Evaluating Sufficient Bootstrapping for Confidence Interval Estimates: A Simulation Approach

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Abstract

Recent studies compare sufficient bootstrapping with conventional bootstrapping using point estimates of parameters, along their biases, relative efficiencies, etc. With numerical illustration and simulation, it claims that sufficient bootstrapping performs better than the conventional bootstrapping in certain situations. In real life, confidence interval estimates are preferable to point estimates. Confidence interval estimates take into account the variability of the point estimates for making better inference. In this paper, we provide algorithm to implement sufficient bootstrapping for constructing confidence interval estimates for several parameters such as mean, variance, standard deviation and coefficient of variation for better evaluating the performance of sufficient bootstrapping as compared to the conventional bootstrapping. A simulation study has been undertaken for evaluating confidence interval estimates using the estimated coverage probability and confidence length. This evaluation makes the recommendation for the sufficient bootstrapping stronger.

Keywords: Bootstrapping, Sufficient bootstrapping, Confidence interval, Coverage probability, Confidence length.

1. Introduction

Introduced by Efron (1979), bootstrapping is a computationally intensive and iterative method that has become increasingly popular in recent years due to the availability of the modern computational facilities. It has made a significant impact in the field of statistics and statistical applications. The users of bootstrapping rely on data-based simulation instead of the traditional algebraic derivations. Because bootstrapping resample from the original sample, it is popularly known as the resampling procedure. For details about bootstrapping, one can consult with Efron and Tibshirani (1993), Chernick (1999), Johnson (2001), Davison et al. (2003), Casella (2003), etc. Efron (2003) discusses a second thought on bootstrapping. Beran (2003), Lele (2003), Shao (2003), Lahiri (2003) and Politis (2003) explain the impact of bootstrap on statistical algorithms and theory, estimating functions, sample surveys, small area estimation and time series, respectively. Ernst and Hutson (2003) and Rueda et al. (1998, 2005, 2006) discussed the application of bootstrapping for quantile estimation. Holmes (2003) and Soltis and Soltis (2003) discuss applications of bootstrapping in phylogenetic trees and phylogeny reconstruction respectively. Holmes et al. (2003) provide an overview of a conversation on bootstrap between Bradley Efron and other good friends. Horowitz (2003) discussed the use of bootstrap in econometrics and Hall (2003) discussed a short prehistory of bootstrap. Bootstrap methods and permutation tests by Hesterberg (2008), and Bootstrap for complex survey data by Kolenikov (2009) are among some significant source for learning and updating about bootstrapping.

Singh and Sedory (2011) proposed a sufficient bootstrapping, where duplication of a sampling unit in the conventional bootstrapping has been ignored. They also develop a theoretical framework for the technique. While the sufficient bootstrapping reduces the computational burden to a greater extent, the gain in relative efficiency of sufficient bootstrapping is significant as compared to the traditional bootstrapping in certain situations. Sufficient bootstrapping is mostly studied and compared for point estimates of parameters, along with biases and relative efficiencies. In real life, however, the confidence interval estimates of parameters are preferable to their point estimates due to the fact that confidence interval estimates take into account the variability of the point estimates, and thereby provide better inference about the parameter of interest. For better recommendation, the performance of confidence interval estimates needs to justified in terms of estimated coverage probability and confidence length.

The main objective of this paper is to consider confidence interval estimates for several parameters such as mean, variance, standard deviation and coefficient of variation using sufficient bootstrapping and traditional bootstrapping. This study leads to an evaluation of the sufficient bootstrapping technique over the traditional bootstrapping technique using a Monte Carlo simulation, and thus leads to a stronger recommendation as to using sufficient bootstrapping.

2. Bootstrapping versus sufficient bootstrapping

Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ be an original sample with mean $\bar{x} = \frac{1}{n}\sum_{k\in\boldsymbol{x}} x_k$ and the sample variance

$$s^2 = \frac{1}{n-1}\sum_{k\in\boldsymbol{x}} (x_k - \bar{x})^2$$

The ith bootstrap sample of $\boldsymbol{x}$ with replacement is given by

$$\boldsymbol{x}^i = \left(x_k^i : k = 1,2,\cdots,n\right); i = 1,2,\cdots,n^n$$

The bootstrap sample mean

$$\bar{x}^i = \frac{1}{n}\sum_{k\in\boldsymbol{x}^i} x_k^i$$

is unbiased for $\bar{x}$, the mean of the original sample and the sample variance of the bootstrapping sample mean is given by

$$V\left(\bar{x}^i\right) = \frac{1}{n}s^2$$

Introduced by Singh and Sedory (2011), the ith sufficient bootstrap sample of $\boldsymbol{x}$ is defined by

$$\boldsymbol{x}^{i(\nu)} = \left(x_k^{i(\nu)} : k = 1,2,\cdots,\nu\right); i = 1,2,\cdots,n^n$$

which consists of ν distinct units of $\boldsymbol{x}^i$. Singh and Sedory (2011) showed that sufficient bootstrap sample mean

$$\bar{x}^{i(\nu)} = \frac{1}{\nu} \sum_{k \in \boldsymbol{x}^{i(\nu)}} x_k^{i(\nu)}$$

is unbiased for $\bar{x}$, the mean of the original sample and the sample variance of the sufficient bootstrapping sample mean is given by

$$V\left(\bar{x}^{i(\nu)}\right) = \left[E_d\left(\frac{1}{\nu}\right) - \frac{1}{n}\right] s^2$$

where E_d denotes the expected value over all possible distinct units. They showed that the percent relative efficiency of the sufficient bootstrapping estimator over the conventional bootstrapping is

$$\left(\frac{n^{n-1}}{\sum_{l=1}^{n} l^{n-1}}\right) \times 100\%$$

3. Parameters of interest

Following Singh and Sedory (2011), we consider the beta distribution, $B(\alpha, \beta)$, for simulating original sample. The population parameters of interest are as follows:

Population mean: $\mu = \frac{\alpha}{\alpha+\beta} = \theta_1$

Population variance: $\sigma^2 = \frac{\alpha}{(\alpha+\beta)^2(\alpha+\beta+1)} = \theta_2$

Population standard deviation: $\sigma = \sqrt{\sigma^2} = \theta_3$

Population coefficient of variation: $CV = \frac{\sigma}{\mu} \times 100\% = \theta_4$

Singh and Sedory (2011) compared performance of sufficient bootstrapping and conventional bootstrapping using point estimates of these parameters along with their biases and relative efficiencies of sufficient bootstrapping as compared to conventional bootstrapping.

We consider confidence interval estimates of these parameters in comparing performance of sufficient bootstrapping over the conventional bootstrapping. Confidence interval estimates of parameters are preferable to their point estimates because confidence interval estimates take the variability of point estimates into account in making inference. While comparing, we consider the estimated coverage probability and length of confidence interval estimates of mean, variance, standard deviation and coefficient of variation. The proportion of bootstrap and sufficient bootstrap confidence interval estimates containing the true parameters of interest refer to the coverage probability. We consider the average length of these confidence interval estimates for comparisons.

4. Confidence interval estimates

We consider two confidence interval estimates of θ_r, $r = 1, 2, 3, 4$, under sufficient and conventional bootstrapping methods. They are (1) percentile confidence interval estimate and (2) t-confidence interval estimate. Introduced and motivated by Efron, below we provide definitions and algorithms to compute percentile and t-confidence interval estimates for conventional and sufficient bootstrap methods:

4.1 The Percentile Method

Given an original sample $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and ith bootstrap sample $\boldsymbol{x}^i = (x_1^i, x_2^i, \dots, x_n^i)$, let $\hat{\theta}_r^i(\boldsymbol{x}^i)$ be the estimate of parameter $\theta_r: r = 1, 2, 3, 4$ on the basis of the ith bootstrap sample. Then, the $100(1-\alpha)\%$ percentile confidence interval estimate of θ_r using bootstrapping is given by

$$pci_b = \left[l(\hat{\theta}_r^i), u\,(\hat{\theta}_r^i))\right]$$

where $l(\hat{\theta}_r^i)$ and $u(\hat{\theta}_r^i)$ are the $\left(\frac{\alpha}{2}\right)th$ and $(1-\frac{\alpha}{2})th$ percentiles of $\hat{\theta}_r^i(\boldsymbol{x}^i)$ over all bootstrap samples $\boldsymbol{x}^i: i = 1, 2, \cdots n^n$.

A $100(1-\alpha)\%$ percentile confidence interval estimate of θ_r using sufficient bootstrapping is given by

$$pci_s = \left[l(\hat{\theta}_r^{i(\nu)}), u\,(\hat{\theta}_r^{i(\nu)}))\right]$$

where $l(\hat{\theta}_r^{i(\nu)})$ and $u(\hat{\theta}_r^{i(\nu)})$ are the $\left(\frac{\alpha}{2}\right)th$ and $(1-\frac{\alpha}{2})th$ percentiles of $\hat{\theta}_r^{i(\nu)}(\boldsymbol{x}^{i(\nu)})$ over all sufficient bootstrap samples $\boldsymbol{x}^{i(\nu)}: i = 1, 2, \cdots n^n (\cong B, \text{say})$.

In real life, it is not always possible to consider all n^n bootstrap samples, and in general, the number of bootstrap replications n^n is approximated by a reasonable value of B between 50 and 1000, and B is popularly known as the number of bootstrap replications. If, for example, $B = 1000$, then for a 95% confidence interval estimate of θ_r, $l(\hat{\theta}_r^i)$ =25th largest value of $\hat{\theta}_r^i(\boldsymbol{x}^i)$'s and $u\,(\hat{\theta}_r^i)$ =975th largest value of the $\hat{\theta}_r^i(\boldsymbol{x}^i)$'s.

4.2 t-confidence interval

Given an original sample $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, let $\hat{\theta}_r(\boldsymbol{x})$ be a point estimate of θ_r. Given ith bootstrap sample $\boldsymbol{x}^i = (x_1^i, x_2^i, \dots, x_n^i)$ of $\boldsymbol{x}$, let $\boldsymbol{x}^{j|i} = \left(x_1^{j|i}, x_2^{j|i}, \dots, x_n^{j|i}\right)$ be the jth bootstrap from $\boldsymbol{x}^i$, $j = 1, 2, \dots, B$; $i = 1, 2, \dots, B$. Let $\hat{\theta}_r^i(\boldsymbol{x}^i)$ and $\hat{\theta}_r^{j|i}(\boldsymbol{x}^{j|i})$ be the point estimates of estimate of θ_r on the basis of $\boldsymbol{x}^i$ and $\boldsymbol{x}^{j|i}$.

Let

$$s_r^{|i} = sd\{\hat{\theta}_r^{j|i}\}_{j=1:B}$$

$$T_r^{|i} = \frac{\hat{\theta}_r^i - \hat{\theta}_r}{s_r^{|i}}; r = 1, 2, 3, 4; i = 1, 2, \cdots, B$$

and

$$s_r = sd\{\hat{\theta}_r^i\}_{i=1:B}$$

The $100(1-\alpha)\%$ t-confidence interval estimate of θ_r using conventional bootstrapping is given by

$$tci_b = \left[\hat{\theta}_r - (s_r)t_r^{\left(1-\frac{\alpha}{2}\right)}, \hat{\theta}_r - (s_r)t_r^{\left(\frac{\alpha}{2}\right)}\right]$$

where $t_r^{\left(\frac{\alpha}{2}\right)}$ and $t_r^{\left(1-\frac{\alpha}{2}\right)}$ are $\left(\frac{\alpha}{2}\right)th$ and $(1-\frac{\alpha}{2})th$ percentiles of $T_r^{|i}$'s.

In order to compute t-confidence interval estimates for sufficient bootstrapping, we compute $\hat{\theta}_r^{i(\nu)}(\boldsymbol{x}^{i(\nu)})$ and $\hat{\theta}_r^{j(\nu)|i)}(\boldsymbol{x}^{j(\nu)|i)})$ using sufficient bootstrap samples $\boldsymbol{x}^{i(\nu)}$ and $\boldsymbol{x}^{j(\nu)|i}$, respectively. Then, compute

$$s_r^{(\nu)|i} = sd\{\hat{\theta}_r^{j(\nu)|i}\}_{j=1:B}$$

$$T_r^{(\nu)|i} = \frac{\hat{\theta}_r^{i(\nu)} - \hat{\theta}_r}{s_r^{(\nu)|i}}; r = 1, 2, 3, 4; i = 1, 2, \cdots, B$$

$$s_r^{(\nu)} = sd\{\hat{\theta}_r^{i(\nu)}\}_{i=1:B}$$

using sufficient bootstrapping samples. Then, a $100(1-\alpha)\%$ t-confidence interval estimate of θ_r using sufficient bootstrapping is given by

$$tci_s = \left[\hat{\theta}_r - \left(s_r^{(\nu)}\right) t(\nu)_r^{\left(1-\frac{\alpha}{2}\right)}, \hat{\theta}_r - \left(s_r^{(\nu)}\right) t(\nu)_r^{\left(\frac{\alpha}{2}\right)}\right]$$

where $t(\nu)_r^{\left(\frac{\alpha}{2}\right)}$ and $t(\nu)_r^{\left(1-\frac{\alpha}{2}\right)}$ are $\left(\frac{\alpha}{2}\right)th$ and $(1-\frac{\alpha}{2})th$ percentiles of $T_r^{(\nu)|i}$'s.

In Table 1, we provide a schematic algorithm for computational details using $\boldsymbol{x}^i$, $\boldsymbol{x}^{j|i}$, $\boldsymbol{x}^{i(\nu)}$ and $\boldsymbol{x}^{j(\nu)|i}$.

Table 1: Algorithm of computational details using bootstrap samples ($\boldsymbol{x}^i$ and $\boldsymbol{x}^{j|i}$) and the corresponding sufficient bootstrap samples ($\boldsymbol{x}^{i(\nu)}$ and $\boldsymbol{x}^{j(\nu)|i}$)

$\boldsymbol{x}^i$ $i = 1,2,\dots,B$	$\boldsymbol{x}^{i(\nu)}$ $i = 1,2,\dots,B$	Given $\boldsymbol{x}^i$, $\boldsymbol{x}^{j\|i}$ $i,j = 1,2,\dots,B$	Given $\boldsymbol{x}^i$, $\boldsymbol{x}^{j(\nu)\|i}$ $i,j = 1,2,\dots,B$
$\boldsymbol{x}^1$ $\hat{\theta}_r^1(\boldsymbol{x}^1)$	$\boldsymbol{x}^{1(\nu)}$ $\hat{\theta}_r^{1(\nu)}(\boldsymbol{x}^{1(\nu)})$	$\boldsymbol{x}^{1\|1}$; $\hat{\theta}_r^{1\|1}(\boldsymbol{x}^{1\|1})$ $\boldsymbol{x}^{2\|1}$; $\hat{\theta}_r^{2\|1}(\boldsymbol{x}^{2\|1})$ ⋮ $\boldsymbol{x}^{j\|1}$; $\hat{\theta}_r^{j\|1}(\boldsymbol{x}^{j\|1})$ ⋮ $\boldsymbol{x}^{B\|1}$; $\hat{\theta}_r^{B\|1}(\boldsymbol{x}^{B\|1})$ $s_r^{\|1} = sd\{\hat{\theta}_r^{j\|1}\}_{j=1:B}$ $T_r^{\|1} = \frac{\hat{\theta}_r^1 - \hat{\theta}_r}{s_r^{\|1}}$	$\boldsymbol{x}^{1(\nu)\|1}$; $\hat{\theta}_r^{1(\nu)\|1}(\boldsymbol{x}^{1(\nu)\|1})$ $\boldsymbol{x}^{2(\nu)\|1}$; $\hat{\theta}_r^{2(\nu)\|1}(\boldsymbol{x}^{2(\nu)\|1})$ ⋮ $\boldsymbol{x}^{j(\nu)\|1}$; $\hat{\theta}_r^{j(\nu)\|1}(\boldsymbol{x}^{j(\nu)\|1})$ ⋮ $\boldsymbol{x}^{B(\nu)\|1}$; $\hat{\theta}_r^{B(\nu)\|1}(\boldsymbol{x}^{B(\nu)\|1})$ $s_r^{(\nu)\|1} = sd\{\hat{\theta}_r^{j(\nu)\|1}\}_{j=1:B}$ $T_r^{(\nu)\|1} = \frac{\hat{\theta}_r^{1(\nu)} - \hat{\theta}_r}{s_r^{(\nu)\|1}}$
$\boldsymbol{x}^2$ $\hat{\theta}_r^2(\boldsymbol{x}^2)$	$\boldsymbol{x}^{2(\nu)}$ $\hat{\theta}_r^{2(\nu)}(\boldsymbol{x}^{2(\nu)})$	$\boldsymbol{x}^{1\|2}$; $\hat{\theta}_r^{1\|2}(\boldsymbol{x}^{1\|2})$ $\boldsymbol{x}^{2\|2}$; $\hat{\theta}_r^{2\|2}(\boldsymbol{x}^{2\|2})$ ⋮ $\boldsymbol{x}^{j\|2}$; $\hat{\theta}_r^{j\|2}(\boldsymbol{x}^{j\|2})$ ⋮ $\boldsymbol{x}^{B\|2}$; $\hat{\theta}_r^{B\|2}(\boldsymbol{x}^{B\|2})$ $s_r^{\|2} = sd\{\hat{\theta}_r^{j\|2}\}_{j=1:B}$ $T_r^{\|2} = \frac{\hat{\theta}_r^2 - \hat{\theta}_r}{s_r^{\|2}}$	$\boldsymbol{x}^{1(\nu)\|2}$; $\hat{\theta}_r^{1(\nu)\|2}(\boldsymbol{x}^{1(\nu)\|2})$ $\boldsymbol{x}^{2(\nu)\|2}$; $\hat{\theta}_r^{2(\nu)\|2}(\boldsymbol{x}^{2(\nu)\|2})$ ⋮ $\boldsymbol{x}^{j(\nu)\|2}$; $\hat{\theta}_r^{j(\nu)\|2}(\boldsymbol{x}^{j(\nu)\|2})$ ⋮ $\boldsymbol{x}^{B(\nu)\|2}$; $\hat{\theta}_r^{B(\nu)\|2}(\boldsymbol{x}^{B(\nu)\|2})$ $s_r^{(\nu)\|2} = sd\{\hat{\theta}_r^{j(\nu)\|2}\}_{j=1:B}$ $T_r^{(\nu)\|2} = \frac{\hat{\theta}_r^{2(\nu)} - \hat{\theta}_r}{s_r^{(\nu)\|2}}$
⋮	⋮	⋮	⋮
$\boldsymbol{x}^i$ $\hat{\theta}_r^i(\boldsymbol{x}^i)$	$\boldsymbol{x}^{i(\nu)}$ $\hat{\theta}_r^{i(\nu)}(\boldsymbol{x}^{i(\nu)})$	$\boldsymbol{x}^{1\|i}$; $\hat{\theta}_r^{1\|i}(\boldsymbol{x}^{1\|i})$ $\boldsymbol{x}^{2\|i}$; $\hat{\theta}_r^{2\|i}(\boldsymbol{x}^{2\|i})$ ⋮ $\boldsymbol{x}^{j\|i}$; $\hat{\theta}_r^{j\|i}(\boldsymbol{x}^{j\|i})$ ⋮ $\boldsymbol{x}^{B\|i}$; $\hat{\theta}_r^{B\|i}(\boldsymbol{x}^{B\|i})$ $s_r^{\|i} = sd\{\hat{\theta}_r^{j\|i}\}_{j=1:B}$ $T_r^{\|i} = \frac{\hat{\theta}_r^i - \hat{\theta}_r}{s_r^{\|i}}$	$\boldsymbol{x}^{1(\nu)\|i}$; $\hat{\theta}_r^{1(\nu)\|i}(\boldsymbol{x}^{1(\nu)\|i})$ $\boldsymbol{x}^{2(\nu)\|i}$; $\hat{\theta}_r^{2(\nu)\|i}(\boldsymbol{x}^{2(\nu)\|i})$ ⋮ $\boldsymbol{x}^{j(\nu)\|i}$; $\hat{\theta}_r^{j(\nu)\|i}(\boldsymbol{x}^{j(\nu)\|i})$ ⋮ $\boldsymbol{x}^{B(\nu)\|i}$; $\hat{\theta}_r^{B(\nu)\|i}(\boldsymbol{x}^{B(\nu)\|i})$ $s_r^{(\nu)\|i} = sd\{\hat{\theta}_r^{j(\nu)\|i}\}_{j=1:B}$ $T_r^{(\nu)\|i} = \frac{\hat{\theta}_r^{i(\nu)} - \hat{\theta}_r}{s_r^{(\nu)\|i}}$

⋮	⋮	⋮	⋮
$\boldsymbol{x}^{B}$ $\hat{\theta}_r^{B}(\boldsymbol{x}^{B})$	$\boldsymbol{x}^{B(\nu)}$ $\hat{\theta}_r^{B(\nu)}(\boldsymbol{x}^{B(\nu)})$	$\boldsymbol{x}^{1\|B}$; $\hat{\theta}_r^{1\|B}(\boldsymbol{x}^{1\|B})$ $\boldsymbol{x}^{2\|B}$; $\hat{\theta}_r^{2\|B}(\boldsymbol{x}^{2\|B})$ ⋮ $\boldsymbol{x}^{j\|B}$; $\hat{\theta}_r^{j\|B}(\boldsymbol{x}^{j\|B})$ ⋮ $\boldsymbol{x}^{B\|B}$; $\hat{\theta}_r^{B\|B}(\boldsymbol{x}^{B\|B})$ $s_r^{\|B} = sd\{\hat{\theta}_r^{j\|B}\}_{j=1:B}$ $T_r^{\|B} = \frac{\hat{\theta}_r^{B} - \hat{\theta}_r}{s_r^{\|B}}$	$\boldsymbol{x}^{1(\nu)\|B}$; $\hat{\theta}_r^{1(\nu)\|B}(\boldsymbol{x}^{1(\nu)\|B})$ $\boldsymbol{x}^{2(\nu)\|B}$; $\hat{\theta}_r^{2(\nu)\|B}(\boldsymbol{x}^{2(\nu)\|B})$ ⋮ $\boldsymbol{x}^{j(\nu)\|B}$; $\hat{\theta}_r^{j(\nu)\|B}(\boldsymbol{x}^{j(\nu)\|B})$ ⋮ $\boldsymbol{x}^{B(\nu)\|B}$; $\hat{\theta}_r^{B(\nu)\|B}(\boldsymbol{x}^{B(\nu)\|B})$ $s_r^{(\nu)\|B} = sd\{\hat{\theta}_r^{j(\nu)\|B}\}_{j=1:B}$ $T_r^{(\nu)\|B} = \frac{\hat{\theta}_r^{B(\nu)} - \hat{\theta}_r}{s_r^{(\nu)\|B}}$

5. Simulation Study

In this section, we carry out a simulation study to compare the performance of confidence interval estimates for sufficient and conventional bootstrapping for varying values of the sample size. All simulations are performed by using the statistical software R. The original sample $\boldsymbol{x}$ is simulated from $B(1.2,1.6)$ population, following Singh and Sedory (2011).

In all simulations, the Monte Carlo size (M) is considered 1,000. The bootstrap replication of $B = 200$ is considered for all simulation. The estimate of coverage probability is obtained from the proportion of confidence intervals containing the true parameters under bootstrap and sufficient bootstrap samples over a Monte Carlo simulation of size 1,000 using 95% confidence coefficient. We consider the average length of CI estimates for which confidence interval estimates contain the true parameters.

The estimated coverage probability for 95% confidence interval of underlying parameters using bootstrapping and sufficient bootstrapping methods are reported in Table 2 and the average length of the corresponding confidence interval estimates over 1000 simulations are reported in Table 3.

Table 2: Estimated coverage probability from the simulation study for varying sample size n.

Sample size $n = 10$		
	Coverage Probability of Percentile CI Estimates	
Parameters	Bootstrap method	Sufficient bootstrap
Mean	0.88	0.80
Variance	0.81	0.79
Standard deviation	0.81	0.79
CV	0.87	0.80
	Coverage Probability of t-CI Estimates	
	Bootstrap method	Sufficient bootstrap
Mean	0.95	0.93
Variance	0.90	0.91
Standard deviation	0.88	0.89
CV	0.90	0.88
Sample size $n = 20$		
	Coverage Probability of Percentile CI Estimates	
Parameters	Bootstrap method	Sufficient bootstrap
Mean	0.91	0.83
Variance	0.88	0.81
Standard deviation	0.88	0.81
CV	0.91	0.82
	Coverage Probability of t-CI Estimates	
	Bootstrap method	Sufficient bootstrap
Mean	0.95	0.89
Variance	0.92	0.89
Standard deviation	0.91	0.87
CV	0.94	0.89
Sample size $n = 30$		
	Coverage Probability of Percentile CI Estimates	
Parameters	Bootstrap method	Sufficient bootstrap
Mean	0.93	0.85
Variance	0.90	0.84
Standard deviation	0.90	0.84
CV	0.92	0.85
	Coverage Probability of t-CI Estimates	
	Bootstrap method	Sufficient bootstrap
Mean	0.95	0.89
Variance	0.94	0.90
Standard deviation	0.94	0.90
CV	0.94	0.90

Sample size $n = 100$		
	Coverage Probability of Percentile CI Estimates	
Parameters	Bootstrap method	Sufficient bootstrap
Mean	0.93	0.86
Variance	0.93	0.85
Standard deviation	0.93	0.85
CV	0.94	0.86
	Coverage Probability of t-CI Estimates	
	Bootstrap method	Sufficient bootstrap
Mean	0.93	0.86
Variance	0.93	0.87
Standard deviation	0.93	0.86
CV	0.94	0.87

Table 3: Average confidence length from the simulation study for varying sample size n.

Sample size $n = 10$		
	Length of Percentile CI Estimates	
Parameters	Bootstrap method	Sufficient bootstrap
Mean	0.29	0.23
Variance	0.08	0.08
Standard deviation	0.17	0.16
CV	52.71	43.86
	Length of t-CI Estimates	
	Bootstrap method	Sufficient bootstrap
Mean	0.38	0.33
Variance	0.20	0.19
Standard deviation	0.29	0.27
CV	74.40	64.94
Sample size $n = 20$		
	Length of Percentile CI Estimates	
Parameters	Bootstrap method	Sufficient bootstrap
Mean	0.21	0.16
Variance	0.06	0.05
Standard deviation	0.12	0.10
CV	35.73	28.36
	Length of t-CI Estimates	
	Bootstrap method	Sufficient bootstrap
Mean	0.23	0.19
Variance	0.08	0.07
Standard deviation	0.14	0.12
CV	40.21	33.05

Sample size $n = 30$		
	Length of Percentile CI Estimates	
Parameters	Bootstrap method	Sufficient bootstrap
Mean	0.17	0.13
Variance	0.05	0.04
Standard deviation	0.09	0.08
CV	28.57	22.43
	Length of t-CI Estimates	
	Bootstrap method	Sufficient bootstrap
Mean	0.19	0.15
Variance	0.06	0.05
Standard deviation	0.11	0.09
CV	31.10	25.06
Sample size $n = 100$		
	Length of Percentile CI Estimates	
Parameters	Bootstrap method	Sufficient bootstrap
Mean	0.10	0.07
Variance	0.03	0.02
Standard deviation	0.05	0.04
CV	15.46	11.94
	Length of t-CI Estimates	
	Bootstrap method	Sufficient bootstrap
Mean	0.10	0.08
Variance	0.03	0.02
Standard deviation	0.05	0.04
CV	16.09	12.46

6. Result discussion and conclusion

The results of Table 2 suggest that coverage probability of conventional bootstrap CI estimates are relatively higher than that of the sufficient bootstrap CI estimates for both percentile and t-confidence intervals. It is also evident that the t-confidence interval has higher coverage probability than the percentile method in all simulation cases. On the other hand, from the results of Table 3 it is evident that the confidence length of the sufficient bootstrap CI estimates are relatively smaller than those of conventional bootstrap CI estimates for both percentile and t-confidence interval methods. The conclusion of results in Table 2 and 3 follows from the fact that there is a substantial reduction in the standard error of the sufficient bootstrap estimates as compared to the conventional bootstrap estimates due to the removal of the duplication of units. This fact conforms to the conclusion of Singh and Sedory (2011), where they note that the relative efficiency of bootstrap estimates improves as compared to the conventional bootstrap estimates. Indeed, the reduction of standard error in sufficient bootstrap estimates leads to the higher relative efficiency of sufficient bootstrap estimates compared to conventional bootstrap estimates, which costs the coverage probability of sufficient bootstrap confidence interval estimates.

This study leads to the conclusion that if higher coverage probability of CI estimates is of concerned, then conventional bootstrapping is preferable to sufficient bootstrap CI estimates. On the other hand, if confidence length is of concern, the sufficient bootstrap CI estimates should be preferred to conventional bootstrap CI estimates.

Appendix. R code for CIs using bootstrap and sufficient bootstrap methods

```
M=1000                        # Monte Carlo size
B=200;                        # number of bootstrap replications from a given sample
n=100;                        # sample size
index=1:n;                    # index of numbers 1 through n
alpha=1.2;                    # shape parameter of beta distribution (bd)
beta=1.6;                     # scale parameter of bd

# population parameters of interest
mu=alpha/(alpha+beta);        # mean of bd
sigma2=alpha*beta/((alpha+beta)^2*(alpha+beta+1));    # var of bd
sigma=sqrt(sigma2);           # sd of bd
cvp=sigma*100/mu;             # cv of bd
par<-c(mu,sigma2,sigma,cvp)   # vector of interested parameters

#define the function est;
est<-function(x){
m<-mean(x);v<-var(x);s<-sqrt(v);cv<-s/m*100;
ests<-c(m,v,s,cv)
return(ests)}

#storage for coverage prob and length over M simulations;
pcb<-array(0,c(M,4))
plb<-array(0,c(M,4))
pcs<-array(0,c(M,4))
pls<-array(0,c(M,4))

tcb<-array(0,c(M,4))
tlb<-array(0,c(M,4))
tcs<-array(0,c(M,4))
tls<-array(0,c(M,4))

for (k in 1:M){
x=rbeta(n,alpha,beta);
e0<-est(x);                    #func est defined on x for point estimate of parameters
                                 from x;
#initialized storage for estimates;
eb=array(0,c(B,4));
es=array(0,c(B,4));
tb=array(0,c(B,4));
ts=array(0,c(B,4));

for (i in 1:B){
indx=sample(index,rep=T);
indxu=unique(indx);
```

```
yb=x[indx];                    #bootstrap sample from x
ys=x[indxu];                   #sufficient bootstrap sample from x
 eb[i,] <- est(yb)             #function est defined on yb;
 es[i,] <- est(ys)             #function est defined on ys;

# initialize storage for second level (i.e., nested) samples
eb2=array(0,c(B,4));
es2=array(0,c(B,4));

for (j in 1:B){
indx=sample(index,rep=T);indxu=unique(indx);
yb2<-yb[indx];
ys2<-yb[indxu]
eb2[j,]<-est(yb2)
es2[j,]<-est(ys2)
}

#sds from innner or 2nd (j) level bootstrap for t-CI estimates;
sdbi=apply(eb2,2,sd)
sdsbi=apply(es2,2,sd)
tb[i,]<-(eb[i,]-e0)/sdbi;
ts[i,]<-(es[i,]-e0)/sdsbi;
}
#sds from 1st (i) level bootstrap;
sr<-apply(eb,2,sd);
srs<-apply(es,2,sd);

#quantile for t-CI;
tr<-apply(tb,2,quantile,c(0.025,0.975))
trs<-apply(ts,2,quantile,c(0.025,0.975))

#percentile CIs;
cib<-apply(eb,2,quantile,c(.025,.975))
cis<-apply(es,2,quantile,c(.025,.975))

cib
cis
#coverage and length for percentile CIs;
pcb[k,]<-cib[1,]<=par&par<=cib[2,];
plb[k,]<-ifelse(pcb[k,],cib[2,]-cib[1,],NA)
pcs[k,]<-cis[1,]<=par&par<=cis[2,];
pls[k,]<-ifelse(pcs[k,],cis[2,]-cis[1,],NA)

#t CIs;
lob<-e0-sr*tr[2,]
upb<-e0-sr*tr[1,]
tcb[k,]<-lob<=par&par<=upb;
tlb[k,]<-ifelse(tcb[k,],upb-lob,NA)

los<-e0-srs*trs[2,]
ups<-e0-srs*trs[1,]
```

```
tcs[k,]<-los<=par&par<=ups;
tls[k,]<-ifelse(tcs[k,],ups-los,NA)

}

apply(pcb,2,mean)
apply(pcs,2,mean)
apply(tcb,2,mean)
apply(tcs,2,mean)

round(apply(plb,2,mean,na.rm = TRUE),digits=3)
round(apply(pls,2,mean,na.rm = TRUE),digits=3)
round(apply(tlb,2,mean,na.rm = TRUE),digits=3)
round(apply(tls,2,mean,na.rm = TRUE),digits=3)
```

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