Analysis of a Process Control Model Subject to Errors in Classification

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Abstract
A process control model that selects items at fixed intervals is studied. In this model, inspection errors can occur. That is, a conforming item might be misclassified as nonconforming or a nonconforming item might be misclassified as conforming. Because of this, an item is subjected to multiple classifications before a final judgment is made. Various short and long-term behaviors and properties and properties of the process are studied

Key Words: quality control, attributes, inspection

1. Introduction

In Taguchi, Elsayed, and Hsiang(1989) and Taguchi, Chowdhury, and Wu (2004) on-line process control by attributes in which every h^{th} item produced is inspected was studied. The process is assumed at the beginning to be in control and to have some high fraction of items conforming to specifications denoted by p_1 close to 1. The process goes out of control at some time and results in a shift to p_2 ($< p_1$) for the fraction conforming. If an inspected item is judged nonconforming, the process is stopped and a search is initiated for an assignable cause.

A number of authors have considered modifications of this or written papers in a similar spirit. For example, Nayebpour and Woodall (1993) considered the random time until the shift from p_1 to p_2 to follow a geometric distribution. Items produced were modeled as independent and identically distributed trials with a constant probability of π for each item to be the first item produced with the new shifted (smaller) fraction conforming. Since only every h^{th} item is inspected, the first item produced under this shifted fraction conforming value might not be inspected and thus there can be some number of items produced before there is even the possibility of detecting this shift.

Borges, Ho, and Turnes (2001) have argued that the inspection process itself can be subject to diagnostic errors so that a classification can result in a conforming item being mistakenly classified as nonconforming. We let p_{CN} be the probability of this misclassification. In addition, a nonconforming item can be classified as conforming. We let p_{NC} be the probability of this misclassification. We also define p_{CC} (p_{NN}) to be the probability of correct classification that a conforming (nonconforming) item is classified as conforming (nonconforming). This naturally suggests the idea that repeated classifications of each inspected item should be made before making the final judgment of whether the item is conforming or nonconforming. When the item has been judged in this final determination to be nonconforming, the process is judged out of control and is stopped for a search for an assignable cause, and if one is found an adjustment is made to put the process back in control. Since there is the possibility of diagnostics errors in the repeated classifications, there is a possibility that an item is judged to be nonconforming and the process is judged out of control, even though actually is not. Still it is stopped for a search for an assignable cause and adjustment is made if one is found. However, if no cause can be found, the process is then restarted and it is assumed that the process has not somehow been put out of control by the stopping and searching for a cause. On the other hand, it is also possible that the process goes out of control, but this is not detected when an item is subjected to inspection through repeated classifications in which case it remains out of control until this is detected with a later item.

In Trindade, Ho, and Quinino (2007), the final determination of whether the inspected item is conforming, and thus whether the process is in control, was based on a pre-specified number of repeated classifications or using majority rule. In Quinino, Colin, and Ho (2009), an item was judged to be conforming and the process to be in control if and only if there were *k* classifications as conforming before *f* classifications as nonconforming, where *k* and *f* are some pre-specified positive integers. We will use the acronym TCTN because the decision is based on the total number of classifications as conforming and nonconforming. Smith and Griffith (2009, 2017) further studied this rule and another rule called CCTN.

In this paper, we continue the study of the alternative rule TCTN in which the final determination that an item is conforming, and thus the process is in control, if and only if a total of *k* classifications as conforming occur before a total of *f* classifications as nonconforming.

2. Probabilistic Analysis

Proposition 1: If the item being inspected is conforming (nonconforming), the probability that it is judged to be conforming is

$$P(judged\ conforming|\ actually\ conforming) = TCTN(p_{CC}) = \sum_{i=0}^{f-1} \binom{k+i-1}{k-1} p_{CC}^{k}(1-p_{CC})^{i}$$

$$P(judged\ conforming|\ actually\ nonconforming) = TCTN(p_{NC}) = \sum_{i=0}^{f-1} \binom{k+i-1}{k-1} p_{NC}^{k}(1-p_{NC})^{i}$$

PROOF: Consider the Markov chain $\{X_n\}$ with state space

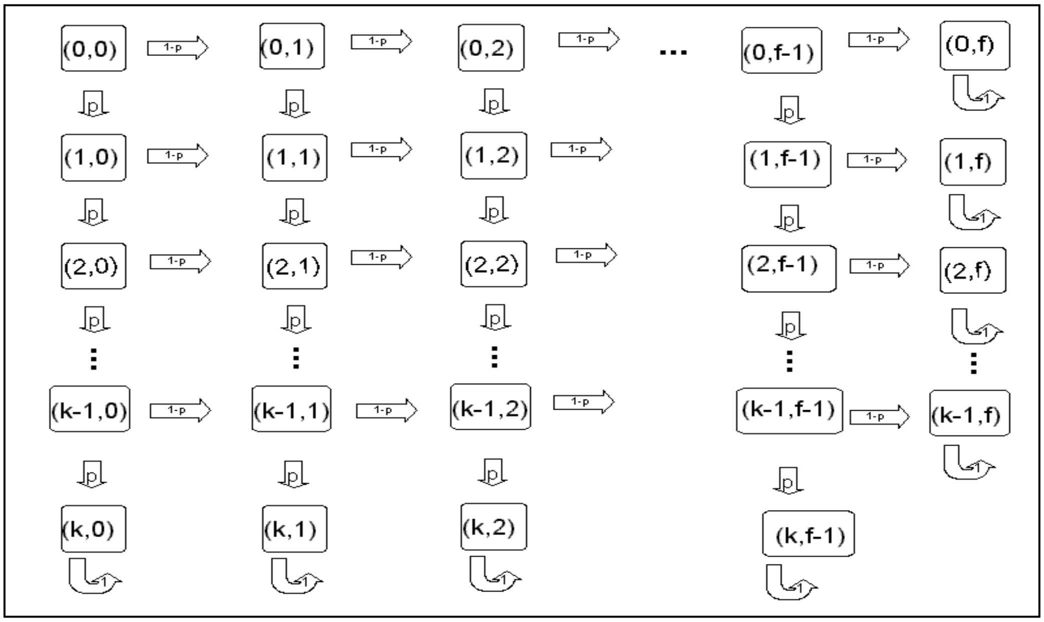
$$\{(r,s): 0 \le r \le k, 0 \le s < f\} \cup \{(r,s): 0 \le r < k,\ s = f\}$$

where $X_n = (r,s)$ means that after the n^{th} test there are r total successes and s total failures. The transition probabilities when beginning in a transient state are of the form

$$P(X_n = (r+1,s)|\ X_{n-1} = (r,s)) = p \text{ and } P(X_n = (r,s+1)|\ X_{n-1} = (r,s)) = q.$$

The situation is depicted in the Figure 1.

Figure 1.



Notice in Figure 1 that whenever we are in a particular state we go down the column with probability p and across the row with probability 1-p. The only exceptions are the absorbing states. The states in the last row are the absorbing states corresponding to acceptance and states in the last column are the absorbing states corresponding to rejection.

The proof of formulas follow by observing that the probabilities that are summed are negative binomial probabilities, with index i denoting the number of failures before the k^{th} success, and the index j being the number of successes before the f^{th} failure.

Proposition 2: If the process is in control, the probability that it is judged to be in control is

$$P_{II} = P(judged\ in\ control|\ actually\ control) = p_1 TCTN(p_{CC}) + \ (1 - p_1)TCTN(p_{NC})$$

Proof: If it is in control, then the inspected item is conforming with probability p_1 and nonconforming with probability 1-p_1. In light of proposition 1 and using the law of total probability the result follows.

Proposition 3: If the process is out of control, the probability that is judged to be in control is

$$P_{OI} = P(judged\ in\ control|\ out\ of\ control) = p_2 TCTN(p_{CC}) + \ (1 - p_2)TCTN(p_{NC})$$

Proof: If out of control, then inspected item conforms with probability p_2 and fails to conform with probability $1 - p_2$. In light of proposition 1 and using the law of total probability the result follows.

Proposition 4: When the process is out of control, the average run length is $\frac{1}{1-P_{OI}}$.

Proof: This is geometric distribution with parameter $1 - P_{OI}$.

Proposition 5: When the process is in control, the average run length is $\frac{1}{1-P_{II}}$.

Proof: This is geometric distribution with parameter $1 - P_{II}$.

3. Short Term Analysis Using Markov Chains

We will use Markov Chains to study the probability of judging the process to be out of control when it is in control as well as judging it to be out of control when it is out of control. We will also study the distribution of the time until the process is declared out of control using first passage probabilities. To this end, we create a Markov Chain whose state space contains four ordered-pairs whose elements are one or zeros. We use a 1 to stand for in control and a 0 to stand for out of control. The first coordinate is the actual state of the process and second coordinate is the judgment. For example, (1,1) means that at a decision point the process is in control and judged to be in control. Whereas, (0,1) means that the process is actually out of control but judged to be in control. Let $\theta = 1-(1-\pi)^h$. So, $1-\theta = (1-\pi)^h$ is the probability that the process has remained in control while those h items have been produced. The one-step probability matrix for the transitions of this Markov Chain is given in the following transition matrix.

$$\begin{array}{c} \\ (1,1) \\ (0,1) \\ (1,0) \\ (0,0) \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (0,1) & (1,0) & (0,0) \end{array} \\ \begin{pmatrix} (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ 0 & P_{OI} & 0 & 1-P_{OI} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

First-passage probabilities can be used to find the probability distribution of the time until the process is declared out of control. To do this one finds the probability of first reaching each absorbing state in n steps and adding these probabilities to obtain the probability that it takes n steps (cycles of item inspections) to declare the process out of control. One can also use first-step analysis to find the probability of absorption into (1,0) and into (0,0). Note: P_{IO} = 1 - P_{II}
and P_{OO} = 1 - P_{OI}.

4. Long Term Analysis Using Markov Chains

We can also study the long-term behavior of this process control. Whenever we reach state (1,0) or state (0,0) the process is judged out of control. When the cause is found and corrected or when it is determined that the process is in control and there is no cause, the process is put back online and the transitions are like the transition from state (1,1). Thus, to analyze the long term behavior of the decision process, we can use a one-step transition probability matrix in which the rows in the matrix that correspond to transitions out of (1,0) and (0,0) are identical to the transitions out of state (1,1). Therefore, the one-step transition probability matrix useful for long term analysis is given below.

$$\begin{array}{c} \\ (1,1) \\ (0,1) \\ (1,0) \\ (0,0) \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (0,1) & (1,0) & (0,0) \end{array} \\ \begin{pmatrix} (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ 0 & P_{OI} & 0 & 1-P_{OI} \\ (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \end{pmatrix} \end{array}$$

This one-step transition probability matrix is that of an irreducible, aperiodic, positive recurrent Markov Chain and the limiting probabilities exist and are independent of the starting state. These limiting probabilities can also be interpreted as the long-term proportion of time spent in each state. These can be found by solving a system of linear equations.

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