Application of Multidimensional Time Model for Probability Cumulative Function to Experimental and Statistical Investigations into Statistical Randomness and Normality of Pi Sqrt2 Etc

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Abstract: Focus on Approach in Investigations of Normality. Application of Multidimensional Time Model for Probability Cumulative Function to Experimental and Statistical investigations into Statistical Randomness and Normality of pi, sqrt(2), and other numbers has its place in Bayesian Statistical analysis and comes after short History of Hypothesis Testing for randomness and normality of different numbers first developed by Kendall and Smith, such as frequency, serial, poker, and gap tests for division between local and "true" randomness. Anderson-Darling, Kolmogorov-Smirnov tests of equidistribution and Siegel-Tukey test, along with mathematical theory underlying computational machine learning, such as the time complexity of computations, language recognition, and string matching followed by statistical reasoning from the Spatiotemporal Analisis and the Theory of Brownian Motion and Random Walk.

Keywords: Statistical Randomness ; Normality ; Bayesian Statistical Analysis

1. Introduction.

108 years ago Emile Borel gave definition of normal numbers as real numbers whose infinite sequence of digits in every base r is distributed uniformly and every digit has the same density 1/r, also all possible r^n digits are equally likely with density r^{-n}.

Definition1 A number y is normal with respect to the base r if for each combination of n digits $l_1l_2\ldots l_n \lim_{n\to} \frac{M(x)}{x} = \frac{1}{r^n}$, where M(x) number of strings of $l_1l_2\ldots l_n$ in first x digits of y

70 years ago I. J. Good gave a definition of normal recurring decimals, such that have normality of order s and all sequences of s digits have normal frequency 10^{-s}. This definition certainly contradicts Borel's definition, because if the number has normality only of order s, then it is not normal.

Following Knill, Ma, and other authors including D.H. Bailey and J. Borwein , it is not known if the π, e, √2, and other numerical constants are normal.

"Remark. While almost all numbers are normal, it is difficult to decide normality for specific real numbers. One does not know for example whether tt - 3 = 0.1415926... or sqrt(2) - 1 = 0.41421... is normal. " (O. Knill Book on Probability and Stochastic processes p. 65)

Dan Ma:"The calculation of 22.4(π^e) trillion digits of pi were completed in November/16. Subsequently, statistical analysis had been performed on these 22.4 trillion decimal digits of π does every digit (from 0 to 9) in the decimal expansion of π appear one-tenth of the time? Does every pair of digits appear one-hundredth of the time? Does every triple of digits in the decimal expansion of π appear one-thousandth of the time and so on? If that is the case, we would say π is a normal number in base 10.

The concept of normal number applies to other bases too. So if each digit in the binary expansion of a number appears half the time, and if each pair of binary digits (00, 01, 10 and 11) appear one-quarter of the time and so on, the number in question is called a normal number in base 2. In general, for a number to be a normal number in a given base, every sequence of possible digits in that base is equally likely to appear in the expansion of that number. A number that is a normal number in every base is called absolutely normal.

Is π normal in the base 10? In Base 2? In base 16 (hexadecimal)?

Is πa normal number? The empirical evidences, though promising, are not enough to prove that $\pi$is a normal number in base 10 or in any other base. Though many mathematicians believe that $\pi$is a normal number in base 10 and possibly other bases, they had not been able to find mathematical proof. It is also not known whether any one of the other special numbers such as natural log constant $e$or other irrational numbers such as $\sqrt{2}$are normal numbers.

Thus determining whether $\pi$is a normal number is a profound and unsolved classic problem in probability."Are digits of pi random? | A Blog on Probability and Statisticshttps://probabilityandstats.wordpress.com/2017/03/14/are-digits-of-pi-random/Mar 14, 2017

"The short answer to this question is that we do not know whether pi is normal or not, either for decimal digits or for digits in any other number base. Indeed, this question is a premier unsolved problem of mathematics. We do not even know the answer to much simpler questions, such as whether a 3 appears one tenth of the time in the decimal expansion of pi, or whether a 1 appears half of the time in the binary expansion of pi. We cannot even prove that there are infinitely many 7s in the decimal expansion of pi. A similarly appalling ignorance applies to most other constants of mathematics, such as e = 2.71828..., the base of natural logarithms, or the square root of 2 = 1.414213..."THE BLOG 04/16/2013 06:01 pm ET | **Updated** Jun 16, 2013 Are the Digits of Pi Random?**By David H. Bailey, Jonathan M. Borwein**http://www.huffingtonpost.com/david-h-bailey/are-the-digits-of-pi-random_b_3085725.html

"In fact, not a single naturally occurring math constant has been proved normal in even one number base, to the chagrin of mathematicians. While many constants are believed to be normal -- including pi, the square root of 2, and the natural logarithm of 2, often written "log(2)" -- there are no proofs."Preuss, Paul (0723/01). "Are The Digits of Pi Random? Lab Researcher May Hold The Key".

"Further research led to proof that a wide class of fundamental constants are mathematically "normal" -- probably including pi, although that remains to be proved." ((A Mathematical Paradigm Shift, January

28/04 Paul PreussThis review is from: Mathematics by Experiment: Plausible Reasoning in the 21st Century)

“ConclusionA prime motivation in computing and analyzing digits of πis to explore the age-oldquestion of whether and why these digits appear \random." Numerous computer basedstatistical checks of the digits of πhave failed to disclose any deviation from reasonable statistical norms….This result was used in a Poisson process model to show that the probability that π is not normal is extraordinarily small, reinforcing the empirical evidence we have presented evidence for the normality of π”David H. Bailey,Jonathan M. Borwein, Cristian S. Calude,Michael J. Dinneen,Monica Dumitrescu, Alex Yee “An Empirical Approach to the Normalityof π” 2/2/12

2 years ago I was concerned with checking some combinations of digits of and for this purpose I traveled toWikiConference that was hosted in Washington, DC.

The staff gave email of Alexander Yee, who wrote me in private correspondence that he did not know about the program that gives the number of 2,3,4,5,or 6 digits combinations, and at least I became assured that there was not sufficiently significant investigation into the Combinations of 2, 3, 4, 5, or 6-digits in pi-digits expansion.

Finally I found quite recently (few weeks ago) an article by
David H. BaileyThe Computation of 0 to 29,360,000 Decimal Digits Using Borweins' Quartically Convergent Algorithm April 21, 1987Ref: Mathematics of Computation, vol. 50, no. 181 (Jan. 1988), pg.283-296

Digit	Count	Deviation	Z-score
0	2935072	-928	-0.5709
1	2936516	516	0.3174
2	2936843	843	0.5186
3	2935205	-795	-0.4891
4	2938787	2787	1.7145
5	2936197	197	0.1212
6	2935504	-496	-0.3051
7	2934083	-1917	-1.1793
8	2935698	-302	-0.1858
9	2936095	95	0.0584

Table 1: Single Digit StatisticsLength of Run

Digit	5	6	7	8	9
0	308	29	3	0	0
1	281	21	1	0	0
2	272	23	0	0	0
3	266	26	5	0	0
4	296	40	6	1	0
5	292	30	4	0	0
6	316	33	3	0	0
7	315	37	6	2	1
8	295	36	3	0	0
9	306	40	7	0	0

Table 2: Single-Digit Run Counts

The frequencies of long runs are all within acceptable limits of randomness. The only phenomenon of any note in table 5 is the occurrence of a 9-long run of sevens. However, there is a 29% chance that a 9-long run of some digit would occur in 29,360,000 digits, so this instance by itself is not remarkable

00	293062	01	293970	02	293533	03	292893	04	294459
05	294189	06	292688	07	292707	08	294260	09	293311
10	294503	11	293409	12	293591	13	294285	14	294020
15	293158	16	293799	17	293020	18	293262	19	293469
20	293952	21	293226	22	293844	23	293382	24	293869
25	293721	26	293655	27	293969	28	293320	29	293905
30	293718	31	293542	32	293272	33	293422	34	293178
35	293490	36	293484	37	292694	38	294152	39	294253
40	294622	41	294793	42	293863	43	293041	44	293519
45	293998	46	294418	47	293616	48	293296	49	293621
50	292736	51	294272	52	293614	53	293215	54	293569
55	294194	56	293260	57	294152	58	293137	59	294048
60	293842	61	293105	62	294187	63	293809	64	293463
65	293544	66	293123	67	293307	68	293602	69	293522
70	292650	71	294304	72	293497	73	293761	74	293960
75	293199	76	293597	77	292745	78	293223	79	293147
80	292517	81	292986	82	293637	83	294475	84	294267
85	293600	86	293786	87	293971	88	293434	89	293025
90	293470	91	292908	92	293806	93	292922	94	294483
95	293104	96	293694	97	293902	98	294012	99	293794

Table 3: Two Digit Frequency Counts

In my observation the biggest deviation are at digits 41 and 80

There are following examples for decimal expansion of   from Weisstein, E. W. "Pi Digits. From MathWorld-A Wolfram Web Resource. http://mathworld.wolfram.com/PiDigits.html:

The sequence 0123456789 occurs beginning at digits 17387594880, 26852899245, 30243957439, 34549153953, 41952536161, and 43289964000.

The sequence 9876543210 occurs beginning at digits 21981157633, 29832636867, 39232573648, 42140457481, and 43065796214.

The sequence 27182818284 (the first few digits of e) occurs beginning at digit 45111908393.
J. Havil in his book “The Irrationals’’ gives this impressive example, that sequence 0123456789 appears for the first time starting at the 17,387,594,880th digit; whereas 0691143420 continues to prove elusive. His book has ISBN 978-069114342-2 and was published 2 years ago by Princeton University Press. The website gives the record for this year as 10 trillion, with current of 13.3 trillion. This problem is very much related to such open problems in the Theory of Brownian Motion as if all Brownian paths are possible, and the like problems of transience and recurrence of Random walk in 2 or more dimensions [Peter Morters and Yuval Peres, Brownian Motion, CUP] [8-12, 14, 17].

Table 4 – First 1 trillion digits of Pi

Digit number

0	99,999,485,134
1	99,999,945,664
2	100,000,480,057
3	99,999,787,805
4	100,000,357,857
5	99,999,671,008
6	99,999,807,503
7	99,999,818,723
8	100,000,791,469
9	99,999,854,780

March 14, 2016 by Steve Humble, Newcastle University, The Conversation
Read more at: https://phys.org/news/2016-03-pi-random-full-hidden-patterns.html#jCp

Conjecture1π is not normal to any base b > 1, but it is normal to some order s that imply that ∃S ∈ N, such that π is normal for ∀s ≤ S and not normal for ∀s> S

Conjecture2 π is not normal to all orders that are greater or equal 19 ∀s> 18 or S =18

Conjecture3 Conjectures 1 and 2 are correct to a wide class of fundamental constants and possibly can be extended to all algebraic numbers.

Previously quoted assurance of Paul Preuss in A Mathematical Paradigm Shift, January 28/04, who is the only science fiction writer would be very astonishing without some arguments supporting Conjectures 1,2, and 3.

What are the ways of approaching the problem?

1. Certainly it is always possible to look at this problem through Random Walk approach in many dimensions.

1. The nearly-Gaussian distributions were first studied by Laplace, followed by Chebyshev and Hermite. And the Edgeworth series , based on Chebyshev-Hermite polynomials, is used for approximation of exactly this type of distributions. One of the articles that concentrate on these subjects is “Expansions for nearly Gaussian distributions” by R.Moessner et al.

The Section of WKB approximation though is placed after Saddlepoint Approximation could be a very good introduction to the nearly-Gaussian distributions as the method was developed some 20 - 40 years before investigations of Chebyshev and Hermite.

2. After this comes the question of measuring the non-Gaussianity of a random variable, which is a central problem in the theory of independent component analysis (ICA) and other fields.“Hermite Polynomials and Measures of Non-Gaussianity” by J. Puuronen et all.
3. I also think that some results related to partial differential equations such as look like Normal waves arising in boundaries of elliptical and parabolical PDE solutions also could be some types of nearly-Gaussian distributions

4. How it is possible to start thinking about nearly-Gaussian distributions. It arises from the exactly of our point of interest, to question the normality in pi-digits expansion distribution. So the consideration comes as consecutive view of frequencies in 10, 100, 1000, 10,000, and so on digits in pi-digits expansion. So we apply without even prior questioning about existence of these properties, as if it is natural to think that their existence is proved, such properties as Bayesian or not, Markov or not, and so on. With the view of sequence of enlarging number of digits in pi-digits expansion we can apply such models as prior and posterior distributions and this exactly leads to the very effect of smoothing as this phenomenon, which is not even considered as phenomenon, is widely discussed in the very wide subject of spatial statistical data. To mention just a few “Robust Filtering and Smoothing with Gaussian Processes” by M. P. Deisenroth et all,“Bayesian Inference and Model Assessment for Spatial Point Patterns Using Posterior Predictive Samples” Alan E. Gelfand et all, “An efficient Markov chain Monte Carlo method for distributions with intractable normalising constants” by J. Moller et all, “Scaling intrinsic Gaussian Markov random field priors in spatial modeling” by H. Rue, “Properties of the cosmological density distribution function” by F Bernardeau. (The author’s private correspondence)

5. The Gaussian distribution is a typical model for signals and noise in many applicationsin science and engineering. However, there are some applications where this Gaussian assumptiondeparts from the actual random behavior. For instance, the samples of a speechsignal are modeled by a Laplacian distribution, and the generalized Gaussian distributionhas been proposed for modeling atmospheric noise “A practical procedure to estimate the shape parameter in the generalized Gaussian distribution”.by Benoit Mandelbrot on the coastline of Britain it was shown that it is inherently nonsensical to discuss certain spatial concepts despite an inherent presumption of the validity of the concept. (The author’s private correspondence)

6. Returning to the question of measure-theoretical approach to the question of normality the Gaussian process measures can be well-defined over such spaces, as an infinite-dimensional Banach or Hilbert space, where no Lebesgue measure can be defined. “Probabilistic Numerics andUncertainty in Computations” M A Osborne et all. So we are coming to the point of lattice measures. However, one of the points can be viewed as even simple one if we consider that an

original lattice of size n×n and so on is divided into k×k and so on grid cells, as we looking for combinations of decimal digits, transience or recurrence of

Random walk on a lattice was studied by Polya, who published a lot on the subject. Polya revealed his secret around 40 years ago, during his visit to Budapest as to insight that he considered random walk on fractal set. Polya also showed that Fourier transform of a probabilistic measure uniquely defines this measure.(The author’s private correspondence)

7. As actually none of the known to the present moment numerical methods for pi-digits calculation is resembling Newton binomial Newton binomial. E.D. Solomentsev (originator), Encyclopedia of Mathematics. URL:
http://www.encyclopediaofmath.org/index.php?title=Newton_binomial&oldid=13002, the limiting distribution of which would be the exact Normal distribution function, so there is no reason to suspect normality.(The author’s private correspondence)

2. Saddle point approximation

The philosophy of application of saddle point approximation as approximation of probability density function of continuing functioning of k components out of n with intractable distribution functions is coming from common multidimensional retrospective approach.

The digits in the expansion of π and further the strings of n digits can be considered as components with intractable distribution functions and the question is in determining the distribution function of continuing functioning of k components out of M strings.

π~3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70679
82148 08651 32823 06647 09384 46095 50582 23172 53594 08128 48111 74502 84102 70193 85211 05559 64462 29489 54930 38196
44288 10975 66593 34461 28475 64823 37867 83165 27120 19091 45648 56692 34603 48610 45432 66482 13393 60726 02491 41273
72458 70066 06315 58817 48815 20920 96282 92540 91715 36436 78925 90360 01133 05305 48820 46652 13841 46951 94151 16094
33057 27036 57595 91953 09218 61173 81932 61179 31051 18548 07446 23799 62749 56735 18857 52724 89122 79381 83011 94912

98336 73362 44065 66430 86021 39494 63952 24737 19070 21798 60943 70277 05392 17176 29317 67523 84674 81846 76694 05132
00056 81271 45263 56082 77857 71342 75778 96091 73637 17872 14684 40901 22495 34301 46549 58537 10507 92279 68925 89235
42019 95611 21290 21960 86403 44181 59813 62977 47713 09960 51870 72113 49999 99837 29780 49951 05973 17328 16096 31859
50244 59455 34690 83026 42522 30825 33446 85035 26193 11881 71010 00313 78387 52886 58753 32083 81420 61717 76691 47303
59825 34904 28755 46873 11595 62863 88235 37875 93751 95778 18577 80532 17122 68066 13001 92787 66111 95909 21642 01989

τ~6.28318 53071 79586 47692 52867 66559 00576 83943 38798 75021 16419 49889 18461 56328 12572 41799 72560 69650 68423 41359
64296 17302 65646 13294 18768 92191 01164 46345 07188 16256 96223 49005 68205 40387 70422 11119 28924 58979 09860 76392
88576 21951 33186 68922 56951 29646 75735 66330 54240 38182 91297 13384 69206 97220 90865 32964 26787 21452 04982 82547
44917 40132 12631 17634 97630 41841 92565 85081 83430 72873 57851 80720 02266 10610 97640 93304 27682 93903 88302 32188
66114 54073 15191 83906 18437 22347 63865 22358 62102 37096 14892 47599 25499 13470 37715 05449 78245 58763 66023 89825

96673 46724 88131 32861 72042 78989 27904 49474 38140 43597 21887 40554 10784 34352 58635 35047 69349 63693 53388 10264
00113 62542 90527 12165 55715 42685 51557 92183 47274 35744 29368 81802 44990 68602 93099 17074 21015 84559 37851 78470
84039 91222 42580 43921 72806 88363 19627 25954 95426 19921 03741 44226 99999 99674 59560 99902 11946 34656 32192 63719
00489 18910 69381 66052 85044 61650 66893 70070 52386 23763 42020 00627 56775 05773 17506 64167 62841 23435 53382 94607
19650 69808 57510 93746 23191 25727 76470 75751 87503 91556 37155 61064 34245 36132 26003 85575 32223 91818 43284 03978

(Wikipedia Six nines in pi https://en.wikipedia.org/wiki/Six_nines_in_pi) “The first 1000 digits of contain ample double consecutive digits (marked yellow), and a few triples (marked green). The presence of the sextuple (marked red) in such a small sample is an intriguing anomaly. A sequence of six 9's occurs in the decimal representation of , starting at the 762nd decimal place.” Wikipedia

As it can be seen from the different authors applying χ^2 tests to strings of repeated digits that the repeated digits are the starting point of analysis including such questions are there million zeros in the expansion of π?

Analyzing them it is possible to suppose that not all of the strings are repeating with the same frequency, and some of the strings or components of the long strings fail, and in this way, they are from the normal recurring decimal and the problem is to find the order, to which the number is normal. Still the distribution of the strings is supposed to be intractable, and some approximation with deviation from the solution is vanishing sufficiently rapidly should be applied. Such approximation is method of Saddle Point Approximations that was introduced by Daniels 2 years before Borel's death.

The first and possibly the closest to check distribution of components is Poisson Point Process as mostly used in Survival Analysis or some mixture of Poisson Distributions. As it is seen from the above mentioned article ofDavid H. Bailey,Jonathan M. Borwein, Cristian S. CaludeMichael J. Dinneen,Monica Dumitrescu, Alex Yee "An Empirical Approach to the Normalityof π" of 2/2/12 they also used Poisson Process without though revealing their motivation.

Method of Saddle Point Approximations.

For cumulant generating function K(t)= log(M(t)), where M(t)is the moment generating function then the saddlepoint approximation to the PDF and CDF of a distribution is given by

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi K\ (\hat{s})}} \exp\left(K(\hat{s}) - \hat{s}\,x\right)$$

$$\hat{F}\,(x) = \quad (\hat{w}) + \varphi(\hat{w})\left(\frac{1}{\hat{w}} - \frac{1}{\hat{u}}\right) \text{ for } x \neq E(x)$$

$$\frac{1}{2} + \frac{K\ \ (0)}{6\sqrt{2\pi}K\ (0)^{3/2}} \qquad \text{for } x = E(x)$$

Where $\hat{s}$ is a solution to $K\ (\hat{s}) = x$, $\hat{w} = \text{sgn}\,\hat{s}\sqrt{2(\hat{s}x - \ K(\hat{s}))}$ and $\hat{u} = \hat{s}\sqrt{K\ (\hat{s})}$

3.WKB Approximation and introduction of time scale approach

The philosophy of application of saddle point approximation as approximation of probability density function of continuing functioning of k components out of n with intractable distribution functions is coming from common multidimensional retrospective approach.

Method of Saddle Point Approximations is used in statistical mechanics(that focuses on very large number of particles. It started possibly some 280 years ago with the argument of Bernoulli of the chaotic movement of very large number of molecules) and in particular for the velocity of chemical reactions and as such play significant role in preliminary analysis leading to transition state theory(TST)that does not require the reactants and products to be in equilibrium, but the activated complexes are assumed to be in quasi- equilibrium with the reactants. It studies activated complexes near the saddle point of a potential energy surface. These complexes and the saddle point itself are considered the transition states. TST as well as Bohr, Kramers, and Slater (BKS that was developed around the same time) theory are the consequences of the controversy in the subject ofquantum mechanics that started, when thesame year ofintroduction by Emile Borel the concept of normal numbers and of introduction of the method of steepest descentby Debye Einstein "first asserted in print that the quantum hypothesis is incompatible with classical assumption about independence of interacting systems"It was the same year when "Ehrlich first proposed the idea that nascent transformed cells arise continuously in our bodies and that the immune system scans for and eradicates these transformed cells before they are manifested clinically, immune surveillance has been a controversial topic in tumour immunology"Hypothesis and surrounding controversy mainly because of little knowledge of cellular structure led to the development of the field of transcriptomics(changes in DNA and RNA structure that is closely related to the subject of discussion).(Robert D. Schreiber et al THE THREE ES OFCANCER IMMUNOEDITINGAnnu. Rev. Immunol. 2004. 22:329–60 doi: 10.1146/annurev.immunol.22.012703.104803

It may happen that the whole entire supposition of normality stems out as mathematical adjustment to the numerous (thousands if not tens of thousands) works on DNA structure, that were coming out 40-30 years ago. Comparisons of DNA sequences were based on introduction of many-dimensional metric spaces, DNA topology, and special measure techniques for probabilities calculations in different types of Laws of Large Numbers. Some of results are in forms of exponential function.

The method of steepest descentor saddlepoint approximation methodworks as an extension of Laplace's method of integral approximation for integrals in the complex plane. To actually make it similar to a real case the contour of integration is changed (by the change of variables) in such a way as to go through or nearly stationary or saddle point, closely to the direction of steepest descent or stationary phase (Imaginary part).

The method is often applied to the functions and integrals of complex variables of the exponential family like

$\int f(z)e^{Ag(z)}$ dz D

and uses asymptotic expansion for single nondegerate saddle point, whereas for multiple nondegerate saddle points the integral becomes the sum ofintegral over open cover of the area with weighted functions, where

$\{ \mathbf{D}_x^{(k)}\}^K_{k=1}$is an open cover of $\mathbf{D}_x$

The previously developed mathematical structure of the models of Boltzmann type kinetic equations for reacting gas mixtures for particles undergoing inelastic interactions with reactions of bimolecular and dissociation-recombination type is very complicated, because of the collisional operators that usually in the full Boltzmann equations are expressed by 5-fold integrals.

Consequently direct numerical applications of these models present several computational difficulties. Besides their own complexity they are certainly further complicated by missing data and additional complication arising from different forces applied to the system. Such complication can be irreversible chemical reaction that the system is undergoing.

As initial consideration can be considered the Bohr—Kramers—Slater theory (BKS theory) that was proposed some 90 years ago in answer to similar problems in physical optics to combine the continuousness of electromagnetic field and the discontinuousness of quantum transitions in atoms("radiation" and "matter") in one unified approach. Their approach stated in their original paperwas further developed by Kramers after 16 years in application of their ideas and methods such as equilibrium and other concepts, including newly discovered concepts of quantum tunneling etc..that introduced his approach to the velocity of chemical reactions:

Next year after Kramers applied Smoluchowski equation to velocity of chemical reactions.(Smoluchowski equation uses such concepts as memory friction, quasi equilibrium etc) Kolmogorov introduced for the equations of turbulence 5/3 law using considerations of dimensionality and time-scale

$E(k) \approx C\epsilon^{2/3}k^{-5/3}$

where ϵ is energy flow and k is a wave number, and C is some dimensionless constant that can be determined from experiments and is close to 1

Turbulent flow is composed from
1]unstable large eddies,energy-containing length scale I that break up into small ones
2]the energy is transferred from large scales to smaller until such length scale (dissipation scale η) that the memory friction of the fluid (as in Smoluchowski equation by Kramers) dissipate the kinetic energy.

Kolmogorov's approach had 3 Hypotheses:

Hypothesis1For very high Re, the motions with length scales $\ll$l are statistically independent (and are locally homogeneous and isotropic) of the components of the motion at the energy-containing scales that may be inhomogeneous and anisotropic. Time scales characteristic for dissipation scale η is $\ll$ than the

time scale of the energy-containing eddies.

The motion of dissipation scale η is close to statistical equilibrium (`equilibrium range').

Hypothesis 2 (1st similarity hypothesis)
For very high Re, statistics in the equilibrium range are uniquely determined by the memory friction $\gamma$and the rate of energy dissipation ϵ

$\eta =(\frac{\gamma^3}{\epsilon})^{1/4}$ lenght $[\gamma] = \frac{m^2}{s}$

$V = (\gamma\epsilon)^{1/4}$ velocity $[\epsilon] = \frac{m^2}{s^3}$

Time-scale in the equilibrium range $\frac{\eta}{v} = (\frac{\gamma}{\epsilon})^{1/2}$

$[E(k)] = \frac{m^3}{s^2}$ and from considerations of dimensionality $E(k) = \epsilon^{2/3}k^{-5/3}F(k\eta)$

Hypothesis 3 (second similarity hypothesis)
At very high Re the statistics of scales in the 'inertial subrange' $I^{-1} \ll k \ll \eta^{-1}$are universally and uniquely determined by thescale k and the rate of energy dissipation ϵ.

$E(k) = C\,\epsilon^{2/3}k^{-5/3}$, which is called Kolmogorov's 5/3 law.

There is presently no fully deductive theory which starts from the Navier-Stokes equations and leads to the Kolmogorov's law.There is no natural closure for the averaged equations (`closure problem'). `Intermittency' is the existing problem.

And the conclusion leads to the importance of multi-scale time analysis.

Proposition1The long runs of digits can be considered as waves with possibly unknown wave distribution contributing to turbulence flow, and therefore, the application of multi-scale time analysis is justified.

The continuous version of the Saddlepoint Approximation is the WKB Approximation

WKB Approximation is a special type of multi-scale time analysis applied to tunneling analysis when the probabilities of the particle to go through are small. It is referred to Wentzel, Kramers, and Brillouin, who introduced it independently 2 years after the publication of Bohr, Kramers, Slaters theory and can be credited back to Francesco Carlini, who developed it 200 years ago, and 20 years later, 99 years after the introduction of the concept of chaotic motion of molecules by Bernoulli, this method was simultaneously introduced by Liouville and applied by Green to waves motion in small tunnels.

If the highest derivative of differential equation is multiplied by small parameter ϵ

$$\epsilon\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)y'(x) + \ldots a_0 y = 0$$

The solution is assumed in the form of asymptotic series $y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}\delta^n H_n(x)\right]$,

with $\delta \to 0$ this determines asymptotic scaling.

For $\epsilon^2\frac{d^2y}{dx^2} = G(x)y \quad \frac{\epsilon^2}{\delta^2}H_0'^2 + \frac{2\epsilon^2}{\delta}H_0'H_1' + \frac{\epsilon^2}{\delta}H_0'' = G(x)$

for $\delta \to 0 \quad \frac{\epsilon^2}{\delta^2}H_0'^2 \sim G(x)$ using scaling and equaling $\epsilon^2 = \delta^2$

$H_0' = \pm\int_{x_0}^{x}\sqrt{G(t)}\,dt$ with the rest is an simple differential equation consider

The concept of quantum tomography
Continuous-variable optical quantum state tomographyA. I. Lvovsky
"As an example, measure the position x of each of
100,000 identically prepared electrons, which can moveonly in one dimension. This yields an estimate of the positionprobability density, or the square-modulus j (x)j2of the Schrodinger wave function. If the wave functionhas the form j (x)j exp[i_(x)], where _(x) is a spatially
dependent phase, then we will need more informationthan simply j (x)j2 in order to know the wave function."

Problems with linear inversion in quantum tomography

One of the primary problems with using linear inversion to solve for the density matrix is that in general the computed solution will not be a valid density matrix. For example, it could give negative probabilities or probabilities greater than 1 to certain measurement outcomes. This is particularly an issue when fewer measurements are made.

Another issue is that in infinite dimensional Hilbert spaces, an infinite number of measurement outcomes would be required. Making assumptions about the structure and using a finite measurement basis leads to artifacts in the phase space density.(Wikipedia)

Consider a particle of mass m and energy $E > 0$ moving through some slowly varying potential $V(x)$. The particle's wave function satisfies

$$\frac{d^2\psi(x)}{dx^2} = -k^2(x)\,\psi(x),$$

where

$$k^2(x) = \frac{2\,m\,[E - V(x)]}{\hbar^2}.$$

Look for solution of the form

$$\psi(x) = \psi_0\,\exp\left(\int_0^x \mathrm{i}\,k(x')\,dx'\right),$$

$$\frac{d\psi(x)}{dx} = \mathrm{i}\,k(x)\,\psi(x),$$

and

$$\frac{d^2\psi(x)}{dx^2} = \mathrm{i}\,k'(x)\,\psi(x) - k^2(x)\,\psi(x),$$

where $k' \equiv dk/dx$ if $|k'| \ll k^2$

it would giveWKB approximation, and the probability density remains constant:i.e.,

$$|\psi(x)|^2 = |\psi_0|^2,$$

if the particle moves through a region in which $E > V(x)$, and $k(x)$ is consequently real (i.e., an allowed region according to classical physics).

If the particle encounters a potential barrier (*i.e.*, a region from which the particle is excluded according to classical physics). By definition, $E < V(x)$ inside such a barrier, and $k(x)$ is consequently imaginary.

The WKB solution inside the barrier is written

$$\psi(x) = \psi_1\,\exp\left(-\int_{x_1}^x |k(x')|\,dx'\right),$$

where

$$\psi_1 = \psi_0\,\exp\left(\int_0^{x_1} \mathrm{i}\,k(x')\,dx'\right).$$

Here was neglected the unphysical exponentially growing solution.

According to the WKB solution the probability density *decays exponentially* inside the barrier: *i.e.*,

$$|\psi(x)|^2 = |\psi_1|^2 \exp\left(-2\int_{x_1}^{x} |k(x')|\,dx'\right),$$

Note that the criterion for the validity of the WKB approximation implies that the transmission probability is very small. Hence, the WKB approximation only applies to situations in which there is very little chance of a particle tunneling through the potential barrier in question.http://www.physicspages.com/2014/07/03/wkb-approximation-tunneling/

4. Multidimensional Time Model for Probability Cumulative Function.

Proposition2. Theorem2Let k and l be arbitrary natural numbers. Then there exists a natural number n(k,l) such that, if an arbitrary segment, of length n(k,l) , of the sequence of natural numbers is divided in any manner into k classes (some of which may be empty), then an arithmetic progression of length l appears in at least one of these classes.(Khinchin ”Three pearls of Number theory”)

Consider Stone representation of Boolean algebra, which is represented by an algebra with known axioms for Boolean algebra and can be characterized by quadruplets B = <X, 0, *, ~>, where 0 is an element from a set X, and * is a binary operation and ~ is an unary operation, which would be a Boolean algebra with 1 as a unit on the operations ∧, ∨ , and ~. Besides that it has four unary operations, two of which are constant operations, another is the identity, and negation and besides the number of n-ary operations, the number of the dimensions that infinite-dimensional model can be reduced to through application of Boolean prime ideal theorem and Stone duality, can be indexed by an index set.

Proposition3. Multidimensional Time Model for Probability Cumulative Function can be reduced to finite-dimensional time model, which can be characterized by Boolean algebra for operations over events and their probabilities and index set for reduction of infinite dimensional time model to finite number of dimensions of time model considering the fractal-dimensional time that is arising from alike supersymmetrical properties of probability,

4.1 First approach to multidimensional time model through Kramers turnover problem in the theory of velocity of chemical reactions.

Consider first the mathematical structure of the models of Boltzmann type kinetic equations for reacting gas mixtures for particles undergoing inelastic interactions with reactions of bimolecular and dissociation-recombination type is very complicated, because of the collisional operators that usually in the full Boltzmann equations, are expressed by 5-fold integrals. Consequently direct numerical applications of these models present several computational difficulties. The search for the simpler solution had its long way till the introduction of the equation for the Brownian motion by Albert Einstein. However, using the theory of Brownian motion for the velocity (rate) of chemical reactions Bohr, Kramers, and Slater used only one-dimensional (1D) model for The Kramers turnover problem, that is, obtaining a uniform expression for the rate of escape of a particle over a barrier for any value of the external frictionuntil it was correctedby Grote-Hynes theory 40 years later, with new improvements following after 6 years by There are certainly other theories followed, all of them distinguish 1D approach from 2D, 3D, and multiD approaches.

It is important and very interesting to consider such point that Kramers in his original work had it as possibility that multidimensional pattern could be related to time dimensions, as he based his introduction theory of Brownian motion on the Einstein's pattern he considered a range of time intervals τ. His discussion of the possibility of a term proportional to τ in the expression for Moments of Brownian motion $B\tau n$(n> 1) related it to the fact that the values, which X takes at moments t1, t2 . . . . tn which lie sufficiently close together are no longer independent; and Moments of Brownian motion $B\tau n$(n> 1) in fact are represented by a volume integral
$\int\ldots\int X$ (t1)X(t2) . . X(tn) d t1d t2 . . . .dtn
over an n-dimensional cube; the contribution to this integral due to a narrow cylinder extending along the diagonal t1= t2 = . . .= tn may give a term proportional to τ.

4.2 Second approach to multidimensional time model through Cumulant Functions and time series analysis.

To strengthen this notion consider cumulants properties for time series analysis that provide measure of Gaussianity. If r.v. X is normal, then cumk{X} = 0 for k > 2, where cumk denotes the joint cumulants of X with itself k times.
For simplicity consider seq of iid Xi with all moments and E{Xi} = 0 and var{Xi} = 1, then for Sn = ΣXi/$\sqrt{n}$cumk{ Sn } = n cumk{X}/nk/2 that tends to 0 for k > 2, as n tends to infinity, so Sn has a limiting normal distribution.
And for time series analysis the moment function
E{X(t+u1)… X(t+uk-1)X(t)} would not depend on t, and on the short time interval centered at point of time t can be approximated by normal distribution.

4.3 Third approach through associated random variables.

Additional to the Brownian motion considerations in the theory of chemical reactions and time series analysis for cumulant functions, the same results can be obtained from the consideration of associated random variables.

Definition1 For n > 1 the set of rv Xi is said to beassociated, if for all given real-valued functions g_i that are increasing ineach component when the other components are held fixed, the inequality
E [$\Pi gj2j$(X)]≥$\Pi E2j$(gj(X))holds, or equivalently, Corr (gi (X), gj(X)) ≥ 0,

Theorem1. (a) A set consisting of a single random variable is a set of associated random variables. (b) Independent random variables areassociated random variables. (c) A subset of a set of associated randomvariables forms a set of associated random variables. (d) Increasing functions of associated random variables are associated random variables [24].

Proposition4. Therefore, the process X(t) with above properties can be represented by composition of Brownian motion processes in finite-dimensional time model.

Conjecture1π is not normal to any base b > 1, but it is normal to some order s that imply that ∃S ∈ N, such that π is normal for ∀s ≤ S and not normal for ∀s> S

Conjecture2 π is not normal to all orders that are greater or equal 19 ∀s> 18 or S =18

Conjecture3 Conjectures 1 and 2 are correct to a wide class of fundamental constants and possibly can be extended to all algebraic numbers.

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