Adjustment for Phase II S^2 Chart Control Limits Based on Tolerance Intervals

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Abstract
Since some works in SPC have revealed that a large amount of Phase I data is required to attain an appropriate specified Phase II performance of control charts, some authors have proposed to adjust the control limits in order to achieve the conditional (given the parameter estimates) in-control performance for a given amount of Phase I data. The use of statistical tolerance limits as adjusted control limits is considered in the recent literature on $\bar{X}$ charts based on the conditional performance perspective, e.g., corrected control limits are obtained for guaranteeing a tolerated conditional false alarm rate ($CFAR$) or, equivalently, the conditional in-control average run length, with a specified high probability. In this paper, assuming normality, we propose the required adjustments for the two-sided S^2 chart control limits to ensure a specified $CFAR$, where the adjusted control limits are based on an equal-tailed tolerance interval for sample variances. This ensures that the interval between the adjusted lower and upper limits contain at least a specified proportion of the center of the S^2 distribution with a certain (specified) high confidence.

Key Words: S^2 control charts, adjusted control limits, estimated parameters, conditional control chart performance, tolerance interval, phase I

1. Introduction

The S^2 control chart is one of the most extensively used tool to monitor the process variability in manufacturing and service processes. The in-control (IC) process variance σ_0^2 is usually estimated from a reference sample consisting of m independent samples or subgroups of size n in what is called the Phase I analysis (for overviews of Phase I, see Chakraborti et al., 2009, and Jones-Farmer et al., 2014). The control limits of the S^2 chart are constructed on the basis of the estimate of the process variance to be used in prospective process monitoring (called Phase II), where samples (also of size n) are collected over a certain period. Each Phase II sample is used to compute a corresponding sample variance that is compared with the control limits in order to detect an out-of-control state. When the IC process parameters are unknown and are estimated in Phase I, the Phase II chart performance may be extremely different from the nominal (unrealistic parameter-known case in which no Phase I is needed). The review of several studies that evaluate the effect of estimation error on control chart performance is presented, e.g., in Jensen et al. (2006) and Psarakis et al. (2014).

To study the performance and design of control charts when process parameters are estimated, most researchers focused on the unconditional run length (RL) distribution of the chart, that is, on the marginal distribution of the number of plotting statistics until a signal, or, equivalently, until a variance sample falls outside the S^2 control limits during Phase II. It is known as the *unconditional perspective*. One of the most customary chart performance measure is the average of such unconditional RL distribution, i.e., the so-called unconditional average run length (ARL). The ARL is impacted by estimation error and then it becomes different from the value of the counterpart known parameter case [see, for the case of S^2 control chart, Chen (1998), Maravelakis et al. (2002) and Castagliola et al. (2009)]. The unconditional RL distribution of the S^2 chart can be obtained by averaging over the distribution of the σ_0^2 estimator. A performance measure under the unconditional perspective (an associated measure of the unconditional RL distribution) is an "average" performance over a too much "large" number of control charts, which are constructed with the same "large" number of parameter estimates, rather than a performance of a specific control chart. The difference of these parameter estimates, which are based on the different reference samples arising from the same Phase I IC process, is called by Saleh et al. (2015) for $\bar{X}$ and X charts the practitioner-to-practitioner variability. However, since there is only one reference sample in practical applications, the "real" RL distribution (and its various attributes) must be always conditioned on only one of these parameter estimates obtained from a specific Phase I reference sample. This is the rising perspective called the *conditional perspective* that considers the practitioner-to-practitioner variability and has been advocated in recent years (see, for the case of S^2 and S charts, Epprecht et al. (2015), Faraz et al. (2015, 2017) and Goedhart et al. (2017)). These authors emphasized the practical importance of focusing on the distribution of conditional performance measures (such as the conditional false-alarm rate $CFAR$ or the conditional in-control average run length $CARL_0$) and some of their properties, (such as the standard deviation of the $CARL_0$ or some extreme quantiles of the $CFAR$) rather than on unconditional performance measures when analyzing the effect of Phase I estimation on control charts performance.

Researches on the performance of the S^2 control chart when σ_0^2 is estimated have concluded that the amount of Phase I reference data must be larger than the amount of recommended data in many books and manuals for achieving similar performance of charts with known process parameter. For instance, Montgomery (2012) suggested number of subgroups $m = 20$ to 30 and sample size $n = 4$ or 5. This finding in the chart performance has been revealed based on the *unconditional perspective* (e.g., Chen (1998), Maravelakis et al. (2002) and Castagliola et al. (2009)) as well as the *conditional perspective* (see, e.g., Epprecht el al. (2015)). Since Epprecht studied only the effect of the amount of Phase I data on the Phase II performance of the one-sided upper S^2 control chart, in this work, we analyze this subject for the case of the two-sided S^2 control chart.

Because the Phase II performance measure of charts with estimated parameter is much less predictable than those with known parameter and the fact that a large amount of Phase I data is required to attain a desired performance, which can be difficult to fulfill in practice, some authors have proposed to adjust the control limits (or, specifically, the chart factors). This adjustment can allow us to achieve either an unconditional performance close to the known parameter case or a specified tolerated conditional IC performance handling a practical amount of Phase I data.

We assume normality of the underlying (process) data, and consider the charts with probability and equal-tailed limits (established for a specified false-alarm rate) instead of "three-sigma" limits, as it has been suggested and justified by Epprecht et al. (2015), Woodall (2017) and Diko et al. (2017). We consider that the Phase I (multi-sample) estimator used for the IC process variance is the pooled variance (S_p^2).

The remainder of this paper is organized as follows: Section 2 presents the relation between the performance of the two-sided S^2 control chart with estimated parameter and the two-sided tolerance intervals for S^2. This section also provided the minimum amounts of Phase I data that guarantee a desired conditional IC performance of the two-sided S^2 chart. In Section 3, the adjustments of the two-sided S^2 chart under the *conditional* perspective are examined. Some conclusions and a summary are given in Section 4.

2. Performance of the S^2 control chart with estimated parameter and two-sided tolerance intervals for $\mathbf{S}^2$

The upper and lower control limits of the two-sided S^2 control chart ($\widehat{UCL}$ and $\widehat{LCL}$, respectively) are given by

$$\widehat{UCL} = \frac{\chi^2_{n-1,1-\alpha^*/2}}{(n-1)}\hat{\sigma}_0^2 = U^*\hat{\sigma}_0^2, \tag{1}$$

$$\widehat{LCL} = \frac{\chi^2_{n-1,\alpha^*/2}}{(n-1)}\hat{\sigma}_0^2 = L^*\hat{\sigma}_0^2, \tag{2}$$

where n is the sample size in Phases I and II, $\hat{\sigma}_0^2$ is the Phase I estimator of the IC process variance (σ_0^2) and $\chi^2_{n-1,q}$ denotes the q-quantile of the distribution of a chi-square variable with $n-1$ degrees of freedom (df). In addition, (L^*,U^*) are the adjusted two-sided lower and upper control limit factors. These adjusted factors depend on the value of the adjusted chart parameter α^* that will be found according to a specified nominal IC chart performance, given the amount of Phase I data (m random samples (subgroups) each of size n) to estimate σ_0^2 and its chosen estimator ($\hat{\sigma}_0^2$). Thus, the computation of the resulting adjusted limits factors takes into account the parameter estimation and the practitioner-to-practitioner variability. On the other hand, when the control limits (Equations (1)-(2)) are set based on the corresponding traditional (unadjusted) control limits factors, α^* is pre-specified and equals the nominal false alarm rate α (frequently, $\alpha^* = \alpha = 0.0027$), and then these factors do not depend on the amount of Phase I data. In the case of adjusted factors, α^* converges to the nominal value α as the amount of Phase I data (mn) increases.

When the IC process variance (σ_0^2) is estimated, the realized values of the Phase II control limits and, therefore, the actual Phase II chart performance depend on the realized value of the estimator $\hat{\sigma}_0^2$ (that is, the particular estimate). To calculate $\hat{\sigma}_0^2$, the user must choose an estimator. In this paper, we consider the unbiased pooled sample variance (S_p^2) to estimate σ_0^2, that is, $S_p^2 = \frac{1}{m}\sum_{i=1}^m S_i^2$, where $S_i^2 = \frac{1}{n-1}\sum_{j=1}^n \left(X_{i,j} - \bar{X}_i\right)^2$ is the i-*th* variance sample, $\bar{X}_i = \frac{1}{n}\sum_{j=1}^n X_{ij}$ is the i-*th* mean sample ($i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$) and X_{ij} is the j-*th* observation of the i-*th* sample in Phase I. X_{ij} is considered normally distributed with mean μ_0 and variance σ_0^2.

2.1 The conditional average run lengths

The probability of a signal is an important property of the Phase II performance of S^2 control chart. The conditional probability of a signal (CPS) is the probability that each l-th sample variance ($l = m+1, m+2, \ldots$), of size n in the monitoring process (Phase II), falls outside the estimated (given the particular estimate $\hat{\sigma}_0^2$) control limits range given by Equations (1)-(2). Thus, the conditional (given $\hat{\sigma}_0^2$) RL distribution [i.e., the distribution of the number of Phase II sample-variance estimates (S_l^2) until a signal] is geometric with probability of success equal to CPS.

To express the CPS in mathematical terms, let's define the variance of the Phase II analysis (process monitoring) as σ^2. Thus, according to Equations (1)-(2), with the estimator S_p^2 used in place of $\hat{\sigma}_0^2$ and considering the fact that $S_l^2(n-1)/\sigma^2$ follows a chi-square distribution with $n-1$ df, the CPS_{two}, for the two-sided S^2 control chart is given by

$$CPS = 1 - P\left(\widehat{LCL} \le S_l^2 \le \widehat{UCL}\right) =$$

$$1 - \left(F_{\chi^2_{n-1}}\left(\frac{S_p^2}{\sigma_0^2}\frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,1-\alpha^*/2}\right) - F_{\chi^2_{n-1}}\left(\frac{S_p^2}{\sigma_0^2}\frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,\alpha^*/2}\right)\right), \quad (3)$$

where $F_{\chi^2_{n-1}}$ denotes the cumulative distribution function (cdf) of a chi-square random variable with $n-1$ df. Next, we represent the change in the process variance by the variance ratio $\gamma^2 = \sigma^2/\sigma_0^2$, i.e., the ratio between the process variances in Phase II and Phase I. Also, note that $Y = m(n-1)\,S_p^2/\sigma_0^2$ follows a chi-square distribution with $m(n-1)$ df. Thus, given a value of the variance ratio (γ^2), the CPS is a function of the random variable Y. Since the conditional RL follows a geometric distribution, the conditional average run length ($CARL(Y;\gamma^2)$) for the two-sided S^2 chart is the reciprocal value of the $CPS(Y;\gamma^2)$ and, from Equation (3), can be found as

$$CARL(Y;\gamma^2) = [CPS(Y;\gamma^2)]^{-1}$$

$$= \left[1 - \left(F_{\chi^2_{n-1}}\left(\frac{Y}{\gamma^2 m(n-1)}\chi^2_{n-1,1-\alpha^*/2}\right) - F_{\chi^2_{n-1}}\left(\frac{Y}{\gamma^2 m(n-1)}\chi^2_{n-1,\,\alpha^*/2}\right)\right)\right]^{-1}. \quad (4)$$

Note that, when $\gamma^2 \neq 1$, the process is out of control, i.e., the variance in Phase II is different from the IC variance in Phase I ($\sigma^2 \neq \sigma_0^2$). Accordingly, when $\gamma^2 = 1$ the process is IC. In this case, the CPS is named the conditional false alarm rate ($CFAR$), i.e., $CFAR(Y) = CPS(Y;\gamma^2 = 1)$ and the $CARL$ is named the conditional in-control average run length ($CARL_0$), i.e., $CARL_0(Y) = CARL(Y;\gamma^2 = 1) = \left(CFAR(Y)\right)^{-1}$.

2.2 Relation between the distribution of the conditional average run lengths and the two-sided tolerance intervals for S^2

The probability of a signal is an important property of the Phase II performance of S^2 control chart. The conditional probability of a signal (CPS) is the probability that each l-th sample variance ($l = m+1, m+2, ...$), of size n in the monitoring process (Phase II), falls outside the estimated (given the particular estimate $\hat{\sigma}_0^2$) control limits range given by Equations (1)-(2). Thus, the conditional (given $\hat{\sigma}_0^2$) $IC\ RL$ distribution [i.e., the distribution of the number of Phase II sample-variance estimates (S_l^2) until a false alarm (signal)] is geometric with probability of success equal to $CFAR$

$$F_{\boldsymbol{CARL_0}}(t) = 1 - F_{CFAR}(t^{-1}) = 1 - P\left(P\left(\widehat{LCL} \le S_l^2 \le \widehat{UCL}\right) \ge 1 - t^{-1}\right) \quad (5)$$

Sarmiento et al. (2018) studied the exact two-sided tolerance intervals for sample variances (S^2) and they provided the lower and upper tolerance factors that are equivalent to the adjusted two-sided lower and upper control limit factors (L^*, U^*) given in Equations (1) and

(2). The $(1-\alpha,\gamma)$ two-sided tolerance interval $\left(S^2_{L_{two}}, S^2_{U_{two}}\right)$ for the sample variance contains at least a specified $(1-\alpha)100\%$ of the population of the sample variances (or, equivalently, the distribution of S^2) with a specified probability γ, and it is given by

$$P_{S_p^2}\left(P_{S^2}\left(\hat{S}^2_{L^*_{two}} \leq S^2 \leq \hat{S}^2_{U^*_{two}} \mid S_p^2\right) \geq 1-\alpha\right)$$
$$= P_Y\left(P_W\left(\frac{Y}{m(n-1)}\chi^2_{n-1,\ \frac{\alpha^*_{two}}{2}} \leq W \leq \frac{Y}{m(n-1)}\chi^2_{n-1,\ 1-\frac{\alpha^*_{two}}{2}} \mid Y\right) \geq 1-\alpha\right) = \gamma \tag{6}$$

where $W = (n-1)S^2/\sigma^2$ and $Y = m(n-1)\,S_p^2/\sigma^2$, so that W and Y each follows a chi-square distribution with $(n-1)$ and $m(n-1)$ d.f., respectively, and W and Y are independent random variables. Also, $\hat{S}^2_{L^*_{two}} = L^*_{two}S_p^2 = \frac{\chi^2_{n-1,\frac{\alpha^*_{two}}{2}}}{n-1}S_p^2$ and $\hat{S}^2_{U^*_{two}} = U^*_{two}S_p^2 = \frac{\chi^2_{n-1,1-\frac{\alpha^*_{two}}{2}}}{n-1}S_p^2$ are the lower and the upper tolerance limits when the variance is estimated, and L^*_{two} and U^*_{two}, respectively, denote the lower and upper tolerance factors. The adjusted value of α is α^*_{two} ($\alpha^*_{two} \leq \alpha$) that needs to be determined to obtain the tolerance limits. Because the two-sided tolerance interval for S^2 (Equation (6)) is equivalent to the cdf of the $CFAR$ (Equation (5)), we can use the formulation of the tolerance factors from Sarmiento et al. (2018) in order to analyze the adjusted control limits of S^2 chart.

2.3 Minimum number of Phase I samples (m) that guarantees a conditional IC performance of the S^2 charts using EPC

In the context of parameter estimation in control chart, a relevant practical question for the practitioner is the amount of the Phase I data that can ensure a "satisfactory" IC Phase II performance, that is, when the unadjusted control limits are used [using α instead of α^* in Equations (1) and (2)]. This problem can be addressed using the Exceedance Probability criterion (EPC), which was proposed by Albers et al. (2005), as follow: the minimum number of Phase I reference samples (m) that guarantees, with a specified high probability $1-p$ (e.g., 0.9), that the $CARL_0$ is at least a tolerated value, which is usually a bit smaller than the $nominal\ ARL_0$ (e.g., the 90% of 370.4). This formulation can be stated as follows: given the values of n, $\alpha^* = \alpha$, ε and p, the minimum value of m is found using a numerical (search) method

$$P\left(CARL_0 \geq \frac{1}{(1+\varepsilon)}\left(\frac{1}{\alpha}\right)\right) = P(CFAR \leq (1+\varepsilon)\alpha) = 1-p, \tag{7}$$

where: α is the nominal false alarm rate; ε ($0 \leq \varepsilon < 1$) is the tolerance factor, meaning that the largest tolerated value for the $CFAR$ is $(100\varepsilon)\%$ larger than the nominal α (trivial algebra shows that this is equivalent to saying that the smallest tolerated $CARL_0$ is $100\left(\frac{\varepsilon}{1+\varepsilon}\right)\%$ smaller than the nominal $ARL_0 = 1/\alpha$). In addition, p is the risk (probability) accepted by the practitioner that the true $CFAR$ be larger than tolerated (or the $CARL_0$ smaller than tolerated). Actually, the "=" sign in Equation (7) must be replaced by the "≥" inequality sign because m is an integer and a perfect match of the probability $(1-p)$ is generally not possible. In this way, in Table 1 we show the minimum number of Phase I samples ($Min\ m$) required to guarantee a conditional IC performance for the two-sided S^2 chart using the EPC. m is expressed as a function of n with $\varepsilon = \{10\%, 20\%\}$, $p = \{0.05, 0.10, 0.20\}$ and $\alpha^* = \alpha = 0.0027$. Figure 1 shows the results given in Table 1.

From Table 1, we found large and, in several cases, impractical and unfeasible minimum values of m, even for large values of n, such as $n = 30$. For instance, for $n = 5$, $\varepsilon = 10$ and $p = 0.05$, the minimum m is 1325 samples. Hence, because of the large amount of Phase I data required to attain a specified nominal IC performance, some authors have proposed adjustments to the control limit(s) in order to achieve this chart performance with less data, which can be collected in practice. Other authors adjusted the limits focusing on the conditional perspective under the EPC [Faraz et al. (2015, 2017) and Goedhart et al. (2017) only for the one-sided chart].

Table 1: The minimum number of Phase I samples ($Min\ m$) required to guarantee a conditional IC performance for the two-sided S^2 when $\varepsilon = \{10\%, 20\%\}$, $p = \{0.05, 0.10, 0.20\}$ and $\alpha^* = \alpha = 0.0027$

	$\varepsilon = 0.10$			$\varepsilon = 0.20$		
	p			p		
n	0.05	0.10	0.20	0.05	0.10	0.20
2	4265	2604	1144	1268	778	347
3	2688	1640	720	814	499	222
4	2028	1237	542	625	383	170
5	1653	1008	442	518	317	141
6	1408	859	376	449	275	122
7	1235	753	330	399	245	108
8	1107	675	295	363	222	98
9	1007	614	269	334	205	91
10	928	566	248	312	191	84
11	864	527	230	293	179	79
12	810	494	216	277	170	75
13	764	466	204	264	162	72
14	725	442	193	253	155	69
15	691	422	184	243	149	66
16	662	403	176	234	143	64
17	636	387	169	227	139	62
18	612	373	163	220	135	60
19	591	360	158	214	131	59
20	573	349	153	208	127	58
25	500	305	134	187	114	53
30	452	275	122	172	106	50

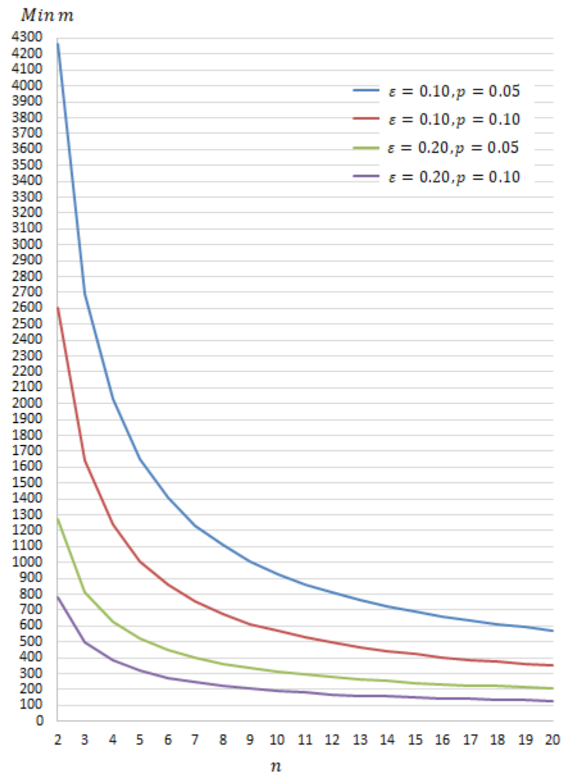

Figure 1: The minimum number of Phase I samples ($Min\ m$) required to guarantee a conditional IC performance for the two-sided S^2 when $\varepsilon = \{10\%, 20\%\}$, $p = \{0.05, 0.10, 0.20\}$ and $\alpha^* = \alpha = 0.0027$

3. Adjusting the control limits of the S^2 charts

Since the $CARL_0$ (or, equivalently, the $CFAR$) is a random variable and large values of the $CARL_0$, such as 370.4 (or, small values of the $CFAR$, such as 0.0027) are desired, the lower (or upper) probability bound may be useful in real Statistical Monitoring Process application (Phase II). Therefore, according to the *conditional* perspective using the EPC (see, for instance, Epprecht et al. (2015), Faraz et al. (2015,2017) and Goedhart et al. (2017)), the user considers the randomness of the $CARL_0$ (or the $CFAR$), via its distribution, for guaranteeing that the $CARL_0$ (or the $CFAR$) is at least (or at most) a minimum (or a maximum) performance threshold with a high probability, say 95%. This threshold is, generally, an exact or a slightly smaller (or larger) value of the nominal IC performance measure, for instance, a minimum $ARL_0 = 370.4$ (or maximum $\alpha = 0.0027$). Using the cdf of $CARL_0$ from Equation (5) and the EPC (using, e.g., in Equation (7)), we can obtain the adjusted chart parameter (α^*) of the two-sided S^2 chart based on exact analytical derivations and non-linear solutions. Hence, α^* can be found solving the Equation (8) by a numerical (search) method

$$\boldsymbol{F_{CFAR}\big((1+\varepsilon)\alpha\,;\,\alpha^*\big) = 1 - F_{CARL_0}\left(\left(\frac{1}{1+\varepsilon}\right)\frac{1}{\alpha};\,\alpha^*\right)}$$

$$= F_{\chi^2_{m(n-1)}}(Y_2) - F_{\chi^2_{m(n-1)}}(Y_1) = 1 - p,\ \ t < max(CARL_0), \qquad (8)$$

where Y_1, Y_2 and $max(CARL_0)$ can be found as

i. Y_1 and Y_2 ($Y_1 < Y_2$) are the solutions (Y) of $CARL_0(Y) = t$ (see Equation (11)). It is due to $CARL_0$ as a function of Y is increasing on $< 0, Y_0]$ and decreasing on $< Y_0, \infty >$, where $Y_0 = m(n-1)\ln(U^*/L^*)/(U^* - L^*)$

ii. $max(CARL_0) = CARL_0(Y = Y_0)$ (see Equation (4))

Since the two-sided tolerance interval for S^2 (Equation (6)) is equivalent to the cdf of the $CFAR$ (Equation (5)), we can use the formulation to obtain the tolerance factors from Sarmiento et al. (2018) in order to adjust the control limits of S^2 charts to guarantee a conditional IC performance using EPC from Equation (8).

With the resulting value of α^* (Equation (8)), the adjusted factors and limits of the two-sided S^2 chart can be found using Equations (1) and (2). Table 2 shows the α^* values and the adjusted lower and upper factors of the two-sided S^2 charts (L^*, U^*) that are obtained using the EPC for $\varepsilon = 0$, $p = 0.05$ and $\alpha = 0.0027$ (i.e., $P(CARL_0 \geq 370.4) = 95\%$) and for $\varepsilon = 0.20$ and $p = 0.20$ (i.e., $P(CARL_0 \geq 308.6) = 80\%$), and for different values of m and n.

Table 2: Adjusted lower and upper factors of the two-sided S^2 control chart required to guarantee a conditional IC performance (using EPC for $\varepsilon = 0, 0.20, p = 0.05, 0.20$ and $\alpha = 0.0027$) for different values of m and n

		$\varepsilon = 0,\ p = 0.05$			$\varepsilon = 0.20,\ p = 0.20$		
m	***n***	α^*	L^*	U^*	α^*	L^*	U^*
	3	0.00038	0.0002	8.5780	0.00153	0.0008	7.1771
25	**5**	0.00062	0.0125	5.2653	0.00184	0.0218	4.6624
	9	0.00085	0.0849	3.5353	0.00210	0.1085	3.2506
	3	0.00085	0.0004	7.7584	0.00204	0.0010	6.8869
50	**5**	0.00112	0.0169	4.9353	0.00228	0.0243	4.5433
	9	0.00136	0.0964	3.3878	0.00248	0.1136	3.1975
	3	0.00115	0.0006	7.4642	0.00228	0.0011	6.7789
75	**5**	0.00140	0.0190	4.8134	0.00248	0.0253	4.4982
	9	0.00162	0.1011	3.3328	0.00264	0.1156	3.1774
	3	0.00134	0.0007	7.3079	0.00241	0.0012	6.7200
100	**5**	0.00158	0.0201	4.7479	0.00259	0.0259	4.4735
	9	0.00178	0.1037	3.3031	0.00273	0.1167	3.1664
	3	0.00158	0.0008	7.1404	0.00257	0.0013	6.6551
150	**5**	0.00179	0.0215	4.6772	0.00272	0.0265	4.4461
	9	0.00197	0.1066	3.2710	0.00284	0.1179	3.1543
	3	0.00174	0.0009	7.0494	0.00267	0.0013	6.6189
200	**5**	0.00192	0.0223	4.6386	0.00280	0.0269	4.4308
	9	0.00208	0.1083	3.2536	0.00290	0.1186	3.1475
	3	0.00184	0.0009	6.9910	0.00273	0.0014	6.5952
250	**5**	0.00201	0.0228	4.6137	0.00285	0.0272	4.4208
	9	0.00215	0.1093	3.2424	0.00294	0.1191	3.1431
	3	0.00270	0.0014	6.6077	0.00324	0.0016	6.4253
∞	**5**	0.00270	0.0264	4.4501	0.00324	0.0290	4.3486
	9	0.00270	0.1163	3.1701	0.00324	0.1163	3.1701

4. Summary and conclusions

- The estimation of IC process variance (σ_0^2) impacts on the Phase II Performance of the S^2 control chart.
- To attain the IC performance of S^2 control chart with estimated variance close to the values of that with known variance, large amount of Phase I reference data is required.
- To guarantee an IC performance of S^2 control chart using a practical (realistic) amount of Phase I reference data, control limits can be adjusted.
- Tolerance limits for the population of sample variances are equivalent to adjusted control limits that guarantee a $CFAR$ smaller than or equal to a maximum $(1+\varepsilon)\alpha$ with a confidence $1-p$.
- The same adjusted factors of S^2 control chart (Table 2) can be used in the context of tolerance interval (e.g., conformity assessment and acceptance of products or processes) as well as in the context of SPC (a guaranteed Phase II conditional Performance).

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