Projecting Age-Specific Death Probabilities at Advanced Ages Using the Mortality Laws of Gompertz and Wittstein

Peter Pflaumer[1]
[1]Department of Statistics, Technical University of Dortmund

Abstract
In this paper, death probabilities derived from the Gompertz and Wittstein models are used to project mortality at advanced ages beginning at the age of 101 years. Life table data of Germany from 1871 to 2012 serve as a basis for the empirical analysis. Projections of the death probabilities and life table survivors will be shown. The increase of the death probabilities slows down at very old ages. Finally, Wittstein´s formula will be regarded as a distribution function. Its reversed hazard rate function, which will be derived together with the median and the modal value, will clarify the significance of the parameters of the Wittstein distribution.

Keywords: Life table, Mortality deceleration, Wittstein distribution, Reversed hazard rate, Centenarians.

1. Introduction

The maximum age for a life table is in general 100 years because there is evidence that life tables for older ages are not reliable for lack of sufficient empirical data (small number of survivors at the most extreme ages). In order to calculate some demographic parameters like the life expectancy at birth, one should know the mortality beyond the age of 100, but the influence of that mortality is so small that it can be neglected, as we will demonstrate later. Other parameters like the median and modal age of death do not need information about that old-age mortality. Earlier, the calculation of mortality rates for the very old was not very important because the numbers involved were very small. However, the numbers of centenarians and supercentenarians (age ≥ 110) are now growing fast and there is much interest within society, government, and insurance companies in their projected numbers – particularly with regard to the impact on pensions and the financial burden of long-term care. Since there is not sufficient data available, mortality at advanced ages has to be determined by a model. Some of the methodologies used to produce mortality rates at advanced ages for life tables are described in Gallop and Macdonald (2005). Death probabilities derived from the Gompertz distribution will be used to project mortality at those old ages. Whereas in the Gompertz model the force of mortality increases exponentially with age, the increase of the death probabilities slows down at very old ages, a phenomenon commonly known as mortality deceleration (see, e.g., Ouellette and Bourbeau, 2014, and the references cited therein). Life table data of Germany from 1871 to 2012 serve as a basis for the empirical analysis. Projections of the death probabilities from the age of 100 will be shown. Finally, an alternative to the Gompertz death probability function, Wittstein´s mortality formula, is proposed, whose rate of increase is less.

2. The Gompertz distribution and its death probabilities

The British actuary, Benjamin Gompertz, proposed a simple formula in 1825 for describing the mortality rates of the elderly. This famous law states that death rates increase exponentially with age.

In terms of actuarial notation, this formula can be expressed as

$$\mu(x)=A\cdot e^{kx},$$

where $\mu(x)$ is the force of mortality, A>0 and k>0. A represents the general mortality level, and k is the age-specific growth rate of the force of mortality. Since

$$\mu(x)=-\frac{d\ l(x)}{dx}\cdot\frac{1}{l(x)}=-\frac{d\ \ln l(x)}{dx},$$

one gets through integration the survivor function of the Gompertz distribution,

$$l(x)=\exp\left(\frac{A}{k}-\frac{A}{k}\cdot e^{k\cdot x}\right)\quad x\geq 0.$$

The mode can be simply determined by differentiation of the density function $-\frac{dl(x)}{dx}$

$$m=-\frac{\ln\left(\frac{A}{k}\right)}{k}.$$

From

$$A=k\cdot e^{-k\cdot m}=\mu(m)\cdot e^{-k\cdot m}$$

follows

$$\mu(x)=k\cdot e^{k\cdot(x-m)}=\mu(m)\cdot e^{k\cdot(x-m)}.$$

In this representation the survivor function is given by

$$l(x)=\exp\left(e^{-k\cdot m}-e^{k\cdot(x-m)}\right) = \exp\left(e^{-m/s}-e^{-(x-m)/s}\right)$$ (cf., e.g., Carriere, 1994).

One recognizes that the survivor function is characterized by the mode m and a spread parameter $s=\frac{1}{k}$.

Since with human populations $e^{-k\cdot m}\approx 0$, the survivor function can be approximated for $-\infty<x<\infty$ by

$$l(x) = \exp\left(-e^{k(x-m)}\right)$$

(see, e.g., Pollard and Valkovics, 1992), which is the survivor function of the Gumbel distribution (minimum) or the extreme value type I distribution for the minimum.

The chance of living one year for a person aged x is

$$p(x)=\frac{l(x+1)}{l(x)}=\frac{l_{Go}(x+1)}{l_{Go}(x)}=\exp\left(-\left(e^{k}-1\right)\cdot e^{k\cdot(x-m)}\right)=\exp\left(-e^{k\cdot(x-m)}\right)^{\left(e^{k}-1\right)}=l(x)^{\left(e^{k}-1\right)}\approx l(x)^{k}$$

where $l(x)=\exp\left(-e^{k\cdot(x-m)}\right)$ (Gumbel Distribution) and $l_{Go}(x)=\exp\left(e^{-k\cdot m}-e^{k\cdot(x-m)}\right)$ (Gompertz Distribution)

Therefore the chance of dying for a person aged x within a year is

$$q(x)=1-p(x)=1-l(x)^{e^{k}-1}\approx 1-l(x)^{k}.$$

The force of mortality increases exponentially at old ages, but the probability of death tends to the limit of 1 with decreasing increments. The force of mortality at the modal age m is $\mu(m) = k$. The probability of death at this age is $q(m)=1-\exp\left(1-e^{k}\right)\approx k-\frac{k^3}{6}$.

The relative change or growth rate of q(x) is

$$\frac{\frac{dq(x)}{dx}}{q(x)} = \frac{d\ln q(x)}{dx} = \left(e^{k}-1\right)\cdot\frac{\mu(x)}{\frac{q(x)}{1-q(x)}}\approx k\cdot\frac{\mu(x)}{\frac{q(x)}{1-q(x)}}=k\cdot\mu(x)\cdot\frac{p(x)}{q(x)}.$$

The difference between the force of mortality $\mu(x)$ and the probability of death q(x) is small at young ages x, so it can be assumed that $\mu(x) = q(x)$. The growth rate of the force of mortality function is k and is nearly identical to the growth rate of the death probability at young ages, since $p(x)$ is near one.

The Gompertz distribution implies that the force of mortality $\mu(x) = A \cdot e^{k \cdot x}$, and not the death probability $q(x) = A \cdot e^{k \cdot x}$, increases exponentially with age. This exponential increase of the force of mortality at very old ages often leads to criticism of the use of the Gompertz distribution in life table analysis, because it is argued that the mortality at old ages is overestimated. But this criticism overlooks the fact that practical life table analysis is done with discrete functions and therefore with death probabilities. Using the Gompertz distribution, one has to calculate the death probabilities q(x) for x=0, 1, 2, 3,… with the previous formula for a given or estimated m and k. The correct use of the Gompertz distribution implies exponentially increasing death probabilities up to a certain age. After this age, the death probabilities increase with decreasing growth rates. Finally, they approach 1. Only if the death probabilities flatten out faster or are substantially below 1 above a certain age, then the Gompertz model will overestimate mortality at old ages. This point is disputed controversially in the literature. But Gavrilov and Gavrilova (2011), for example, found that the Gompertz law is a good fit up to the age of 106.

Remarks:

1. If one knows l(x) and p(x) of a life table, it is easy to estimate k by means of simple linear regression forced through the origin, since $\ln p(x)=\left(e^{k}-1\right)\cdot\ln l(x)\approx k\cdot\ln l(x)$.

2. The probability of death function q(x) is a distribution function having an inflection point at $x_i = m - \frac{\ln\left(e^{k}-1\right)}{k}$ with $q\left(x_i\right) = 1 - e^{-1} \approx 0.632$. Its α-quantiles are $x_\alpha = m + \frac{\ln\left(\frac{\ln(1-\alpha)}{1-e^{k}}\right)}{k}$ with $q(m) = 1 - e^{1-\exp(k)} \approx k$.

3. Empirical analysis

We will begin our analysis with the German life table of 2012/2014. Some key dates are given in Table 1. In life table construction, it is impossible to calculate death probabilities at certain old ages, because of a lack of deaths or at-risk population. In these circumstances, the creators of the German life table used a model for estimating old-age mortality rates. The problem now is that we partly fit our model to their model and not to values derived from empirical data. We partly explain one model with another model. We have therefore to assume that their model is an accurate representation of the mortality pattern. This restriction will not really be relevant for the analysis, because the projections should rather demonstrate the application of the proposed models and serve less as a resource for real forecasts.

Table 1: German life table 2012/2014

Parameter	Male	Female
Life expectancy at birth	78.13	83.05
Median age	81.3	85.9
Modal age	84.64	88.45
Life expectancy at age 100	1.83	2.09
Number surviving to age 100 l_{100} *)	622	1873
Death probability at age 100 q_{100}	0.4046	0.35596

*) $l_0 = 100,000$

We used non-linear least squares to estimate the parameters (the non-linear regression procedure of STATISTIX 10.0, a statistical software package). The estimation results are seen in Table 2a and 2b. Because of the low mortality at young ages the results do not depend on the age range x (x=0, 1, 2.. or x=70, 71,….). The estimators are significant and the (approximative) coefficient of variation is near 1. The estimated modal values are similar to the modal values in the life tables of 2012 (cf. Table 1 and Table 2). The life expectancy of the model is $e(0) = \int_0^\infty \exp\left(-e^{k\cdot(x-m)}\right)dx$. Through numerical integration we obtain for the female life expectancy a value of 82.7 years and for the male one a value of 78.7 years. Again, the model values are similar to those of the life table. This similarity shows the goodness of the fit. It is possible to adequately describe the German life table with only two parameters m and k. The influence of mortality beyond 100 is not of great importance for the life expectancy at birth, since

$\int_{100}^\infty \exp\left(-e^{k\cdot(x-m)}\right)dx = l(0)\cdot e(100) = 0.033$ (model value) or

$l(100)\cdot e(100) = 0.01873 \cdot 2.09 = 0.039$ (life table value).

Table 2 a: Regression results for male death probabilities

```
MODEL: qxm = (1-exp(-exp(k*(x-m)))^(exp(k)-1)) for x=0, 1, 2 …100

                                            Lower      Upper
Parameter    Estimate     Std Error     95% C.I.    95% C.I.
k            0.102974      0.0000798   0.101389    0.104558
m            84.23541      0.162900    83.91218    84.55864

Standard Deviation        0.0040
Degrees of Freedom           99
Pseudo R²                 0.9982

Age x > 70
Parameter    Estimate     Std Error    L95% C.I.   U95% C.I.
k            0.103068      0.00154     0.099928    0.106209
m            84.25454      0.31294     83.61450    84.89457
```

Table 2 b: Regression results for female death probabilities

```
MODEL: qxw = (1-exp(-exp(k*(x-m)))^(exp(k)-1)) for x=0,1,2,…100

                                              Lower      Upper
Parameter    Estimate    Std Error    95% C.I.    95% C.I.
 k           0.113375    0.0013541    0.110688    0.116062
 m           87.76842     0.206590    87.35850    88.17834

Standard Deviation   0.005289
Degrees of Freedom         99
Pseudo R²              0.9959

Age x >70

Parameter    Estimate    Std Error    L95% C.I.   U95% C.I.
k            0.113171     0.002558    0.107940    0.118402
m            87.73680     0.391932    86.93521    88.53839
```

Table 3 shows the projected death probabilities up to the age of 120. The probabilities of the female population are less than those of the male population up to the age of 113. Then, a mortality crossover can be observed. The death probability of the female population is now higher than that of the male population. If this is not a measurement or estimation error, then possible explanations of this phenomenon can be found, e.g., in Wrigley-Field and Elwert, 2016.

Table 3: Projected female (qxwd) and male (qxmd) probabilities of death

x	*100*	*101*	*102*	*103*	*104*	*105*	*106*	*107*	*108*	*109*	
qxwd	0.3560*	0.4161	0.4527	0.4909	0.5305	0.5712	0.6127	0.6544	0.6958	0.7363	
qxmd	0.4046*	0.4564	0.4912	0.5271	0.564	0.6016	0.6394	0.6772	0.7144	0.7507	
x	*110*	*111*	*112*	*113*	*114*	*115*	*116*	*117*	*118*	*119*	*120*
qxwd	0.7753	0.8121	0.8463	0.8772	0.9046	0.928	0.9475	0.9632	0.9752	0.9841	0.9903
qxmd	0.7856	0.8186	0.8492	0.8772	0.9022	0.924	0.9425	0.9578	0.9701	0.9796	0.9866

*) probabilities of the life table 2012/2014

We can now calculate the number of survivors from age 100 using the projected death probabilities. The results are given in Table 4. In the last two columns the number of survivors at age 100 has been set to a radix of 100,000 in order to visualize the size of the centenarians.

Since $\int_x^\infty l(u)du = e(x)\cdot l(x)$ and at very high ages $e(x) \approx \frac{1}{\mu(x)}$, we get the proportion of those people in a stationary population as $\pi(x) = \frac{\int_x^\infty l(u)du}{e(0)} \approx \frac{\frac{l(x)}{\mu(x)}}{e(0)} = \frac{\frac{\exp\left(-e^{k\cdot(x-m)}\right)}{k\cdot e^{k(x-m)}}}{m-\frac{\gamma}{k}}$ with γ=0,577221566... (Euler-Mascheroni-Constant). For example, one expects about 3 supercentenarians in a stationary population of 80 million (k=0.113, m=87.7).

Table 4: Projected number of female (lxwd) and male (lxmd) survivors at age x for the German life table 2012/2014

x	lxwd	lxmd	lxwd	lxmd
0	100,000	100,000	5,455,537	16,077,170
100	1,833*	622*	100,000	100,000
101	1,181	370	64,404	59,543
102	689	188	37,603	30,297
103	377	89	20,581	14,326
104	192	39	10,478	6,246
105	90	15	4,920	2,489
106	39	6	2,109	897
107	15	2	817	290
108	5	1	282	83
109	2	0	86	21
110	0	0	23	4
111	0	0	5	1
112	0	0	1	0.1
113	0	0	0.1	0
114	0	0	0	0
115	0	0	0	0
116	0	0	0	0

*) Values of the life table 2012/2014

In the next step, the analysis has been expanded to German life tables for females between 1871 and today. The Gompertz model is only valid at the ages of 30 and more, if there is high infant mortality. Therefore all data with age x<30 were omitted. The estimation results are given in Table 5. The last two columns show the inverse probability of a newborn reaching the ages 100 and 110, where the following formula has been used: $l(x) = \exp\left(-e^{k\cdot(x-m)}\right)$ for x=100 and 110.

For example, in 2012 at least 1 in every 55 newborns will reach the age of 100, whereas in 1871 only 1 in every 22,256 achieved that age. The probability of becoming a supercentenarian according to our projected life table is still very low today. Only 1 in about 250,000 will celebrate their 110^{th} birthday. The Gompertz model yields l(100)=0.01828, whereas the life table 2012/14 is l(100)=0.01873. This small difference again shows the good fit of the model to the data.

Figure 1 shows death probabilities for females between the ages of 30 and 120 calculated and projected from different German life tables. The graph clearly demonstrates the deceleration of mortality at advanced ages. Mortality has declined enormously during the last 150 years. In the first phase, infant mortality decreased, which resulted in an increase of the life expectancy at birth. In later phases, adult mortality, especially in the age classes 70 to 90 fell strongly. The mortality also decreased in midrange age classes, but this was not so remarkable, because it fell from an already low level. In the current phase, notably mortality at very high ages decreases, which yields an increase of centenarians. As shown earlier, the mortality decline at those ages, however, will hardly influence the expectancy at birth.

Table 5: Estimation results for death probabilities of females

Year	k	m	R-Square	1/l(100)	1/l(110)
1871	0.071527	67.7937	0.9976	22,256	775,144,003
1881	0.068369	67.12033	0.9925	12,943	140,170,764
1891	0.070106	68.31711	0.9926	10,077	117,565,666
1901	0.069963	68.93485	0.989	6,556	48,193,542
1910	0.07465	70.26028	0.995	9,976	273,053,620
1924	0.071535	71.42316	0.994	2,260	7,227,715
1932	0.081696	74.43494	0.9968	3,208	86,426,203
1949	0.080676	75.54479	0.9914	1,329	9,962,347
1960	0.080922	76.47939	0.9863	819	3,498,068
1970	0.086222	78.44302	0.993	611	3,969,222
1986	0.097385	83.40172	0.9967	154	617,490
2012	0.113376	87.76853	0.9955	55	251,450

(Age $x \geq 30$, number of observations n=70, all estimators are significant)

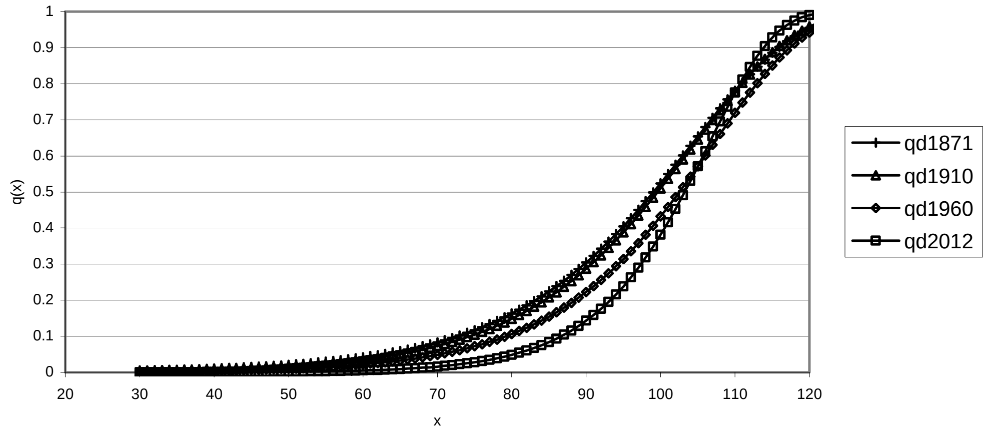

Figure 1: Death probabilities calculated and projected from German life tables for females between 1871 and 2012.

4. Wittstein´s Mortality Formula

The question of how realistic the projection results are using the Gompertz distribution cannot be answered easily, for lack of sufficient empirical observations. However, Ouellette and Bourbeau (2014) used a highly reliable set of data on French-Canadian centenarians in the province of Quebec in order to investigate mortality at the ages between 100 and 112. Their data set included 2198 females born between 1870 and 1896 who died between 1970 and 2009. Their results provided evidence of a late-life mortality deceleration at very old ages for this population. Comparing their results (cf. Fig. 2b and Fig. 3 in Ouellette and Bourbeau, 2014) with our results in Table 3, it is observed that our death probabilities are around 0.15–0.2 percentage points higher. In view of this fact, we consider as an alternative model Wittstein´s mortality formula, whose mortality increase at advanced age is smaller. Wittstein´s formula was employed, for example, by the US Bureau of the Census for calculating the life table of 1910, where the rate of mortality was taken as unity at the age of 115 (see US Bureau of the Census, 1916, p. 12).

The Wittstein formula (Wittstein, 1883) is given by

$$q(x) = a^{-(M-x)^n} + \frac{1}{m} \cdot a^{-(m \cdot x)^n} = e^{-k(M-x)^n} + \frac{1}{m} e^{-k(m \cdot x)^n}$$

with the parameters $n > 0$, m>0, M>0, $0 < a < 1$, k>0: It is $q(0) \approx \frac{1}{m}$; $q(M) = 1$ (see also Appendix 1).

If n>1, then q(x) is increasing with decreasing growth rates. If n=1, then q(x) is exponentially increasing with a constant growth rate. If $0 < n < 1$, then q(x) is increasing with an increasing growth rate.

The function has a minimum at the age

$$x = \frac{M}{m+1}.$$

Since the value of the last term of this equation is practically zero at the older ages, only the first term will be employed

$$q(x) = a^{-(M-x)^n} = e^{-k(M-x)^n}$$

The parameter M should not be considered as the maximum age, as is sometimes done, because it is extremely unlikely that this age will be ever reached.

The estimation results are seen in Table 6a and 6b using data of the German life table 2012/14. Note that the estimator k is close to zero and is not significant. The main problem we experienced with the Wittstein model (with unknown M) is the fact that the starting values for the nonlinear estimation procedure must be close to the as yet unknown parameter estimates or the optimization procedure will not converge. Table 7 shows the projected death probabilities. However, the main differences between the Gompertz and Wittstein models are seen in Figure 2. The projected increase of the death probabilities is lower when employing Wittstein´s formula. The difference is also expressed in Table 8, where the probabilities of reaching extreme age are compared. The projected rates are similar to the rates obtained by Ouellette and Bourbeau (2014).

Table 6a: Regression results for Wittstein´s formula (female)

MODEL: qxw = exp(-k*(M-x)^n) $x \geq 70$

Parameter	Estimate	Std Error	Lower 95% C.I.	Upper 95% C.I.
k	6.64019E-07	2.0798E-06	-3.59617E-06	4.92421E-06
M	152.7083	10.46822	131.2651	174.1515
n	3.590523	0.611484	2.337954	4.843093

Residual SS (SSE)	4.743E-04
Residual MS (MSE)	1.694E-05
Standard Deviation	4.116E-03
Degrees of Freedom	28
Pseudo R^2	0.9987

If M=152.8 were known, then the following would be obtained:

Parameter	Estimate	Std Error	L. 95% C.I.	U. 95% C.I.
k	6.46036E-07	8.6448E-08	4.69231E-07	8.22841E-07
n	3.595883	0.032675	3.529054	3.662711

Table 6b: Regression results for Wittstein´s formula (male)

MODEL: qx = exp(-k*(M-x)^n) $x \geq 70$

Parameter	Estimate	Std Error	Lower 95% C.I.	Upper 95% C.I.
k	8.81775E-04	7.3209E-04	-6.17852E-04	2.38140E-03
M	127.6605	3.167486	121.1722	134.1488
n	2.082986	0.179941	1.714394	2.451578

Residual SS (SSE)	5.052E-04
Residual MS (MSE)	1.804E-05
Standard Deviation	4.248E-03
Degrees of Freedom	28
Pseudo R^2	0.9988

Table 7: Projected female (qxdwW) and male (qxdmW) probabilities of death (Wittstein´s formula)

x	qxdwW	qxdmW	x	qxdwW	qxdmW
101	0.3893	0.4391	121	0.8498	0.9552
102	0.4150	0.4676	122	0.8650	0.9679
103	0.4409	0.4968	123	0.8792	0.9785
104	0.4671	0.5263	124	0.8924	0.9869
105	0.4933	0.5562	125	0.9047	0.9933
106	0.5196	0.5863	126	0.9160	0.9975
107	0.5456	0.6164	127	0.9263	0.9996
108	0.5715	0.6464	128	0.9358	
109	0.5970	0.6761	129	0.9444	
110	0.6221	0.7054	130	0.9522	
111	0.6467	0.7341	131	0.9592	
112	0.6706	0.7621	132	0.9655	
113	0.6939	0.7891	133	0.9710	
114	0.7165	0.8151	134	0.9759	
115	0.7383	0.8399	135	0.9802	
116	0.7592	0.8633	136	0.9839	
117	0.7792	0.8852	137	0.9871	
118	0.7983	0.9054	138	0.9898	
119	0.8165	0.9239	139	0.9921	
120	0.8336	0.9406	140	0.9940	

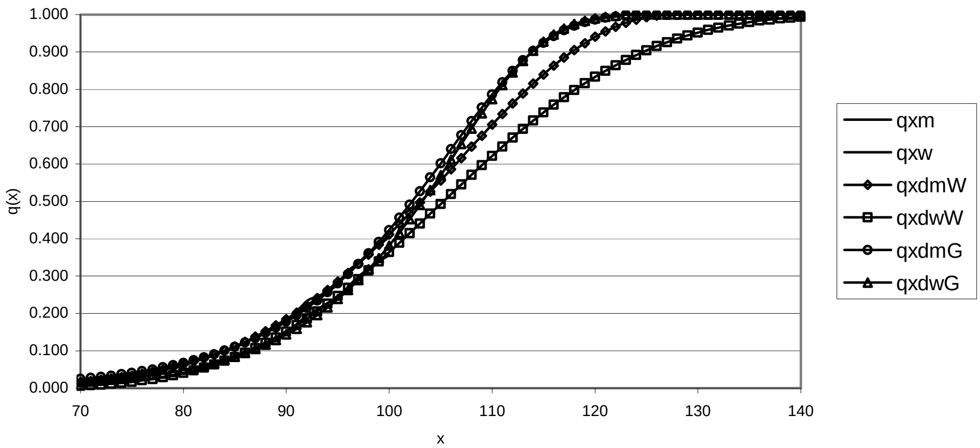

Figure 2: Death probabilities for males (qxm) and females (qxw) and their projections with the Gompertz model (qxdmG, qxdwG) and the Wittstein model (qxdmW, qxdwW)

Table 8: Probabilities (female)

	Gompertz	Wittstein
1/l(100)	55	55
1/l(110)	251,450	42,036

Table 9 shows the estimation results which have been obtained using the Wittstein formula and different life tables. Over time, M and n have increased, whereas k has decreased. Note, however, that the estimators of k are not significant. It was not possible to obtain estimation results for the other German life tables, because the iterative estimation procedure failed to converge and produced "strange", i.e. very large or very small, parameter estimates. The mortality pattern of these life tables in the age range 70 to 100 differed from that in Table 9. The death probabilities in the life tables in Table 9 exhibited an exponentially growing trend, whereas the other life tables showed a trend with decreasing growth rates (S-shape pattern). Thus, the estimation finally depends on the assumptions on the pattern of old age mortality which have been made by the creators of the German life tables. As pointed out earlier, the creators of the German life table used a model to estimate old-age mortality rates.

In Figure 3 projections of death probabilities in different years for females are shown. The noticeable mortality reduction at all ages over time is clearly visible. However, a mortality crossover occurs at age 109 for the life tables of 2012 and 1986, i.e., the (projected) mortality after the age 109 is higher in 2012 than in 1986. The turning point m (modal age) of the q(x)-functions and the slopes at the turning points did not change substantially. However, the death probabilities fell from 0.582 to 0.486. The median $x_{0.5}$, the age at which q(x)=0.5, increased from about 100 in the older life tables to about 105 in the more current life tables (see Table 9). Although n increased, the differences between M and m respectively M and the median $x_{0.5}$ did not change because there was a decrease of k. The influence of parameter changes on the q(x)-function can be observed in Figure 4 (in Appendix 2). Beginning with the death probability function of 1871, one parameter after the other is changed in order to finally obtain the current death probability function of 2012.

Table 9: Estimation results for the Wittstein formula and some parameters (age x >69)

	Parameter			Standard deviation		
Year	k	M	n	sd(k)	sd(M)	sd(n)
1871	2.81E-04	135.8	2.18	2.08E-04	3.31	0.1523
1932	1.30E-05	146.7	2.85	0.00001	1.92	0.0944
1986	2.91E-06	152.3	3.21	2.13E-06	2.76	0.1425
2012	6.64E-07	152.7	3.59	2.08E-06	10.47	0.6115

All pseudo" R-squareds” are near 1

Year	k	M	n	m	q(m)	slope at m	$x_{0.5}$	difference M-m	difference M- $x_{0.5}$
1871	2.81E-04	135.8	2.18	103.7	0.582	0.0214	99.8	32.1	36
1932	1.30E-05	146.7	2.85	102.2	0.525	0.0216	101.1	44.5	45.6
1986	2.91E-06	152.3	3.21	105.1	0.502	0.0215	105	47.2	47.3
2012	6.64E-07	152.7	3.59	104.7	0.486	0.0262	105.2	48	47.5

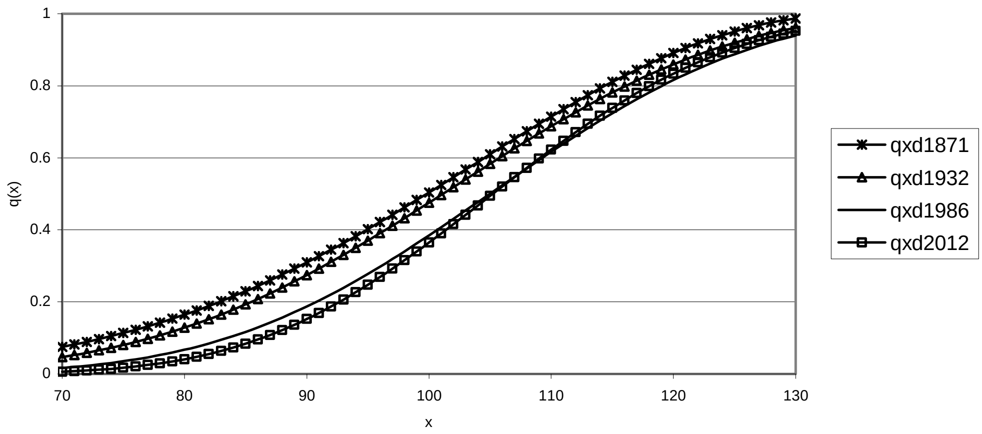

Figure 3: Projections of death probabilities for females (qxdwW) with the Wittstein model

5. Concluding Remarks

Death probabilities functions derived from the Gompertz and Wittstein models are an alternative to the currently widely used logistic functions to fit observed probabilities at the oldest ages. In order to find the best model, it is necessary to have more data. Thus, a solution will be found in the future, when the number of persons at advanced ages will have increased significantly.

Literature

Carriere, JF (1994): An investigation of the Gompertz law of mortality, Actuarial Research Clearing House, 2, 161-177.

Gallop, A and Macdonald, AS (2005): Mortality at advanced ages in the United Kingdom, in: Living to 100 and Beyond Monograph [online edition]. Shaumburg: Society of Actuaries.

Gavrilov, LA and Gavrilova, NS (2011): Mortality measurement at advanced ages: A study of the Social Security Administration Death Master File, North American Actuarial Journal 15 (3), 432-447.

Ouellette, N and Bourbeau, R (2014): Measurement of mortality among centenarians in Canada, Paper presented at the Living to 100 Symposium Orlando, FL., January 8–10, online in the archive of the Society of Actuaries (soa.org).

Pollard, JH and Valkovics, EJ (1992): The Gompertz distribution and its applications, Genus, XLVIII-n3-4, 15-28.

Statistisches Bundesamt (2016): Sterbetafel 2012/2014, Wiesbaden.

US Bureau of the Census (2016): United States Life Tables 1910, Washington.

Wittstein, T (1883): Das mathematische Gesetz der menschlichen Sterblichkeit, 2. Aufl., Hannover.

Wittstein, T and Bumsted, DA (1883): The mathematical law of mortality, Journal of the Institute of Actuaries and Assurance Magazine, Vol. 24, 153-173.

Wrigley-Field, E and Elwert, F (2016): Mortality crossovers from dynamic subpopulation reordering, Robert Schoen, ed., in: Dynamic Demographic Analysis, 177-199.

Appendix 1: Wittstein Distribution

The mortality pattern in the Wittstein formula can be regarded as a distribution function:
$F(x) = e^{-k(M-x)^n} \quad -\infty < x \le M$, k>0, $n > 0$
The density function is given by
$f(x) = k \cdot n \cdot \left(M - x\right)^{n-1} \cdot e^{-k(M-x)^n} \quad -\infty < x \le M$.

The median and modal value can easily be determined:
$x_{0.5} = M - \left(\frac{\ln 2}{k}\right)^{\frac{1}{n}}$ and
$m = M - \left(\frac{n-1}{k \cdot n}\right)^{\frac{1}{n}} \qquad n > 1$.
The slope or gradient of the distribution function at x=m increases with increasing n.
We find that $x_{0.5} > m \;\; if \;\; n > \frac{1}{1-\ln 2} = 3.2588$.
The reversed hazard rate function is given by:

$r(x) = \frac{f(x)}{F(x)} = k \cdot n \cdot \left(M - x\right)^{n-1}$.
The partial derivatives of r(x) are
$\frac{\frac{dr(x)}{dk}}{r(x)} = \frac{1}{k}$ (k is responsible for an exponential increase)
$\frac{\frac{dr(x)}{dn}}{r(x)} = \frac{1}{n} + \ln(M - x)$ (n is responsible for the S-shape, if n>1). The higher n is, the higher the growth rate at age x is.

The difference in years between M and m (decreasing with increasing n):
$\left(\frac{n-1}{k \cdot n}\right)^{\frac{1}{n}}$ with $\lim_{n\to\infty} \left(\frac{n-1}{k \cdot n}\right)^{\frac{1}{n}} = 1$.
The difference in years between M and m (decreasing with increasing n):
$\left(\frac{\ln 2}{k}\right)^{\frac{1}{n}}$ with $\lim_{n\to\infty} \left(\frac{\ln 2}{k}\right)^{\frac{1}{n}} = 1$.

Appendix 2: Graphical presentation of parameter changes in Wittstein´s formula

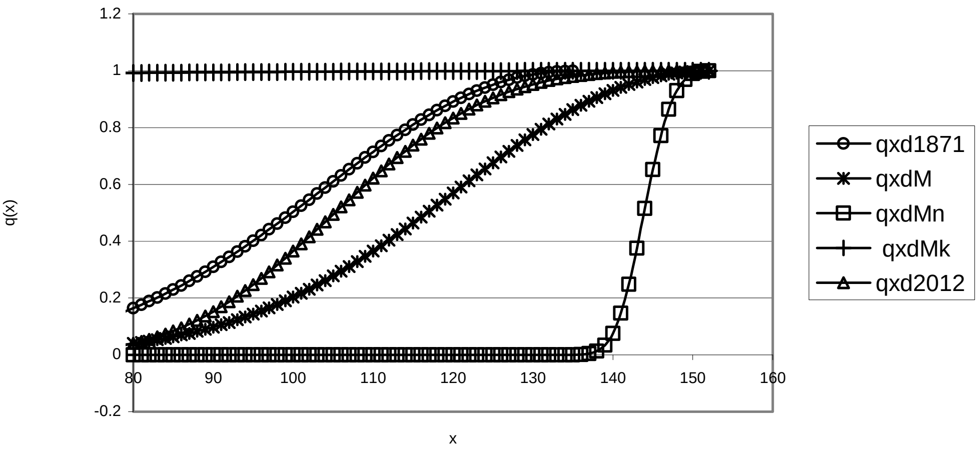

Figure 4: Parameter changes and their influence on the shape of the q(x)-function

Definitions and R-codes:

$$q(x) = e^{-k(M-x)^n}$$

```
qxd1871<-exp(-2.81E-04*(135.8-x)**2.18)
qxdM<-exp(-2.81E-04*(152.7-x)**2.18)
qxdMn<-exp(-2.81E-04*(152.7-x)**3.59)
qxdMk<-exp(-6.64E-07*(152.7-x)**2.18)
qxd2012<-exp(-6.64E-07*(152.7-x)**3.59)
```