Xbar Chart with Estimated Parameters: New Formulas to Guarantee a Conditional In-Control Performance

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Abstract
Recently, to design control charts with estimated parameters, a new perspective has emerged where the usual chart performance measure, the average run length (ARL), is recognized as a random variable that depends on the parameter estimates. In this context, a recent idea is to measure the chart performance with the probability of the ARL be greater than a specified value. This is called the exceedance probability criterion (EPC). In recent years, bootstrap method and approximate formulas were proposed to adjust the Xbar chart under normality to guarantee an in-control performance in terms of the EPC. These methods provide accurate, but not exact, results. Given this, in this paper, we present a summary of two recent papers written by us, under review, where we propose the use of an equation in which exact adjustments can be quickly calculated with a software such as RStudios. Furthermore, realizing the complexity of all these methods for a regular user, in these papers, we also derive a new simpler approximate formula which depends only on the well-known chi-square distribution. We show that this new simpler formula provides accurate results compared to the already existing ones.

Key Words: Xbar Control Chart Performance, Conditional Performance, Exceedance Probability Criterion, Control Limits Adjustments, Guaranteed In-Control Performance, Average Run Length

1. Introduction

The Xbar (or $\bar{X}$) Control Chart is still one of the most used tools for monitoring the mean of quality characteristics of processes in many industries. To design this chart, its parameters (the in-control process mean and the in-control process standard deviation) must be estimated from m reference (historical) samples each of size n when process is in control. This is called Phase I. For a detailed literature review on Phase I, see Chakraborti et al. (2009) and Jones-Farmer et al. (2014).

It is well-known that when parameters are estimated, the performance of the $\bar{X}$ Control Chart is severely affected. For example, its main performance measure, the in-control average run length (ARL_0) will be conditioned on the estimated parameters (and, because of this, it is often called $CARL_0$). The $CARL_0$ varies depending on the estimate's values. This is often called the practitioner-to-practitioner variability [see, Saleh et al. (2015a,b)], because each practitioner will estimate the chart´s parameters only once and these estimates will be different from another practitioner.

Saleh et al. (2015a), Jardim et al. (2018a) and others showed that when the amount of Phase I data used to calculate the 3-Sigma limits of the $\bar{X}$ chart is small (such as 25 samples of size 5), the variability of the $CARL_0$ is large, meaning that the $CARL_0$ may be much different from the 370.4 (which is the ARL_0 value in the "known parameters case" for the $\bar{X}$ Chart with the 3-Sigma Limits). The variability of the $CARL_0$ results in a small probability of the $CARL_0$ being smaller than the 370.4 (or a value close to 370.4).

In this context, a recent idea is to adjust the $\bar{X}$ Control Chart in order to have a large probability of the $CARL_0$ being greater than a specified value (equal or close to the nominal when parameters are known). This is called the exceedance probability criterion (EPC), proposed by Albers et al. (2005). To this end, Saleh et al. (2015b) proposed EPC adjustments using the bootstrap approach of Gandy and Kvaloy (2013) and Goedhart et al. (2017 and 2018) derived approximate formulas for the EPC adjusted limits based on the Taylor Series approximation.

In this note, we present a summary of two recent papers written by us, one accepted in the *Journal of Quality Technology* and another one under second review in the Production and Operation Management journal). In one paper [henceforth Jardim et al (2018b)], we propose the use of an equation in which exact EPC adjustments can be quickly calculated with a software such as RStudios. In the second paper [henceforth Jardim et al (2018a)], realizing the complexity of all the methods presented in the literature for a regular user, we derive a new simpler approximate formula which depends only on the well-known central chi-square distribution. We show that this new simpler formula provides accurate results compared to the already existing ones.

Before proceeding, it is important to note that when parameters are estimated, or more precisely regarding the $\bar{X}$ Control Chart, when the in-control process mean μ_0 and the in-control process standard deviation σ_0 are estimated by $\hat{\mu}_0$ and $\hat{\sigma}_0$, all these authors recommend to calculated the Upper Control Limits ($\widehat{UCL}$) and the Lower Control Limits ($\widehat{LCL}$) as

$$\widehat{UCL} = \hat{\mu}_0 + L^* \frac{\hat{\sigma}_0}{\sqrt{n}}, \tag{1}$$

$$\widehat{LCL} = \hat{\mu}_0 - L^* \frac{\hat{\sigma}_0}{\sqrt{n}}. \tag{2}$$

Where L^* is the corrected limited factor (or the adjusted limited factor) that must be found to achieve an EPC performance. μ_0 is estimated by the well-established estimator for the mean: the sample grand mean ($\bar{\bar{X}}$) defined as

$$\hat{\mu}_0 = \bar{\bar{X}} = \frac{1}{m} \sum_{i=1}^{m} \bar{X}_i, \tag{3}$$

where $\bar{X}_i = \frac{1}{n} \sum_{j=1}^{n} X_{ij}$, $i = 1,2, \dots, m$, $j = 1,2, \dots, n$ and X_{ij} denotes the j-th observation of the i-the Phase I sample. For σ_0, we choose the highly recommended unbiased pooled sample standard deviation estimator ($S_p/c_{4,b}$), see Mahmoud et al. (2010), which is given by

$$\hat{\sigma}_0 = \frac{S_p}{c_{4,b}} = \frac{\sqrt{\frac{1}{m}\sum_{i=1}^{m} S_i^2}}{c_{4,b}}, \tag{4}$$

where $S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{i,j} - \bar{X}_i\right)^2$ and $c_{4,b}$ is the unbiasing constant for $b = m(n-1) + 1$, where $c_{4,b} = \Gamma(b)/\Gamma((b-1)/2)$ and Γ is the gamma function. A good approximation for $c_{4,b}$ can be obtained by $c_{4,b} \approx 4(b-1)/(4b-3)$ [see Montgomery (2009)].

2. Control Limits Adjustments based on the EPC

As noted before, the $CARL_0$ is a random variable when the process parameters are estimated. The distribution of the $CARL_0$ has a large variance when the amount of data used to estimate the parameters is small to moderate, such as $m = 25$ and $n = 5$ [see Saleh et al. (2015a) and Jardim el al. (2018a)]. According to the EPC adjustments, the focus is on limiting the risk that the $CARL_0$ value be smaller than a specified tolerated value, in other words, the idea is to ensure that the $CARL_0$ is at least 370 (or perhaps a value slightly smaller), with a high probability. Formally, the EPC adjustments consists in finding the L^* factor so that the

$$P\left(CARL_0 \geq \frac{1}{(1+\varepsilon)}(1/\,\alpha)\right) = 1 - p, \tag{5}$$

for a small value p (such as 0.05), where α is the nominal false alarm rate specified by the chart user ($\alpha = 2\big(1 - \Phi(L)\big)$), which is the false alarm rate in the known parameters case when the unadjusted limit factor (L) is used. The quantity ε is called the tolerance factor (meaning that we are willing to tolerate a $CARL_0$ that is at least $100\left(\frac{\varepsilon}{1+\varepsilon}\right)$% smaller than the nominal $1/\alpha$ with a high probability (that is with a small specified p). This is the so-called "Exceedance Probability" criterion [see Albers et al. (2005)].

From Equation (5), it is clear that the exceedance probability depends on the c.d.f of the random variable $CARL_0$. Taking $\hat{\mu}_0 = \bar{\bar{X}}$ and $\hat{\sigma}_0 = S_p/c_{4,b}$ presented in the Introduction, we in Jardim et al. (2018b), have shown that the exact c.d.f. of the $CARL_0$ is given by

$$P(CARL_0 \leq t) = \int_{-\infty}^{\infty} F_{\chi^2_{m(n-1)}}\left(\frac{m(n-1)F^{-1}_{\chi^2_{1,}\left[\frac{z^2}{m}\right]}\left(1-\frac{1}{t}\right)}{\left(\frac{L^*}{C_{4,b}}\right)^2}\right)\phi(z)dz\,, \qquad t \geq 1 \tag{6}$$

where $F^{-1}_{\chi^2_{1,}\left[\frac{z^2}{m}\right]}(1-t)$ denotes the $(1-t)$-quantile of the distribution of a non-central chi-squared random variable with 1 d.f. and non-centrality parameter $\frac{z^2}{m}$ and $F_{\chi^2_{m(n-1)}}$ is the c.d.f. of a central chi-square random variable with $m(n-1)$ d.f. So, using (5), (6) and substituting $[(1+\varepsilon)\alpha]^{-1}$ for t, the adjusted control limit factor (L^*) can be obtained by solving the following equation for L^* for a given value of α, m, n, ε and p.

$$\int_{-\infty}^{\infty} F_{\chi^2_{m(n-1)}}\left(\frac{m(n-1)F^{-1}_{\chi^2_{1,\left[\frac{z^2}{m}\right]}}(1-(1+\varepsilon)\alpha)}{\left(\frac{L^*}{C_{4,b}}\right)^2}\right)\phi(z)dz = p. \tag{7}$$

This solution is denoted L^*_{CE} and can be found with a software like RStudios. Note that, Equation (7) is exact, but the solution must be found using a computer, so some have referred to this method as the "numerical method".

Goedhart et al. (2017) derived an approximated formula for L^* using a two-step Taylor approximation. Here, their approximate EPC formula for L^* is denoted by L^*_{CA1} and is given by

$$L^*_{CA1} = L + \frac{\Phi^{-1}(1-p) - g(L)}{g'(L)}. \tag{8}$$

Where $g(L)$ and $g'(L)$ are functions of the expectation and the variance of the conditional false alarm rate, $CFAR$ (which is the reciprocal of the $CARL_0$). The expressions of $g(L)$ and $g'(L)$ are presented in Appendix A. From (7), it is possible to derive an alternative and simpler approximate formula for $L^* = L^*_{CA2}$ which is given by

$$L^*_{CA2} = C_{4,b}\sqrt{m(n-1)\frac{F^{-1}_{\chi^2_{1,\left[\frac{1}{m}\right]}}(1-(1+\varepsilon)\alpha)}{F^{-1}_{\chi^2_{m(n-1)}}(p)}}, \tag{9}$$

where $F^{-1}_{\chi^2_{m(n-1)}}(p)$ denotes the p-quantile of a central chi-square distribution with $m(n-1)$ degrees of freedom and $F^{-1}_{\chi^2_{1,\left[\frac{1}{m}\right]}}(1-(1+\varepsilon)\alpha)$ denotes the $(1-(1+\varepsilon)\alpha)$-quantile of a non-central chi-square distribution with 1 degree of freedom and non-centrality parameter $\frac{1}{m}$. Formula (9) is given by Goedhart et al. (2018). Note that L^*_{CA2} requires the information of a non-central chi-square distribution, which is not tabulated in many text books in Statistics and not available in popular software such as Excel, so its calculation will require a relatively advanced statistical skills of the practitioner. Given this, we in Jardim et al. (2018a) derived the following even simpler approximate formula for L^* (denoted here by L^*_{CA3}).

$$L^*_{CA3} = C_{4,b}\sqrt{(n-1)(m+1)\frac{F^{-1}_{\chi^2_1}(1-(1+\varepsilon)\alpha)}{F^{-1}_{\chi^2_{m(n-1)}}(p)}} \tag{10}$$

Note that there is no non-centrality parameter in (10), since $F^{-1}_{\chi^2_1}(1-(1+\varepsilon)\alpha)$ is the $(1-(1+\varepsilon)\alpha)$-quantile of a central chi-square distribution with 1 degree of freedom.

Finally, Saleh et al. (2015b) suggested finding L^* using the parametric bootstrap simulation approach of Gandy and Kvaløy (2013). In order to do this, the users, with the help of a software (Like SAS, RStudios, etc.), should generate B bootstrap estimates of the in-control process mean and standard deviation (μ_k^* , σ_k^*), $k = 1,2,\dots,B$, with $\mu_k^* \sim N\left(\bar{\bar{X}}, S_p^2/nmc_{4,b}^2\right)$ and $\sigma_k^* \sim \sqrt{S_p^2 \frac{\chi_v^2}{vc_{4,b}^2}}$ with $v = m(n-1)$. Considering that $\bar{\bar{X}}$ and $S_p/c_{4,b}$ are respectively the real in-control process mean and standard deviation and μ_k^* and σ_k^* are respectively the estimator of $\bar{\bar{X}}$ and $S_p/c_{4,b}$ (following the bootstrap method), for each μ_k^* and σ_k^{*2}, the user has to find the value of L_k^* using the following equation:

$$L_k^* = \frac{S_p \sqrt{F^{-1}_{\chi^2_{1,\left[\frac{\mu_k^* - \bar{\bar{X}}}{S_p/c_{4,b}}\sqrt{n}\right]^2}}(1-(1+\varepsilon)\alpha)}}{C_{4,b}\sigma_k^*}, \quad k = 1,2,\dots,B \tag{11}$$

where $F^{-1}_{\chi^2_{1,\left[\frac{\mu_k^* - \bar{\bar{X}}}{S_p/c_{4,b}}\sqrt{n}\right]^2}}(1-(1+\varepsilon)\alpha)$ denotes the $(1-(1+\varepsilon)\alpha)$ th quantile of the distribution of a non-central qui-square random variable with 1 degree of freedom and non-centrality parameter given by $\left(\frac{\mu_k^* - \bar{\bar{X}}}{S_p/c_{4,b}}\sqrt{n}\right)^2$. Finally, the required $L^* = L_{boot}^*$ is the $(1-p)$th quantile of the collection of bootstrap estimators $(L_1^*, L_2^*, \dots, L_B^*)$.

3. Results and Discussion

For brevity, we will only present the results for $\alpha = 0.0027$ (the value that, in the parameters-known case and under the assumption of data normality, corresponds to a ARL_0 of 370.4), $n = 5$, $\varepsilon = 0\%$, $p = 5\%$ (i.e., the following desired exceedance probabilities: $P(CARL_0 \geq 370.4(1+\varepsilon)^{-1}) = 1-p$, or equivalently $P(CARL_0 \geq 370.4) = 95\%$). Also, we considered the estimator presented in the Introduction and the following values of m: $m = 25,50,75,100,150,200,250$. Results for different values of α, ε, p, m and n can be easily generated using the equations and methods presented in this note.

Table 1 - Values of L^* and the resulting values of $P(CARL_0 \geq 370.4(1+\varepsilon)^{-1})$ for $n = 5$, $\alpha = 0.0027$, $\varepsilon = 0\%$, $p = 5\%$

	Unadjusted Limits (3-Sigma Limits)		Bootstrap [Saleh et al. (2015b)]		Approximation 1 [Goedhart et al. (2017)]		Approximation 2 [Goedhart et al. (2018)]		Our Approximation [Jardim et al. (2018a)]		Exact Equations [Jardim et al. (2018b)]	
m	L	$P(CARL_0 \geq 370.4)$	L^*_{boot}	$P(CARL_0 \geq 370.4)$	L^*_{CA1}	$P(CARL_0 \geq 370.4)$	L^*_{CA2}	$P(CARL_0 \geq 370.4)$	L^*_{CA3}	$P(CARL_0 \geq 370.4)$	L^*_{CE}	$P(CARL_0 \geq 370.4)$
13	3.00	38.23%	3.73	94.75%	3.70	94.56%	3.69	94.36%	3.70	94.59%	3.72	95.00%
15	3.00	38.74%	3.69	95.43%	3.65	94.80%	3.63	94.36%	3.64	94.55%	3.65	95.00%
20	3.00	39.74%	3.56	95.10%	3.54	95.06%	3.52	94.38%	3.53	94.51%	3.54	95.00%
25	3.00	40.50%	3.49	95.60%	3.47	95.15%	3.45	94.41%	3.46	94.51%	3.47	95.00%
50	3.00	42.69%	3.32	95.56%	3.31	95.11%	3.30	94.57%	3.30	94.61%	3.31	95.00%
75	3.00	43.82%	3.24	94.98%	3.24	95.03%	3.24	94.67%	3.24	94.69%	3.24	95.00%
100	3.00	44.54%	3.20	94.66%	3.20	94.99%	3.20	94.73%	3.20	94.75%	3.20	95.00%
150	3.00	45.45%	3.16	94.74%	3.16	94.94%	3.16	94.81%	3.16	94.81%	3.16	95.00%
200	3.00	46.02%	3.13	94.25%	3.14	94.92%	3.14	94.85%	3.14	94.85%	3.14	95.00%
250	3.00	46.41%	3.12	95.42%	3.12	94.91%	3.12	94.88%	3.12	94.88%	3.12	95.00%

Table 1 shows that the five EPC adjustment methods generate extremely similar results, i.e., for all values of L^*_{boot} , L^*_{CA1}, L^*_{CA2}, L^*_{CA3} and L^*_{CE}, the $P(CARL_0 \geq 370.4)$ is very close to 95%, being the method proposed by Jardim et al. (2018b) the most precise one.

As noted in the Introduction, for the unadjusted 3-Sigma Limits ($L = 3$), the probability of the $CARL_0$ be larger than 370.4 is very small (see the first column in orange in Table 1), meaning that chances are higher that the average number of Phase 1 samples until a false alarm will be smaller than the target 370.4 with unadjusted limits.

The bootstrap method proposed by Saleh et al. (2015b) has the disadvantage of generating different values of L^*_{boot} every time the bootstrap runs. As can be checked in Appendix A, the approximate method 1 of Goedhart et al. (2017) requires several numerical integrals and derivative, which makes this method the most complex one. The second approximation proposed by Goedhart et al. (2018) is much simpler than the first, but it still requires the computation of non-central chi-squared distribution which is not presented in some software, such the Excel, and it is not tabulated in the major statistical books. The method proposed by Jardim et al. (2018b) has the advantage of providing extremely accurate results but it still requires the computation of the quantile of a non-central chi-squared distribution, the computation of a numerical integral and a search algorithm. Because of this, we indicate L^*_{CE} to be implemented in Statical Quality Conbtrol Softwares. Finally, the simplest method which also provide good results is the approximate formula of Jardim et al. (2018a) which is only a function of central chi-squared distribution presented in many software and tabulated in all statistical books.

6. Conclusions

In the present note, we summarized two recent papers written by us which propose adjustments to the $\bar{X}$ control chart limits with estimated parameters. These adjustments guarantee an in-control performance in terms of what is known as the Exceedance Probability Criterion (EPC), which basically guarantee that the in-control average run length conditioned on the estimated parameters will be greater than a nominal/target value with a large probability.

We compare the EPC adjustment methods proposed in our two papers with other methods presented in the literature. Our conclusion is that all the EPC adjustment methods generate very similar results. We consider the Approximate Method derived in one of our papers [Jardim et al. (2018a)], the simplest method, because it just depends on the quantiles of a central chi-square distribution which is tabulated in all statistical text books and presented in most basic software (such as Excel). All the other adjustment methods will require a more advanced skill in statistics, like the calculation of the quantile of a non-central chi-squared distribution presented in the Approximate Method 2, given by Goedhart et al. (2018), or numerical integrals presented in the Exact Method given by us in Jardim et al. (2018b) and in the Approximate Method 1 given by Goedhart et al. (2017). However, we think that our exact solution in Jardim et al. (2018b) is indicated to be incorporated in Statistical Quality Control Software which generate the control limits automatically, since this method generate extremely precise results without requiring much computational time.

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Appendix A – Expressions of $g(L)$ and $g'(L)$ on Equation (8)

Here we present the expressions of $g(L)$ and$g'(L)$ to calculate the approximation showed in (8).

$g(L) = 3\sqrt[3]{(1+\varepsilon)\alpha}\,\frac{f_E(L)^{2/3}}{f_V(L)^{1/2}} - 3\,\frac{f_E(L)}{f_V(L)^{1/2}} + \frac{1}{3}\frac{f_V(L)^{1/2}}{f_E(L)}$ and

$g'(L) = 3\sqrt[3]{(1+\varepsilon)\alpha}B - 3C + \frac{1}{3}D$ with $C = \frac{f_E'(L)f_V(L)^{1/2} - f_E(L)\frac{1}{2}f_V(L)^{-1/2}f_V'(L)}{f_V(L)}$,

$B = \frac{\frac{2}{3}f_E(L)^{-1/3}f_E'(L)f_V(L)^{1/2} - f_E(L)^{2/3}\frac{1}{2}f_V(L)^{-1/2}f_V'(L)}{f_V(L)}$, $D = \frac{\frac{1}{2}f_V(L)^{-1/2}f_V'(L)f_E(L) - f_V(L)^{1/2}f_E'(L)}{f_E(L)^2}$.

Finally, $f_E(L) = E(CFAR)$, $f_V(L) = V(CFAR)$, $f'_E(L) = \frac{dE(CFAR)}{dL}$ and $f'_V(L) = \frac{dV(CFAR)}{dL}$.

Following below, we present the expressions of $E(CFAR)$, $V(CFAR)$, $f'_E(L)$ and $f'_V(L)$.

Since $CFAR$ is expressed by

$$CFAR = 1 - \left(\Phi\left(\frac{Z}{\sqrt{m}} + \frac{L^*}{c_{4,b}}\sqrt{\frac{Y}{m(n-1)}}\right) - \Phi\left(\frac{Z}{\sqrt{m}} - \frac{L^*}{c_{4,b}}\sqrt{\frac{Y}{m(n-1)}}\right)\right),$$

where $Y = m(n-1)\, S_p^2/\sigma_0^2$ follows a central chi-square distribution with $m(n-1)$ and $Z = \left(\frac{\bar{\bar{X}}-\mu_0}{\sigma_0}\right)\sqrt{mn}$ follows a standard normal distribution, the $E(CFAR)$ can be calculated by

$$E(CFAR) = \int_{-\infty}^{\infty}\int_{0}^{\infty} (CFAR)\,\phi(z) f_Y(y)\,dy\,dz.$$

$V(CFAR)$ is given by $V(CFAR) = E(CFAR^2) - E^2(CFAR)$, where

$$E(CFAR^2) = \int_{-\infty}^{\infty}\int_{0}^{\infty} (CFAR)^2\,\phi(z) f_Y(y)\,dy\,dz.$$

Since $f'_E(L) = \frac{dE(CFAR)}{dL}$, one has:

$$f'_E(L) = \int_{-\infty}^{\infty}\int_{0}^{\infty} -\frac{1}{C_{4,b}}\sqrt{\frac{Y}{m(n-1)}}\,G\,\phi(z) f_Y(y)\,dy\,dz,$$

where $G = \phi\left(\frac{Z}{\sqrt{m}} + \frac{L}{C_{4,b}}\sqrt{\frac{Y}{m(n-1)}}\right) + \phi\left(\frac{Z}{\sqrt{m}} - \frac{L}{C_{4,b}}\sqrt{\frac{Y}{m(n-1)}}\right)$.

Finally,

$$f'_V(L) = \frac{dV(CFAR)}{dL} = \frac{dE(CFAR^2)}{dL} - 2\,E(CFAR) f'_E(L),$$

where

$$\frac{dE(CFAR^2)}{dL} = \int_{-\infty}^{\infty}\int_{0}^{\infty} 2CFAR\left(-\frac{1}{C_{4,b}}\sqrt{\frac{Y}{m(n-1)}}\,G\right)\phi(z) f_Y(y)\,dy\,dz.$$