Playoff Series and the Incomplete Beta Function

Roger W. Johnson
South Dakota School of Mines & Technology, Department of Mathematics,
501 East St. Joseph Street, Rapid City, SD 57701, roger.johnson@sdsmt.edu

Abstract
An alternate form for the chance of winning a playoff series allows for simpler verification of intuitive properties.

Key Words: Binomial probability mass function, Expected value, Probability, Sports

1. The Probability of Winning a Playoff Series

Consider a best of $2k + 1$ playoff series in which your team wins individual games against its opponent with a fixed chance p with game outcomes independent. Using the binomial probability mass function, the chance that your team wins the playoff series, W_{2k+1}, is

$$\begin{aligned} W_{2k+1}(p) &= \left[\sum_{i=0}^{k} \binom{k+i}{k} p^k (1-p)^i \right] \cdot p \\ &= p^{k+1} \sum_{i=0}^{k} \binom{k+i}{k} (1-p)^i \end{aligned}$$

as your team wins the last game and k of the previous games, which range in number from k to $2k$.

This particular form for W_{2k+1}, however, is not amenable to verifying simple, intuitive properties about it. We believe, for instance, that if the teams are evenly matched in individual games, then the chance our team wins the series is ½. That is, $W_{2k+1}(\tfrac{1}{2}) = \tfrac{1}{2}$. Likewise, as the chance of our team winning any particular game increases, we believe the chance it wins the series should also increase. That is, W_{2k+1} should be increasing in p.

To this end, alternate forms for W_{2k+1} were developed in Johnson (2017). In particular, it was shown that

$$W_{2k+1}(p) = I_p(k+1, k+1)$$

where

$$I_p(a,b) = \frac{\int_0^p t^{a-1}(1-t)^{b-1}\,dt}{\int_0^1 t^{a-1}(1-t)^{b-1}\,dt}$$

is the *incomplete beta function*.

Now, using

$$W_{2k+1}(p) = \frac{\int_0^p [t(1-t)]^k \, dt}{\int_0^1 [t(1-t)]^k \, dt}$$

and the fact that $t(1-t) = 1/4 - (t-1/2)^2$ is symmetric about t = ½, it follows that the numerator integral is half the denominator integral when p = ½. That is, W_{2k+1}(½) = ½. Furthermore, using the Fundamental Theorem of Calculus, it follows that

$$\frac{d}{dp} W_{2k+1}(p) = \frac{[p(1-p)]^k}{\int_0^1 [t(1-t)]^k \, dt} > 0$$

so W_{2k+1} is indeed increasing on (0,1). Other results concerning W_{2k+1} appear in Johnson (2017).

2. Expected Playoff Series Length

The expected number of games played in our best of $2k + 1$ playoff series, call it $E_{2k+1}(p)$, may also be expressed in terms of the incomplete beta function. In particular,

$$E_{2k+1}(p) = (k+1)\left[\frac{I_p(k+1,k+1)}{p} + \frac{I_{1-p}(k+1,k+1)}{(1-p)} - \binom{2k+1}{k} p^k (1-p)^k\right]$$

This was derived analytically and is, using an identity for the incomplete beta function, equivalent to a similar expression given by Gibbons, Olkin and Sobel (1978). The above expression, necessarily symmetric about ½ in p, is maximized when p = ½, the value of which is

$$E_{2k+1}(½) = (k+1)\left[2 - \binom{2k+1}{k} / 4^k\right]$$

For a best of 7 game series (k = 3), for example, the expected length is 5.8125 games.

References

Johnson, Roger (2017), Bet(ch)a my team wins the playoffs, *The College Mathematics Journal*, **48**(5), 347-353.

Gibbons, Jean D., Ingram Olkin, Milton Sobel (1978), Baseball competitions – Are enough games played?, *The American Statistician*, **32**(3), 89-95.