-\frac{108}{73}$ is irrational.

Note that if c_1 is of the form in (4), then

$$\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}=\frac{\left(\pm\frac{2\sqrt{\frac{n'}{m'}}\frac{a_0'}{b_0'}}{1\mp\sqrt{\frac{n'}{m'}}\frac{a_1'}{b_1'}}\right)^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}\left(\pm\frac{2\sqrt{\frac{n'}{m'}}\frac{a_0'}{b_0'}}{1\mp\sqrt{\frac{n'}{m'}}\frac{a_1'}{b_1'}}\right)}=4\frac{a_0'}{b_0'}\frac{1}{\frac{m'}{n'}-(\frac{a_1'}{b_1'})^2}$$

is the rational value of $\frac{c_1^2}{\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$, where $\frac{m'}{n'}=\left(\frac{2\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}{c_1}\right)^2$.

[16] The integers n' and m' must not be perfect squares, for then $\frac{c_1}{2\frac{a_0'}{b_0'}+\frac{a_1'}{b_1'}c_1}$ would be rational, which is a contradiction, since c_1 is irrational.

8. Appendix 2: Decimal Representations Of Trirational Numbers

Claim 33 *The decimal representation of a number is either finite or periodic if and only if the number is a reduced rational.*

Proof. ($\Longrightarrow$) Suppose the decimal representation of a number x is finite. Then

$$x = \sum_{k=-\infty}^{N} a_k 10^k$$

for integers $0 \leq a_k \leq 9$, where $a_k = 0$ for all $k < L$ for some integer $L > -\infty$. Hence, we have

$$x = \sum_{k=-L}^{N} a_k 10^k$$

which means

$$x = \frac{\sum_{k=0}^{N+L} a_{k-L} 10^k}{10^L}$$

which is rational, since $\sum_{k=0}^{N+L} a_k 10^k$ and 10^L are integers, and $10^L \neq 0$, and every rational has a unique reduced form.

Now suppose the decimal representation of a number x is periodic. Then

$$x = \sum_{k=-\infty}^{N} a_k 10^k$$

for integers $0 \leq a_k \leq 9$, and

$$10^L \sum_{k=-\infty}^{-M} a_k 10^k - \left\lfloor 10^L \sum_{k=-\infty}^{-M} a_k 10^k \right\rfloor = \sum_{k=-\infty}^{-M} a_k 10^k$$

for some integers $M \geq 1$ and $L \geq 1$. This is the same as

$$10^L \sum_{k=-\infty}^{-1} a_{k-M+1} 10^k - \left\lfloor 10^L \sum_{k=-\infty}^{-1} a_{k-M+1} 10^k \right\rfloor = \sum_{k=-\infty}^{-1} a_{k-M+1} 10^k$$

Hence, we have

$$\begin{aligned} 10^{L+M-1} x + \sum_{k=0}^{N} a_{k-M+1} 10^k &= \left(\sum_{k=-M+1}^{N} a_k 10^{k+L+M-1} + \left(10^L \sum_{k=-\infty}^{-1} a_{k-M+1} 10^k \right) \right) + \sum_{k=0}^{N} a_{k-M+1} 10^k \\ &= \left(\sum_{k=-M+1}^{N} a_k 10^{k+L+M-1} + \left(\left\lfloor 10^L \sum_{k=-\infty}^{-1} a_{k-M+1} 10^k \right\rfloor + \sum_{k=-\infty}^{-1} a_{k-M+1} 10^k \right) \right) + \sum_{k=0}^{N} a_{k-M+1} 10^k \\ &= \sum_{k=-M+1}^{N} a_k 10^{k+L+M-1} + \left\lfloor 10^L \sum_{k=-\infty}^{-1} a_{k-M+1} 10^k \right\rfloor + x \end{aligned}$$

so that

$$x = \frac{\sum_{k=-M+1}^{N} a_k 10^{k+L+M-1} + \left\lfloor 10^L \sum_{k=-\infty}^{-1} a_{k-M+1} 10^k \right\rfloor - \sum_{k=0}^{N} a_{k-M+1} 10^k}{10^{L+M-1} - 1}$$

which is rational, since $\sum_{k=-M+1}^{N} a_k 10^{k+L+M-1} + \left\lfloor 10^L \sum_{k=-\infty}^{-1} a_{k-M+1} 10^k \right\rfloor - \sum_{k=0}^{N} a_{k-M+1} 10^k$ and $10^{L+M-1} - 1$ are integers, and $10^{L+M-1} - 1 \neq 0$ since $L + M \geq 2$, and every rational has a unique reduced form.

($\Longleftarrow$) Let $\frac{n}{m}$ be a reduced rational number, i.e., n and m are relatively prime integers where $m \neq 0$, and let

$$\frac{n}{m} = \sum_{k=-\infty}^{N} a_k 10^k$$

for integers $0 \leq a_k \leq 9$. If all $a_{k \leq -N_0 \leq -1} \equiv 0$, then we have

$$\frac{n}{m} = \sum_{k=-N_0+1}^{N} a_k 10^k$$

which means the fractional part of $\frac{n}{m}$ has a finite decimal representation.

Therefore, suppose not all $a_{k \leq -N_0}$ are zero for any N_0. This means $\frac{n}{m}$ is the sum of a part with a finite decimal representation and a part with an infinite non-zero decimal representation. Without loss of generality, the latter part of $\frac{n}{m}$ is taken to be the entire fractional value of $\frac{n}{m}$, with no leading zeros in the decimal representation.

Since n and m are relatively prime, there exists an integer $d \geq 0$ such that

$$\left(10^d - 1\right) n \bmod m = 0$$

which means

$$10^d n - m \left\lfloor 10^d \frac{n}{m} \right\rfloor = n - m \left\lfloor \frac{n}{m} \right\rfloor$$

or

$$10^d \frac{n}{m} - \left\lfloor 10^d \frac{n}{m} \right\rfloor = \frac{n}{m} - \left\lfloor \frac{n}{m} \right\rfloor$$

Furthermore, for integer $q \geq 1$, if

$$10^{qd} \frac{n}{m} - \left\lfloor 10^{qd} \frac{n}{m} \right\rfloor = \frac{n}{m} - \left\lfloor \frac{n}{m} \right\rfloor$$

then

$$\left\lfloor 10^{(q+1)d} \frac{n}{m} \right\rfloor = 10^d \left\lfloor 10^{qd} \frac{n}{m} \right\rfloor + \left\lfloor 10^d \frac{n}{m} \right\rfloor - 10^d \left\lfloor \frac{n}{m} \right\rfloor$$

and

$$\begin{aligned}
10^{(q+1)d} \frac{n}{m} - \left\lfloor 10^{(q+1)d} \frac{n}{m} \right\rfloor &= 10^{(q+1)d} \frac{n}{m} - \left(10^d \left\lfloor 10^{qd} \frac{n}{m} \right\rfloor + \left\lfloor 10^d \frac{n}{m} \right\rfloor - 10^d \left\lfloor \frac{n}{m} \right\rfloor\right) \\
&= 10^d \left(10^{qd} \frac{n}{m} - \left\lfloor 10^{qd} \frac{n}{m} \right\rfloor\right) + \left(10^d \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor 10^d \frac{n}{m} \right\rfloor\right) \\
&= 10^d \left(\frac{n}{m} - \left\lfloor \frac{n}{m} \right\rfloor\right) + 10^d \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor 10^d \frac{n}{m} \right\rfloor \\
&= 10^d \frac{n}{m} - \left\lfloor 10^d \frac{n}{m} \right\rfloor \\
&= \frac{n}{m} - \left\lfloor \frac{n}{m} \right\rfloor
\end{aligned}$$

so that $\frac{n}{m}$ has a periodic decimal representation. ∎