A Spectral-Based Kolmogorov-Smirnov Method for Detecting the Information Loss of Temporal Aggregation

Bu Hyoung Lee*

Abstract
In this article, we develop a spectral method for identifying information loss on the process characteristics of an aggregate series. Even though temporal aggregation is a simple and efficient technique summarizing sequential observations, it causes substantial structural changes in a process because a non-aggregate series of a relatively high frequency is transformed into an aggregate series of a relatively low frequency. The effects of temporal aggregation can be explained with changes in the spectral density function. Then, we propose a spectral-based Kolmogorov-Smirnov test for detecting an aggregation resulting significant structural changes to white noise.

Key Words: temporal aggregation, information loss, spectral density function, Kolmogorov-Smirnov test, time series

1. Introduction

Univariate temporal aggregation, defined as the periodic non-overlapping sums of a time series process, is a simple and efficient technique accumulating sequential observations and so reducing their size. As a result, the technique has been widely used in various fields such as economics and environmental science where long temporal data are treated. Nevertheless, the aggregation is known to cause substantial changes in a process structure, called loss of process information (Abraham, 1982; Rossana and Seater, 1995), when a non-aggregate series of a relatively high frequency and a short period is transformed into an aggregate series of a relatively low frequency and a long period. Those structural changes due to temporal aggregation are associated with the fact that the transformed spectral density function of the aggregates is closely linked but completely different to the initial spectral density function of the non-aggregates. That is, the information loss on a process structure can be explained by similarities and differences between the two spectral density functions. Thus, we can characterize the effects of temporal aggregation as "changes in spectral densities" and "information loss of model structure."

The aggregation effects on various time series processes have been investigated in the literature, for example, Tiao (1972) of an integrated moving average (IMA) process, Amemiya and Wu (1972) of an autoregressive (AR) process, Brewer (1973) of an autoregressive moving average (ARMA) process and an ARMA with exogeneous variables (ARMAX) process, Tiao and Wei (1976) of a dynamic input-output process, Wei (1978) and Weiss (1984) of an autoregressive integrated moving average (ARIMA) process and a seasonal ARIMA process, Stram and Wei (1986) of an ARIMA process with hidden periodicity, Ltkepohl (1984) and Marcellino (1999) of a vector ARMA (VARMA) process, Drost and Nijman (1993) of a generalized autoregressive conditional heteroscedasticity (GARCH) process, and Hafner (2008) of a multivariate GARCH (MGARCH) process. Teles et al. (2008) derived the exact model expression for the temporal aggregation of an AR(1) process. Lee and Wei (2017) extended the exact model expression to the temporal aggregation of an AR(p) process where p is a positive integer. Also, temporal aggregation is known to affect other process characteristics. For example, the aggregation reduces

*Loyola University Maryland, 4501 N. Charles Street, Baltimore, MD 21210

the non-linearity (Granger and Lee, 1999; Teles and Wei, 2000), the non-Gaussianity (Teles and Wei, 2002), and the non-stationarity (Teles et al., 2008) of a univariate time series. Breitung and Swanson (2002) examined the impact of temporal aggregation on instantaneous causality in a vector AR process.

Even though those researches state in common that the initial process information of a time series disappears through temporal aggregation, they do not clarify when the periodic condition of temporal aggregation begins to wipe out the information significantly. In this paper, we therefore propose a spectral method for identifying a periodic order m of temporal aggregation where the mth order aggregates of a stationary series lose all the initial information and start to behave like a white noise series. This paper is organized as follows. In Section 2, we review the concepts of temporal aggregation and the limiting property of a stationary aggregate time series. Section 3 presents the spectral-based Kolmogorov-Smirnov test. In Section 4, Monte Carlo simulations and an illustrated example are given. Section 5 concludes the article.

2. Temporal Aggregation

2.1 Temporal Aggregation in a Stationary Process

We consider a second-order weakly stationary process $\{x_t, t = 1, 2, \ldots\}$ with $\mathrm{E}\,(x_t) = 0$. The stationary series x_t can be expressed as a moving average (MA) model of a finite order q,

$$x_t = \left(\sum_{j=0}^{q} \psi_j B^j \right) e_t, \tag{1}$$

where $\psi_0 = 1$, $\{e_t\}$ is a white noise process of mean zero and variance σ_e^2, and B is the backshift operator such that $B^j x_t = x_{t-j}$. From the autocovariance generating function of x_t

$$C_x(B) = \sum_{k=-q}^{q} \gamma_x(k) \cdot B^k = \sigma_e^2 \left(\sum_{j=0}^{q} \psi_j B^j \right) \left(\sum_{j=0}^{q} \psi_j B^{-j} \right), \tag{2}$$

we obtain the autocovariance of x_t at lag k,

$$\gamma_x(k) = \mathrm{E}\left(x_t x_{t-|k|} \right) = \begin{cases} \sigma_e^2 \sum_{j=|k|}^{q} \psi_j \psi_{j-|k|}, & \text{for } |k| = 0, 1, \ldots, q, \\ 0, & \text{for } |k| > q, \end{cases} \tag{3}$$

and the autocorrelation of x_t at lag k,

$$\rho_x(k) = \frac{\gamma_{x,k}}{\gamma_{x,0}} = \begin{cases} \frac{\sum_{j=|k|}^{q} \psi_j \psi_{j-|k|}}{\sum_{j=0}^{q} \psi_j^2}, & \text{for } |k| = 0, 1, \ldots, q, \\ 0, & \text{for } |k| > q. \end{cases} \tag{4}$$

For a positive integer m, the mth-order temporal aggregate X_T of x_t is defined to be the m-period non-overlapping sum of x_t,

$$X_T = \sum_{t=m(T-1)+1}^{mT} x_t = \left(\sum_{i=0}^{m-1} B^i \right) x_{mT}, \tag{5}$$

for $T = 1, 2, \ldots$. When multiplying both sides of (1) at $t = mT$ by the link polynomial $\sum_{i=0}^{m-1} B^i$, we obtain

$$\left(\sum_{i=0}^{m-1} B^i \right) x_{mT} = \left(\sum_{i=0}^{m-1} B^i \right) \left(\sum_{j=0}^{q} \psi_j B^j \right) e_{mT} \equiv X_T. \tag{6}$$

Thus, the autocovariance generating function of X_T can be written as

$$C_X(B) = \sigma_e^2 \left(\sum_{i=0}^{m-1} B^i\right) \left(\sum_{i=0}^{m-1} B^{-i}\right) \left(\sum_{j=0}^{q} \psi_j B^j\right) \left(\sum_{j=0}^{q} \psi_j B^{-j}\right), \tag{7}$$

which implies that the autocovariance (or autocorrelation) of X_T is closely linked but completely different to the autocovariance (or autocorrelation) of x_t. That is, the autocovariance of X_T loses some of the process information of x_t.

Equation (6) indicates that the aggregate series X_T is also stationary and so the series can follow an MA(Q) model,

$$X_T = \left(\sum_{j=0}^{Q} \Psi_j \mathcal{B}^j\right) E_T \tag{8}$$

with

$$Q \leq \left\lfloor 1 + \frac{q-1}{m} \right\rfloor \tag{9}$$

for fixed m, where $\Psi_0 = 1$, $\{E_T\}$ is another white noise process of mean zero and variance σ_E^2, $\mathcal{B} = B^m$ is the backshift operator such that $\mathcal{B}^j X_T = X_{T-j}$, and $\lfloor x \rfloor$ denotes the largest integer not greater than a real number x (See Tiao, 1972; Brewer, 1973; Weiss, 1984; Stram and Wei, 1986). In the same manner as (3) and (4), the autocovariance and autocorrelation of X_T at lag K are defined to be

$$\gamma_X(K) = \mathrm{E}\left(X_T X_{T-|K|}\right) = \begin{cases} \sigma_E^2 \sum_{j=|K|}^{Q} \Psi_j \Psi_{j-|K|}, & \text{for } |K| = 0, 1, \ldots, Q, \\ 0, & \text{for } |K| > Q, \end{cases} \tag{10}$$

and

$$\rho_X(K) = \frac{\gamma_X(K)}{\gamma_X(0)} = \begin{cases} \frac{\sum_{j=|K|}^{Q} \Psi_j \Psi_{j-|K|}}{\sum_{j=0}^{Q} \Psi_j^2}, & \text{for } |K| = 0, 1, \ldots, Q, \\ 0, & \text{for } |K| > Q, \end{cases} \tag{11}$$

respectively. The exact expressions of $\gamma_X(K)$ and $\rho_X(K)$, in terms of the aggregation order m and the model parameters ψ_j of x_t, are given in Section 2.2 of Tiao (1972).

Neither (10) nor (11) of X_T captures all the process information of x_t. Moreover, Tiao (1972) shows that the aggregate series X_T uniformly approaches a white noise process of uncorrelated random shocks with the autocorrelation at lag K,

$$\rho_X(K) = \begin{cases} 1, & \text{for } K = 0 \\ 0, & \text{for } K \neq 0, \end{cases} \tag{12}$$

if $m \to \infty$ as $n \to \infty$ but $m/n \to 0$. We interpret the limiting property as the complete loss of the process information due to temporal aggregation in a stationary series x_t.

2.2 Spectral Properties of an Aggregate Process

The standardized spectral density function of the aggregate series X_T in (8) is given by

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} \rho_X(K) \cdot e^{-i\lambda K} = \frac{1}{2\pi} \left[1 + 2 \sum_{K=1}^{\infty} \rho_X(K) \cdot \cos(\lambda K)\right] \tag{13}$$

for $|\lambda| \leq \pi$. Since $f_X(\lambda)$ is symmetric about $\lambda = 0$, the standardized spectral distribution function of X_T can be defined as

$$F_X(\lambda) = 2\int_0^{\lambda} f_X(\omega)d\omega = \frac{1}{\pi}\left[\lambda + 2\sum_{K=1}^{\infty}\rho_X(K)\cdot\frac{\sin(\lambda K)}{K}\right] \tag{14}$$

for $0 \leq \lambda \leq \pi$ (Anderson, 1993). We remark that $F_X(\lambda)$ can be regarded as a cumulative distribution function of $\lambda \in [0, \pi]$ because it is monotonically nondecreasing, bounded with $F_X(0) = 0$, and $F_X(\pi) = 1$.

The Fourier transform of the density function $f_X(\lambda)$ equals the autocorrelation function of X_T,

$$\rho_X(K) = \int_{-\pi}^{\pi} f_X(\lambda)\cdot\cos(\lambda K)d\lambda, \tag{15}$$

which implies that the process information captured by $f_X(\lambda)$ is equivalent to the process information captured by $\rho_X(K)$ (For more details, see Durlauf, 1991; Anderson, 1993; Shumway and Stoffer, 2011, pp.180–186). Thus, we can explain the information loss of X_T with characteristic changes in either $f_X(\lambda)$ or $\rho_X(K)$.

Now we consider a special case of $f_X(\lambda)$. As X_T follows an MA(0) model or, equivalently, a white noise process with the autocorrelation function (12), the standardized spectral density function $f_X(\lambda)$ in (13) simplifies to

$$f_X(\lambda) = \frac{1}{2\pi} \tag{16}$$

and the standardized spectral distribution $F_X(\lambda)$ in (14)

$$F_X(\lambda) = \frac{\lambda}{\pi} \tag{17}$$

for $0 \leq \lambda \leq \pi$.

3. The Spectral Kolmogorov-Smirnov Test

Based on the limiting property of X_T shown in (12), we realize that an aggregate series X_T starts to behave like a white noise process and significantly loses the initial process information about x_t when a periodic order m in (5), which is proportional to the magnitude of temporal aggregation, exceeds a certain critical point. Now, we develop a spectral method for detecting the critical point of m where the mth order aggregates of a stationary series turn to white noise.

The problem of interest is to decide whether the standardized spectral density function $f_X(\lambda)$ of X_T is equal to $\frac{1}{2\pi}$ in (13). It can be reworded as testing for the null hypothesis

$$H_0: \ f_X(\lambda) = \frac{1}{2\pi}, \tag{18}$$

against the alternative

$$H_a: \ f_X(\lambda) \neq \frac{1}{2\pi}. \tag{19}$$

We consider the mth aggregate series X_T of a stationary series x_t for $t = 1, \ldots, n$, $T = 1, \ldots, N$, and $N = \lfloor n/m \rfloor$. The standardized sample spectral density function of X_T

is given by

$$\begin{aligned}\hat{f}_X(\lambda) &= \frac{1}{2\pi\hat{\gamma}_X(0)}\left|\frac{1}{\sqrt{N}}\sum_{T=1}^{N} X_T \cdot e^{-i\lambda T}\right|^2 \\ &= \frac{1}{2\pi\hat{\gamma}_X(0)}\sum_{K=-(N-1)}^{N-1}\left(\frac{1}{N}\sum_{T=|K|}^{N} X_T X_{T-|K|}\right)e^{-i\lambda K} \\ &= \frac{1}{2\pi}\sum_{K=-(N-1)}^{N-1}\left(\frac{\hat{\gamma}_X(K)}{\hat{\gamma}_X(0)}\right)e^{-i\lambda k} \\ &= \frac{1}{2\pi}\left(1+2\sum_{K=1}^{N-1}\hat{\rho}_X(K)\cdot\cos(\lambda K)\right),\end{aligned} \tag{20}$$

and the standardized sample spectral distribution function of X_T,

$$\hat{F}_X(\lambda) = 2\int_0^{\lambda}\hat{f}(\omega)d\omega = \frac{1}{\pi}\left(\lambda + 2\sum_{K=1}^{N-1}\hat{\rho}_X(k)\cdot\frac{\sin(\lambda K)}{K}\right), \tag{21}$$

for $0 \le \lambda \le \pi$, where $\hat{\gamma}_X(K) = \frac{1}{N}\sum_{T=|K|}^{N} X_T X_{T-|K|}$ and $\hat{\rho}_X(K) = \hat{\gamma}_X(k)/\hat{\gamma}_X(0)$.

Here, we adopt the framework of the Kolmogorov-Smirnov (KS) test. From (17) and (21), we have

$$\sqrt{N}\left[\hat{F}_X(\lambda) - F_X(\lambda)\right] = \frac{2\sqrt{N}}{\pi}\sum_{K=1}^{N-1}\hat{\rho}_X(K)\cdot\frac{\sin(\lambda K)}{K} \tag{22}$$

under the null hypothesis (18). As discussed in Durlauf (1991) and Anderson (1993), (22) can be treated as a stochastic process on $\lambda \in [0, \pi]$ and, as $N \to \infty$, it converges weakly to a Gaussian process with the autocovariance function

$$\begin{aligned}&4\pi G(\pi)\left[\frac{G(\lambda_1)}{G(\pi)}\left(1-\frac{G(\lambda_2)}{G(\pi)}\right) + \left(\frac{G(\lambda_1)}{G(\pi)} - F(\lambda_1)\right)\left(\frac{G(\lambda_2)}{G(\pi)} - F(\lambda_2)\right)\right] \\ &= 2\left[\frac{\lambda_1}{\pi}\left(1-\frac{\lambda_2}{\pi}\right) + \left(\frac{\lambda_1}{\pi} - \frac{\lambda_1}{\pi}\right)\left(\frac{\lambda_2}{\pi} - \frac{\lambda_2}{\pi}\right)\right] \\ &= \frac{2\lambda_1}{\pi}\left(1-\frac{\lambda_2}{\pi}\right)\end{aligned} \tag{23}$$

for $0 \le \lambda_1 \le \lambda_2 \le \pi$, where $F_X(\lambda_i) = 2\int_0^{\lambda_i} f_X(\omega)d\omega = \frac{\lambda_i}{\pi}$ and $G_X(\lambda_i) = 2\int_0^{\lambda_i} f_X^2(\omega)d\omega = \frac{\lambda_i}{2\pi^2}$. Then, we propose the spectral KS test statistic D_N,

$$\begin{aligned}D_N &= \frac{1}{\sqrt{4\pi G(\pi)}}\sup_{0\le\lambda\le\pi}\left|\sqrt{N}\left[\hat{F}_X(\lambda) - F_X(\lambda)\right]\right| \\ &= \frac{\sqrt{2N}}{\pi}\sup_{0\le\lambda\le\pi}\left|\sum_{K=1}^{N-1}\hat{\rho}_X(K)\cdot\frac{\sin(\lambda K)}{K}\right|.\end{aligned} \tag{24}$$

The asymptotic null distribution of the test statistic D_N in (24) is shown below.

Theorem 1. *The asymptotic null distribution of the spectra KS test statistic D_N is*

$$D_N = \frac{\sqrt{2N}}{\pi}\sup_{0\le\lambda\le\pi}\left|\sum_{K=1}^{N-1}\hat{\rho}_X(K)\cdot\frac{\sin(\lambda K)}{K}\right| \xrightarrow{d} \sup_{0\le\nu\le1}|B(\nu)|. \tag{25}$$

Table 1: Simulated critical values c_α

$(1-\alpha)100\%$	90%	95%	97.5%	99%	99.5%	99.9%
c_α	1.20546	1.33940	1.46212	1.60737	1.71155	1.92921

Proof. Consider the monotonic transformation $\nu = \lambda/\pi$ for $0 \le \lambda \le \pi$. Here, the stochastic process (22) and the autocovariance function (23) are rewritten as

$$\sqrt{N}\left[\hat{F}_X(\pi\nu) - F_X(\pi\nu)\right] = \frac{2\sqrt{N}}{\pi}\sum_{K=1}^{N-1}\hat{\rho}_X(K)\cdot\frac{\sin(\pi\nu K)}{K} \tag{26}$$

for $0 \le \nu \le 1$ and

$$\frac{2\pi\nu_1}{\pi}\left(1-\frac{\pi\nu_2}{\pi}\right) = 2\nu_1(1-\nu_2) \tag{27}$$

for $0 \le \nu_1 \le \nu_2 \le 1$, respectively. Let $B(\nu)$ be the Brownian bridge with $\mathrm{E}\left[B(\nu)\right] = 0$ for $0 \le \nu \le 1$ and $\mathrm{E}\left[B(\nu_1)B(\nu_2)\right] = \nu_1(1-\nu_2)$ for $0 \le \nu_1 \le \nu_2 \le 1$. $B(\nu)$ is Gaussian and sample paths are continuous with probability 1. Then, it can be shown that

$$\sqrt{\frac{N}{2}}\left[\hat{F}_X(\pi\nu) - F_X(\pi\nu)\right] = \frac{\sqrt{2N}}{\pi}\sum_{K=1}^{N-1}\hat{\rho}_X(K)\cdot\frac{\sin(\pi\nu K)}{K} \to B(\nu) \tag{28}$$

in probability. (See Durlauf, 1991; Anderson, 1993; van der Vaart, 1998, pp.277–279; Lehman and Romano, 2005, pp.584–590). Therefore,

$$D_N = \frac{\sqrt{2N}}{\pi}\sup_{0\le\lambda\le\pi}\left|\sum_{K=1}^{N-1}\hat{\rho}_X(K)\cdot\frac{\sin(\lambda K)}{K}\right| \to \sup_{0\le\nu\le1}|B(\nu)|$$

in distribution. □

4. Simulation Study and Application

4.1 Simulation Study

We generate 10^6 different $B(\nu)$ processes, each of size 1001, assuming $\nu = 0, \frac{1}{1000}, \frac{2}{1000}$, …, $\frac{999}{1000}$, and 1. The probability density of their supreme values $\sup_{0\le\nu\le1}|B(\nu)|$ is plotted in Figure 1. Table 1 presents the percentiles $(1-\alpha)100\%$ of the probability density for $\alpha = 0.1$, 0.5, 0.25, 0.01, 0.005, and 0.001. We now use those percentiles as the critical values c_α of the spectral KS test in (24).

4.2 Data Application

The Center for Research in Security Prices of the University of Chicago (CRSP) historical index is a useful benchmark for understanding the US financial markets. We consider the monthly log returns of the CRSP equal-weighted index from January 1926 to December 2008 (Tsay, 2010).

The non-aggregate series of the CRSP log returns with $m = 1$ (monthly) and mth aggregate series with calendrical aggregation orders $m = 2$ (bimonthly), $m = 3$ (quarterly), $m = 6$ (semiannual), $m = 12$ (annual), and $m = 24$ (biennial) are displayed in Figure 2. Through those plots, we observe that the original data of high frequency is transformed into

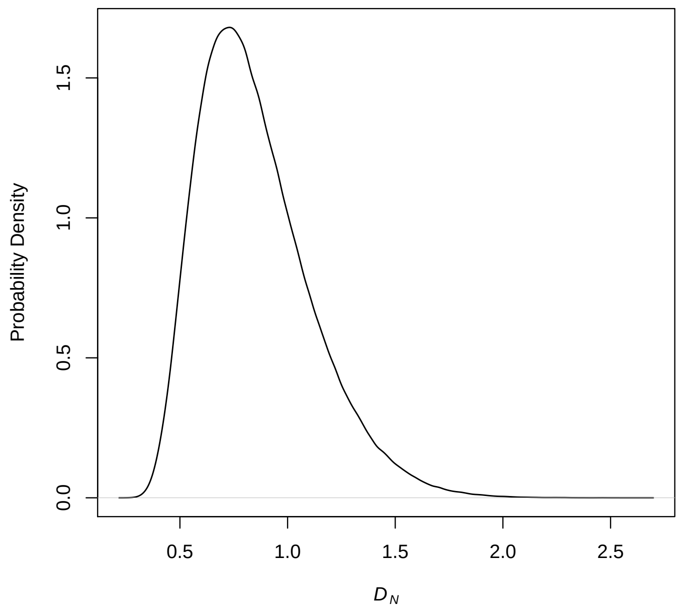

Figure 1: Empirical null distribution for the spectral KS test

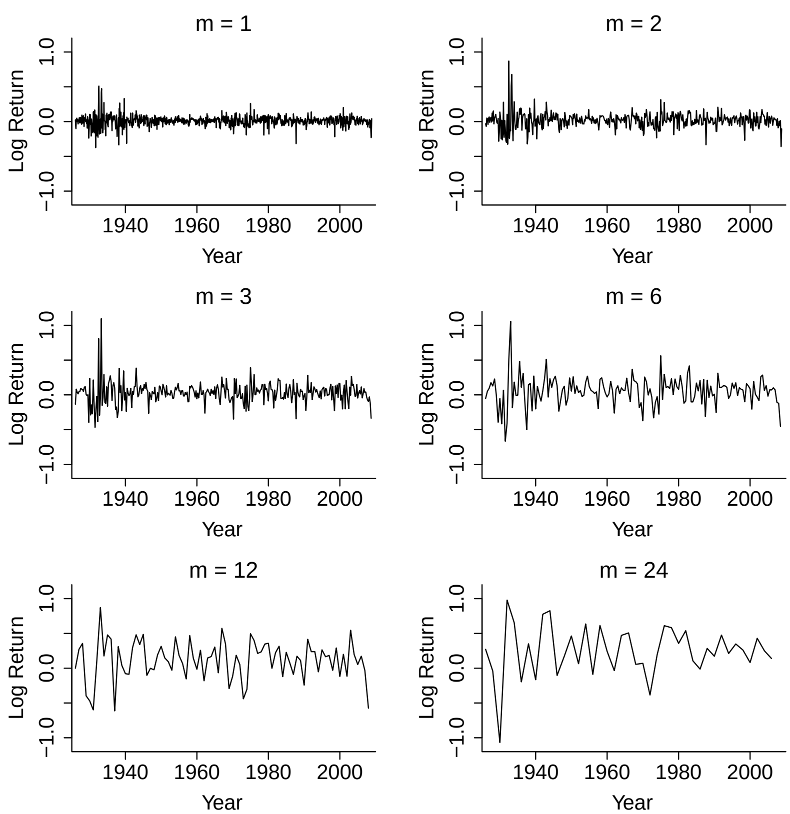

Figure 2: The calendrical aggregations of the CRSP log returns from January 1926 to December 2008

Table 2: The spectral KS tests for the calendrical aggregations of the CRSP log returns

m	D_m
1	3.3581775
2	1.5354085
3	1.3879360
6	0.5773513
12	0.5541316
24	0.4206513

an aggregated series of low frequency. The data frequency of an aggregate series becomes lower as order m increases.

We perform the spectral KS test for the CRSP data and present the test statistic for each m in Table 2. Then, we are confident that the aggregations of $m \geq 6$ show a significant information loss of the original series of $m = 1$ at $\alpha = 0.05$. In other words, we claim that the quarterly series of $m = 3$ is the optional aggregation in term of process information retainment.

5. Concluding Remarks

We propose a spectral-based Kolmogorov-Smirnov test for identifying a significant information loss of the mth aggregation of a stationary series. Through the Monte Carlo simulations in Section 4.1, we find the empirical null distribution and the critical values of the test statistic. Therefore, using the proposed test procedure, we can find a marginal m for a significant information loss to white noise and an optional aggregation in terms of process information retainment.

REFERENCES

Abraham, B. (1982), "Temporal aggregation and time series," *International Statistical Review*, 50, 285–291.

Amemiya, T., and Wu, R. Y. (1972), "The effect of aggregation on prediction in the autoregressive model," *Journal of the American Statistical Association*, 67, 628–632.

Anderson, T. W. (1993), "Goodness of fit tests for spectral distributions,"*Annals of Statistics*, 21, 803–847.

Breitung, J., and Swanson, N. R. (2002), "Temporal aggregation and spurious instantaneous causality in multiple time series models," *Journal of Time Series Analysis*, 23, 651–665.

Brewer, K. R. W. (1973), "Some consequences of temporal aggregation and systematic sampling for ARMA and ARMAX models," *Journal of Econometrics*, 1, 133–154.

Drost, F. C., and Nijman, T. E. (1993), "Temporal aggregation of GARCH processes," *Econometrica*, 61, 909–927.

Durlauf, S. N. (1991), "Spectral based testing of the martingale hypothesis," *Journal of Econometrics*, 50, 355–376.

Granger, C. W. J., and Lee, T. H. (1999), "The effect of aggregation on nonlinearity," *Econometric Reviews*, 18, 259–269.

Hafner, C. M. (2008), "Temporal aggregation of multivariate GARCH processes," *Journal of Econometrics*, 142, 467–483.

Lee, B. H., and Wei, W. W. S. (2017), "The use of temporally aggregated data on detecting a mean change of a time series process," *Communications in Statistics - Theory and Methods*, 46, 5851–5871.

Lehmann, E. L., and Romano, J. P., (2005), *Testing Statistical Hypotheses* (3rd ed.), New York: Springer.

Ltkepohl, H. (1984), "Linear aggregation of vector autoregressive moving average processes," *Economics Letters*, 14, 345–350.

Marcellino, M. (1999), "Some consequences of temporal aggregation in empirical analysis," *Journal of Business and Economic Statistics*, 17, 129–136.

Rossana, R. J., and Seater, J. J. (1995), "Temporal aggregation and economic time series," *Journal of Business & Economic Statistics*, 13, 441–451.

Shumway, R. H. and Stoffer, D. S. (2011), *Time Series Analysis and its Application with R Examples* (3rd ed.), New York: Springer.

Stram, D. O., and Wei, W. W. S. (1986), "Temporal aggregation in the ARIMA process," *Journal of Time Series Analysis*, 7, 279–292.

Teles, P., and Wei, W. W. S. (2000), "The effects of temporal aggregation on tests of linearity of a time series," *Computational Statistics and Data Analysis*, 34, 91–103.

Teles, P., and Wei, W. W. S. (2002), "The use of aggregate time series in testing for Gaussianity," *Journal of Time Series Analysis*, 23, 95–116.

Teles, P., Wei, W. W. S, and Hodgess, E. N. (2008), "Testing a unit root based on aggregate time series," *Communications in Statistics - Theory and Methods*, 37, 565–590.

Tiao, G. C. (1972), "Asymptotic behaviour of temporal aggregates of time series," *Biometrika*, 59, 525–531.

Tiao, G. C., and Wei, W. W. S. (1976), Effect of temporal aggregation on the dynamic relationship of two time series variables. *Biometrika*, 63, 513–523.

Tsay, R. S. (2010), *Analysis of Financial Time Series* (3rd ed.), Hoboken, NJ: John Wiley & Sons.

Vaart, A. W. van der (1998), *Asymptotic Statistics*, New York: Cambridge University Press.

Wei, W. W. S. (1978), "Some consequences of temporal aggregation in seasonal time series models," In A. Zellner, ed. *Seasonal Analysis of Economic Time Series* (pp. 433–444), Washington, DC: U.S. Department of Commerce, Bureau of the Census.

Wei, W. W. S. (2006), *Time Series Analysis: Univariate and Multivariate Methods* (2nd ed.), Boston: Addison Wesley.

Weiss, A. A. (1984), "Systematic sampling and temporal aggregation in time series models," *Journal of Econometrics*, 26, 271–281.