Optimal Step-stress Accelerated Life Test under Type-I Censoring with Flexible Stress Durations

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Abstract
To collect the information about the lifetime distribution of a product, a standard life testing method at normal operating conditions is not practical when the product has a substantially long lifespan. Accelerated life tests solve this problem by subjecting the test items at higher stress levels for quicker and more failure data. This paper investigates the optimal stress durations for a simple step-stress accelerated life test under Type-I censoring. The determination of the optimal step durations is discussed under several well-known optimality criteria, including C-optimality, D-optimality, and A-optimality. For an exponential population with a single stress variable, the efficiencies of the optimal designs with flexible step durations and uniform step durations are compared through a numerical study and a real case study.

Key Words: accelerated life test, cumulative exposure model, flexible stress duration, optimal regression design, order statistics, step-stress loading, Type-I censoring

1. Introduction

With increasing reliability and substantially long life-spans of products, it is often difficult for standard life testing methods under normal operating conditions to obtain sufficient information about the failure time distribution of the products. This difficulty is overcome by Accelerated Life Test (ALT) where the test units are subjected to higher stress levels than normal for rapid failures. By applying more severe stresses, ALT collects information on the parameters of lifetime distributions more quickly. The lifetime at the design stress is then estimated through extrapolation using a suitable regression model.

Over the decades, a variety of inferential procedures has been developed for ALT. Zhao and Elsayed (2005) proposed a general accelerated life model for step-stress ALT using both Weibull and lognormal distributions in which the stress level only affects the scale parameter. Meeter and Meeker (1994) developed the statistical models and ALT plans with a non-constant shape parameter, and later, Seo *et al.* (2009) investigated the optimal ALT sampling plans for deciding the lot acceptability under Weibull distribution with a non-constant shape parameter and Type-I/II censorings. Exact conditional inference for a step-stress model with exponential competing risks was studied by Balakrishnan and Han (2008), Han and Balakrishnan (2010). Lee and Pan (2010) discussed the parameter estimation for multiple step-stress ALT by employing the generalized linear model based on Poisson distribution while Hu *et al.* (2012) proposed a proportional hazards-based nonparametric model for a simple step-stress ALT to obtain upper confidence bounds for the cumulative failure probability. A Bayesian inferential method for ALT was also developed by Van Dorp and Mazzuchi (2005) under Weibull lifetimes where a multivariate prior distribution was indirectly defined for scale parameters at various stress levels. Khamis (1997) compared constant-stress ALT and step-stress ALT under Weibull lifetime distribution for units subjected to stress. Under complete sampling, Hu *et al.* (2013) studied the statistical equivalency of a simple step-stress ALT to other stress loading designs while Han and Ng (2013) compared the efficiencies of general k-level constant-stress and step-stress ALT under complete sampling and Type-I censoring.

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In order to conduct ALT efficiently, the decision variables such as stress durations should be chosen carefully since they affect not only the experimental cost but also the estimate precision of the lifetime parameters of interest. For this reason, the optimal ALT design has attracted great attention in the reliability literature. Nelson and Kielpinski (1976) initiated research in this area by considering the optimally censored ALT for normal and lognormal distributions. By minimizing the asymptotic variance of the Maximum Likelihood Estimator (MLE) of the acceleration factor, Bai *et al.* (1993) determined the optimal stress change time point of partially ALT under lognormal lifetime distribution. By minimizing the asymptotic variance of a lot acceptability statistic, Bai *et al.* (1995) also designed the sampling plans for failure-censored ALT under Weibull distribution subject to the expected test time constraint. For a general step-stress ALT, Gouno *et al.* (2004), Balakrishnan and Han (2009) discussed the problem of determining the optimal stress duration under progressive Type-I censoring; see also Han *et al.* (2006) for related comments. Ma and Meeker (2008) developed the optimal step-stress ALT plans for general log-location-scale distributions. The optimum combinations of multiple stresses for ALT were investigated by Elsayed and Zhang (2007) based on the proportional hazards model. Zhu and Elsayed (2013) also investigated the optimal ALT plans under progressive censoring when test units experience competing failure modes and are subjected to either single or multiple stress types. To improve the statistical efficiency and reduce the total energy consumption of ALT experiments, Zhang and Liao (2013) developed a design methodology, which depends on the product reliability and the physical characteristics of the testing equipment along with its controller.

The current works on the optimal ALT designs have mainly focused on the variance minimization of a certain estimator ($i.e.$, C-optimality). In addition, for a multiple stress ALT, only a uniform stress duration has been considered although the operation cost could substantially increase with the physical constraints and limitations of the testing instruments when the stress level increases. In this work, we investigate the optimal simple step-stress ALT plans under Type-I censoring with varying stress durations. Assuming a log-linear relationship between the lifetime parameter and stress level, along with the cumulative exposure model for the effect of changing stress in a step-stress ALT, the optimal step durations are determined under several well-known optimality criteria, including C-optimality, D-optimality, and A-optimality. For an exponential population with a single stress variable, the efficiencies of the optimal designs with flexible step durations and uniform step durations are compared through a numerical study and a real case study. The rest of the paper is organized as follows. Section 2 presents the model description and formulation, derives the MLEs of the model parameters and the associated Fisher information for the step-stress ALT. In Section 3, the optimality criteria are defined based on the Fisher information ($viz.$, variance, determinant, and trace) and the existence of optimal design points is discussed in each case under time censoring. Section 4 presents the results of the numerical study and Section 5 illustrates the proposed methods using a case study and compares the efficiencies of the optimal designs with flexible step durations and uniform step durations.

2. Model Description and MLEs

Let us define $x_1 < x_2 < \cdots < x_k$ to be the ordered stress levels to be used in the test. Then, for the stress level x_i, $i = 1, 2, \cdots, k$, let n_i denote the number of units failed in the time interval $(\tau_{i-1}, \tau_i]$ with the usual convention of $\tau_0 = 0$ and $y_{i,l}$ denote the l-th ordered failure time of n_i units at stress level x_i, $l = 1, 2, \cdots, n_i$. Also, let $\Delta_i = \tau_i - \tau_{i-1}$ denote the step duration at stress level x_i. Furthermore, let N_i denote the number of units operating

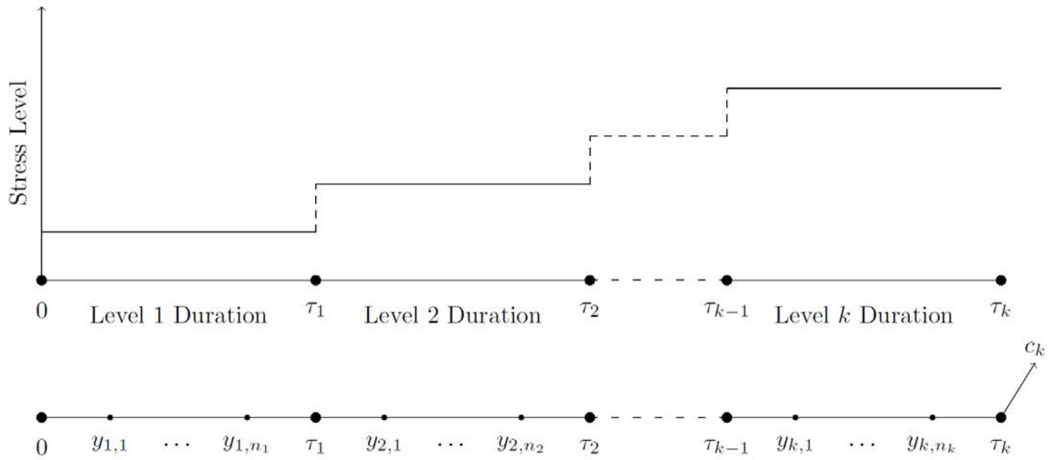

Figure 1: A schematic illustration of the step-stress ALT under Type-I censoring

and remaining on test at the start of stress level x_i. Then, a step-stress ALT under Type-I censoring proceeds as follows. A total number of $N_1 = n$ test units are simultaneously placed at the stress level x_1 and run until τ_2 when the stress level is changed to x_2. The remaining $N_2 = N_1 - n_1$ units are continued to be run until τ_3 at which point the stress level is changed to x_3, and so on. Finally, at the termination time point τ_k, all the surviving units $c_k = n - \sum_{i=1}^{k} n_i$ are withdrawn. Note that when there is no right censoring (viz., $\tau_k = \infty$ or $c_k = 0$), this situation corresponds to the k-level step-stress under complete sampling as a special case. The procedure is depicted pictorially in Figure 1. Now, the following assumptions are the basis of construction subsequent step-stress models.

Assumptions:

- A cumulative exposure model holds;
- For any stress level x_i, $i = 1, 2, \cdots, k$, the lifetime of a test unit follows an exponential distribution. That is, the probability density function (PDF) and the corresponding cumulative distribution function (CDF) of a test unit at stress level x_i are

$$f_i(t) = \frac{1}{\theta_i} \exp\left(-\frac{t}{\theta_i}\right), \quad 0 < t < \infty, \tag{1}$$

and

$$F_i(t) = 1 - \exp\left(-\frac{t}{\theta_i}\right), \quad 0 < t < \infty,$$

respectively. The survival function of a test unit at stress level x_i is readily given by

$$S_i(t) = 1 - F_i(t), \quad 0 < t < \infty. \tag{2}$$

- A log-linear relationship between the stress level x_i and the mean time to failure (MTTF) θ_i holds, That is,

$$\log \theta_i = \alpha + \beta x_i, \quad i = 1, 2, \cdots, k.$$

Then, under the assumptions, the PDF and CDF of a test unit are obtained as

$$f(t) = \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] f_i(t - \tau_{i-1}) \quad \text{if} \begin{cases} t \in (\tau_{i-1}, \tau_i], \text{ for } i = 1, 2, \cdots, k-1, \\ t \in (\tau_{k-1}, \infty), \text{ for } i = k \end{cases}$$

$$F(t) = 1 - \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] S_i(t-\tau_{i-1}) \quad \text{if } \begin{cases} t \in (\tau_{i-1}, \tau_i], \text{ for } i = 1, 2, \cdots, k-1, \\ t \in (\tau_{k-1}, \infty), \text{ for } i = k \end{cases}$$

where $f_j(t)$ and $S_j(t)$ are as defined in (1) and (2), respectively. Therefore, the likelihood function of $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_k)$ based on $\boldsymbol{n} = (n_1, n_2, \cdots, n_k)$ and $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_k)$ with $\boldsymbol{y}_i = (y_{i,1}, y_{i,2}, \cdots, y_{i,n_i})$ is obtained as

$$L(\boldsymbol{\theta} \mid \boldsymbol{y}, \boldsymbol{n}) = \left[\prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!}\right] \left[\prod_{i=1}^{k} \theta_i^{-n_i}\right] \exp\left(-\sum_{i=1}^{k} \frac{U_i}{\theta_i}\right), \tag{3}$$

where the Total Time on Test statistic U_i in (3) is defined as

$$U_i = \sum_{l=1}^{n_i} (y_{i,l} - \tau_{i-1}) + (N_i - n_i)\Delta_i, \quad i = 1, 2, \cdots, k.$$

The log-likelihood function is given by

$$l(\boldsymbol{\theta} \mid \boldsymbol{y}, \boldsymbol{n}) = -\sum_{i=1}^{k} n_i \log\theta_i - \sum_{i=1}^{k} \frac{U_i}{\theta_i}. \tag{4}$$

With the log-linear link, the log-likelihood function in (4) can be written for α and β as

$$l(\alpha, \beta \mid \boldsymbol{y}, \boldsymbol{n}) = -\alpha \sum_{i=1}^{k} n_i - \beta \sum_{i=1}^{k} n_i x_i - \sum_{i=1}^{k} U_i e^{-\alpha-\beta x_i}. \tag{5}$$

Upon differentiating (5) with respect to α and β, the likelihood equations are obtained as

$$\frac{\partial}{\partial\alpha} l(\alpha, \beta \mid \boldsymbol{y}, \boldsymbol{n}) = -\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} U_i e^{-\alpha-\beta x_i} = 0, \tag{6}$$

$$\frac{\partial}{\partial\beta} l(\alpha, \beta \mid \boldsymbol{y}, \boldsymbol{n}) = -\sum_{i=1}^{k} n_i x_i + \sum_{i=1}^{k} U_i x_i e^{-\alpha-\beta x_i} = 0. \tag{7}$$

Based on (6) and (7), the MLEs of α and β are obtained by solving the following two equations simultaneously.

$$\hat{\alpha} = \log\left(\frac{\sum_{i=1}^{k} U_i \exp(-\hat{\beta} x_i)}{\sum_{i=1}^{k} n_i}\right), \tag{8}$$

$$\frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i} = \frac{\sum_{i=1}^{k} U_i x_i \exp(-\hat{\beta} x_i)}{\sum_{i=1}^{k} U_i \exp(-\hat{\beta} x_i)}. \tag{9}$$

Theorem 1. *In the case of the step-stress ALT under Type-I censoring with the log-linear assumption, there exist the unique MLEs of α and β.*

Since $\hat{\alpha}$ and $\hat{\beta}$ are nonlinear functions of random variables, statistical inference for (α, β) using the MLEs is based on the asymptotic property that the vector $(\hat{\alpha}, \hat{\beta})$ is approximately distributed as a bivariate normal with mean vector (α, β) and variance-covariance matrix $I_n^{-1}(\alpha, \beta)$, the inverse of Fisher information matrix.

Theorem 2. *In the case of the step-stress ALT under Type-I censoring with the log-linear assumption, the Fisher information matrix is given by*

$$I_n(\alpha, \beta) = n \begin{pmatrix} \sum_{i=1}^k A_i & \sum_{i=1}^k A_i x_i \\ \sum_{i=1}^k A_i x_i & \sum_{i=1}^k A_i x_i^2 \end{pmatrix}$$

with

$$A_i = \left[\prod_{j=1}^{i-1} S_j(\Delta_j) \right] F_i(\Delta_i), \quad i = 1, 2, \cdots, k.$$

Proof of Theorem 1 and 2 is provided in the appendix.

3. Optimality Criteria

In this section, three optimality criteria are considered for designing the optimal step-stress ALT with flexible step durations $\mathbf{\Delta}^* = (\Delta_1^*, \Delta_2^*, \cdots, \Delta_k^*)$. Later, we will compare the resulting optimal designs with those with uniform step durations $\Delta^* = \Delta_1^* = \Delta_2^* = \cdots = \Delta_k^*$. Note that unlike the optimal designs with uniform durations, the optimal step-stress ALT under Type-I censoring with flexible step durations requires a pre-fixed termination time of the test τ_k such that $\sum_{i=1}^k \Delta_i^* = \tau_k$.

3.1 C-optimality

Researchers often wish to maximize the precision and to minimize the variability of the estimate of the parameter of interest. In the step-stress ALT, the MTTF of a test unit at use-condition θ_0 is such a parameter of interest. Motivated by this, our first optimality criterion is the C-optimality, which minimizes the asymptotic variance of the estimator of $\log \theta_0$, defined by

$$\begin{aligned} \phi(\mathbf{\Delta}) &= n\text{AVar}(\log \hat{\theta}_0) = n[1, x_0] I_n^{-1}(\alpha, \beta)[1, x_0]' \\ &= \frac{2 \sum_{i=1}^k A_i (x_i - x_0)^2}{\sum_{i=1}^k \sum_{j=1}^k A_i A_j (x_i - x_j)^2}, \end{aligned} \tag{10}$$

where Avar denotes the asymptotic variance and x_0 is the normal use-stress level. The C-optimal $\mathbf{\Delta}^*$ is obtained by minimizing (10). For the case of simple step-stress ALT ($viz.$, $k = 2$), the objective function $\phi(\mathbf{\Delta})$ in (10) can be reduced to

$$\phi(\Delta_1, \Delta_2) = \frac{(1 + \xi)^2}{A_1} + \frac{\xi^2}{A_2}, \tag{11}$$

where $\xi = (x_1 - x_0)/(x_2 - x_1)$.

Theorem 3. *In the case of a simple step-stress test under Type-I censoring ($viz.$, $k = 2$), there exists a unique optimal step duration (Δ_1^*, Δ_2^*) which minimizes the objective function $\phi(\Delta_1, \Delta_2)$ in (11).*

In the special case of complete sampling as $\tau_k \to \infty$, the k-level step-stress ALT under C-optimality reduces to a simple step-stress ALT with stress levels x_1 and x_k. In this situation, the optimal step duration of the first stage Δ_1^* and the corresponding optimum $\phi(\Delta_1^*)$ are

$$\Delta_1^* = -\theta_1 \log \left(\frac{\xi}{1 + 2\xi} \right) \quad \text{and} \quad \phi(\Delta_1^*) = (1 + 2\xi)^2,$$

respectively, where $\xi = (x_1 - x_0)/(x_k - x_1)$.

3.2 D-optimality

Another optimality criterion is concerned with the joint precision of $(\hat{\alpha}, \hat{\beta})$. Note that the determinant of the Fisher information matrix $|I_n(\alpha, \beta)|$ is proportional to the reciprocal of the volume of the asymptotic joint confidence region of (α, β) at a fixed level of confidence. Hence, the smallest asymptotic joint confidence ellipsoid of (α, β) is obtained by maximizing $|I_n(\alpha, \beta)|$ through selecting the appropriate step durations $\boldsymbol{\Delta}$. Motivated by this, the D-optimal objective function is defined as

$$\delta(\boldsymbol{\Delta}) = n^{-2}|I_n(\alpha, \beta)| = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} A_i A_j (x_i - x_j)^2. \tag{12}$$

The D-optimal step duration $\boldsymbol{\Delta}^*$ is obtained by maximizing $\delta(\boldsymbol{\Delta})$ in (12) for the maximal joint precision of $(\hat{\alpha}, \hat{\beta})$. In the case of simple step-stress ALT ($viz.$, $k = 2$), the objective function $\delta(\boldsymbol{\Delta})$ in (12) reduces to

$$\delta(\Delta_1, \Delta_2) = A_1 A_2 (x_2 - x_1)^2. \tag{13}$$

Theorem 4. *In the case of a simple step-stress test under Type-I censoring (viz., $k = 2$), there exists a unique optimal step duration (Δ_1^*, Δ_2^*) which maximizes the objective function $\delta(\Delta_1, \Delta_2)$ in (13).*

In the special case of complete sampling as $\tau_k \to \infty$, the k-level step-stress ALT under D-optimality reduces to a simple step-stress ALT with stress levels x_1 and x_k. In this situation, the optimal step duration of the first stage Δ_1^* and the corresponding optimum $\delta(\Delta_1^*)$ are

$$\Delta_1^* = \theta_1 \log 2 \quad \text{and} \quad \delta(\Delta_1^*) = \frac{1}{4}\left(x_k - x_1\right)^2,$$

respectively.

3.3 A-optimality

The third optimality criterion considered in this study is based on the first-order approximation of the variance-covariance matrix of the MLEs. It is identical to the sum of the diagonal elements of the inverse of the Fisher information matrix. This criterion provides an overall measure of the average variance of the parameter estimates and gives the sum of the eigenvalues of the inverse of the Fisher information matrix. The A-optimal objective function is defined by

$$a(\boldsymbol{\Delta}) = n\text{tr}\left(I_n^{-1}(\alpha, \beta)\right) = \frac{2\sum_{i=1}^{k} A_i(1 + x_i^2)}{\sum_{i=1}^{k}\sum_{j=1}^{k} A_i A_j (x_1 - x_j)^2}. \tag{14}$$

The A-optimal step-duration $\boldsymbol{\Delta}^*$ is obtained by minimizing $a(\boldsymbol{\Delta})$ in (14) for the minimal average variance of $(\hat{\alpha}, \hat{\beta})$. In the case of simple step-stress ALT ($viz.$, $k = 2$), the objective function $a(\boldsymbol{\Delta})$ in (14) reduces to

$$\begin{aligned} a(\Delta_1, \Delta_2) &= \frac{A_1(1 + x_1^2) + A_2(1 + x_2^2)}{A_1 A_2 (x_2 - x_1)^2} \\ &= \frac{\xi_2^2}{A_1} + \frac{\xi_1^2}{A_2}, \end{aligned} \tag{15}$$

where $\xi_i = \sqrt{1 + x_i^2}/(x_2 - x_1)$ for $i = 1, 2$.

Theorem 5. *In the case of a simple step-stress test under Type-I censoring* ($viz.$, $k = 2$)*, there exists a unique optimal step duration* (Δ_1^*, Δ_2^*) *which minimizes the objective function* $a(\Delta_1, \Delta_2)$ *in (15).*

In the special case of complete sampling as $\tau_k \to \infty$, the k-level step-stress ALT under A-optimality reduces to a simple step-stress ALT with stress levels x_1 and x_k. In this situation, the optimal step duration of the first stage Δ_1^* and the corresponding optimum $a(\Delta_1^*)$ are

$$\Delta_1^* = -\theta_1 \log\left(\frac{\xi_1}{\xi_1 + \xi_k}\right) \quad \text{and} \quad a(\Delta_1^*) = (\xi_1 + \xi_k)^2,$$

respectively, where $\xi_i = \sqrt{1 + x_i^2}/(x_k - x_1)$ for $i = 1, k$.

4. Illustrative Example

Greven *et al.* (2004) provided the step-stress ALT data of 14 fish in their control group. The fish were swum at 15 cm/sec for 90 minutes and then, the flow rate was increased every 20 minutes by 5 cm/sec until fatigue. Table 1 reproduces the experimental data with 2 observations right censored at 150 minutes. Here, the water velocities are the stress levels and the fatigue time of fish is regarded as the unit failure time. It is desired to find the optimal step-stress ALT designs under the C-, D-, and A-optimality criteria discussed in the previous section.

Table 1: The fatigue times of the control group fish ($n = 14$) from the four-level step-stress ALT censored at 150 minutes

Time Interval (minutes)	Flow Rate (cm/sec)	Fatigue Times (minutes)
[0, 90)	15	83.50
[90,110)	20	91.00, 91.00, 97.00, 107.00, 109.50
[110,130)	25	114.00, 115.41, 128.61
[130,150)	30	133.53, 138.58, 140.00, 150+, 150+

First, using (8) and (9), the MLE of (α, β) was computed as $\hat{\alpha} = 9.18$ and $\hat{\beta} = -0.22$. Then, the estimates of MTTF at each stress level were obtained as $\hat{\theta}_1 = 380.29$ min, $\hat{\theta}_2 = 128.99$ min, $\hat{\theta}_3 = 43.75$ min, and $\hat{\theta}_4 = 14.84$ min. With the parameter of interest being the MTTF at the flow rate of 0 cm/sec for the C-optimality, the optimal stress change time points of this step-stress ALT under each optimality criterion are shown in Table 2, Table 3, and Table 4 along with the corresponding values of the objective functions. In comparison to the original step-stress ALT (design A) specified in Table 1, 4 different optimal ALT designs were considered. They are

design B: the step-stress ALT with flexible stress durations under Type-I censoring at the same termination time (150.00 min) of the original ALT (design A);

design C: the step-stress ALT with flexible stress durations under Type-I censoring with the value of the objective function equal to that of the original ALT (design A);

design D: the step-stress ALT with uniform stress durations under Type-I censoring;

design E: the step-stress ALT with flexible stress durations under Type-I censoring at the same termination time of the step-stress ALT under the design D

Table 2: The optimal stress change time points of the step-stress ALT and the corresponding value of $\phi(\cdot)$ under the C-optimality

Design Type	τ_1^*	τ_2^*	τ_3^*	τ_4^*	$\phi(\cdot)$
A. Original Design	90.00	110.00	130.00	150.00	19.66
B. Flexible Duration with $\tau_4 = 150.00$	134.27	–	–	150.00	15.62
C. Flexible Duration with $\phi = 19.66$	100.55	–	–	111.90	19.66
D. Uniform Duration	68.17	136.34	204.51	272.68	26.22
E. Flexible Duration with $\tau_4 = 272.68$	241.20	–	–	272.68	10.66

Table 3: The optimal stress change time points of the step-stress ALT and the corresponding value of $\delta(\cdot)$ under the D-optimality

Design Type	τ_1^*	τ_2^*	τ_3^*	τ_4^*	$\delta(\cdot)$
A. Original Design	90.00	110.00	130.00	150.00	27.10
B. Flexible Duration with $\tau_4 = 150.00$	110.74	–	–	150.00	39.47
C. Flexible Duration with $\delta = 27.10$	69.53	–	–	99.30	27.10
D. Uniform Duration	44.54	89.08	133.62	178.16	20.94
E. Flexible Duration with $\tau_4 = 178.16$	133.91	–	–	178.16	44.58

Table 4: The optimal stress change time points of the step-stress ALT and the corresponding value of $a(\cdot)$ under the A-optimality

Design Type	τ_1^*	τ_2^*	τ_3^*	τ_4^*	$a(\cdot)$
A. Original Design	90.00	110.00	130.00	150.00	19.69
B. Flexible Duration with $\tau_4 = 150.00$	134.24	–	–	150.00	15.65
C. Flexible Duration with $a = 19.69$	100.50	–	–	111.85	19.69
D. Uniform Duration	68.07	136.14	204.21	272.28	26.27
E. Flexible Duration with $\tau_4 = 272.28$	240.81	–	–	272.28	10.69

Figure 2 below visualizes the relationships among these 5 step-stress ALT designs in terms of their termination/censoring times and the corresponding values of the objective functions under each optimality criterion. In each plot, the solid line represents the objective function for the optimal step-stress ALT with flexible stress durations under Type-I censoring while the dashed line represents the objective function for the optimal step-stress ALT with uniform stress durations under Type-I censoring. Regardless of the optimality criteria, it is observed that longer the test duration is, more efficient the optimal step-stress ALT with flexible stress durations is even though the gain in efficiency seems to diminish. It is not so for the optimal step-stress ALT with uniform stress durations, and there exists a unique optimal censoring time point. For this particular study, the optimal objective functions under the C- and A-optimalities are behaving in the same way, producing the almost identical results in Table 2, Table 3, and Table 4.

From the tables, we see that although the original study design (design A) is not an optimal one, it is more efficient and takes shorter to complete compared to the optimal step-stress ALT with uniform stress durations (design D). When the censoring time is fixed to be that of the original ALT (design A), the optimal step-stress ALT with flexible stress durations produces a more efficient test design (design B) as expected. On the other hand, when the value of the objective function is kept equal to that of the original ALT (design A), the optimal step-stress ALT with flexible stress durations produces a test design which takes shorter to complete (design C). The optimal step-stress ALT with uniform stress durations (design D) takes longer but is less efficient despite the fact that it is an optimal one.

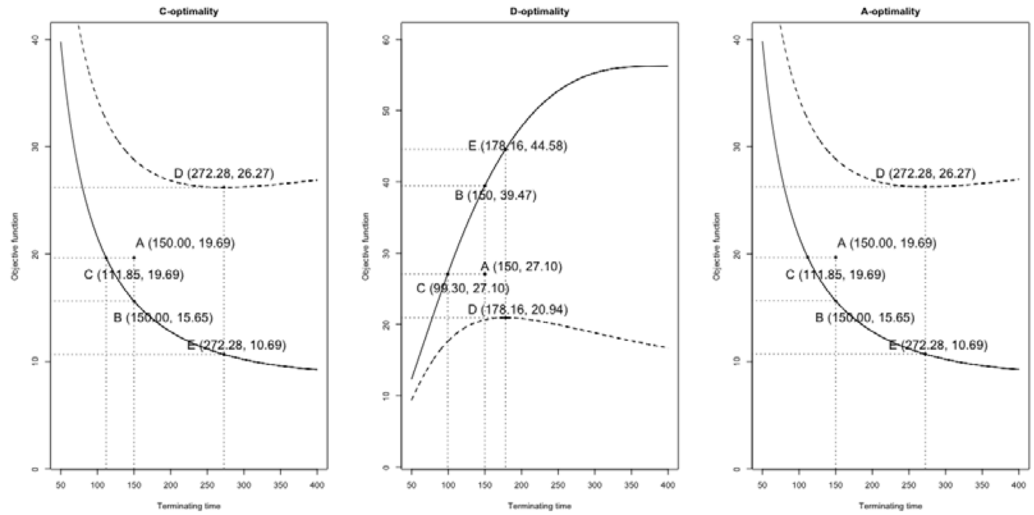

Figure 2: The objective functions under each optimality criterion with respect to the termination time of the step-stress ALT

Hence, under the same censoring time, it is better to implement the optimal step-stress ALT with flexible stress durations (design E). Figure 2 confirms that at the identical censoring time, the optimal step-stress ALT with flexible stress durations is more efficient than the ALT with uniform stress durations in general. Also, regardless of the optimality criteria, at the identical value of the objective function, the optimal step-stress ALT with flexible stress durations takes a shorter time to complete compared to the ALT with uniform stress durations.

Furthermore, the tables exhibit that regardless of the optimality criteria, the optimal step-stress ALT with flexible stress durations (designs B,C,E) requires only two stress levels, which are the lowest and highest stress levels used in the study. This implies that no matter how many stress levels are to be implemented in a study, the optimal test design is a simple step-stress ALT when the log-linear assumption holds between the stress level and the corresponding MTTF. The resulting simple step-stress ALT also assigns a shorter duration to the higher stress level compared to the lower stress level.

Appendix

Proof of Theorem 1: Let us denote the RHS of (9) by $H(\beta; \boldsymbol{x}, \boldsymbol{n}, \boldsymbol{U}) = \frac{\sum_{i=1}^{k} U_i x_i \exp(-\beta x_i)}{\sum_{i=1}^{k} U_i \exp(-\beta x_i)}$ with $\boldsymbol{x} = (x_1, x_2, \cdots, x_k)$ and $\boldsymbol{U} = (U_1, U_2, \cdots, U_k)$. Then, for given $\boldsymbol{x}$, $\boldsymbol{n}$, and $\boldsymbol{U}$, we need to show that $H(\beta; \boldsymbol{x}, \boldsymbol{n}, \boldsymbol{U})$ is a monotone decreasing function of β, and

$$\lim_{\beta \to -\infty} H(\beta; \boldsymbol{x}, \boldsymbol{n}, \boldsymbol{U}) > \frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i} > \lim_{\beta \to \infty} H(\beta; \boldsymbol{x}, \boldsymbol{n}, \boldsymbol{U}).$$

Since the LHS of (9) is a constant, it then follows that $\frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i}$ and $H(\beta; \boldsymbol{x}, \boldsymbol{n}, \boldsymbol{U})$ would intersect exactly once at the MLE of β. To verify that $H(\beta; \boldsymbol{x}, \boldsymbol{n}, \boldsymbol{U})$ is a monotone decreasing function of β, it is sufficient to show that

$$\frac{\partial}{\partial \beta} H(\beta; \boldsymbol{x}, \boldsymbol{n}, \boldsymbol{U}) = \frac{h(\beta; \boldsymbol{x}, \boldsymbol{n}, \boldsymbol{U})}{\left[\sum_{i=1}^{k} U_i \exp(-\beta x_i)\right]^2} \leq 0, \tag{16}$$

where the numerator in (16) is written as

$$h(\beta;\boldsymbol{x},\boldsymbol{n},\boldsymbol{U}) = -\left(\sum_{i=1}^{k} U_i x_i^2 \exp(-\beta x_i)\right)\left(\sum_{i=1}^{k} U_i \exp(-\beta x_i)\right) + \left(\sum_{i=1}^{k} U_i x_i \exp(-\beta x_i)\right)^2. \tag{17}$$

Setting $a_i = x_i\sqrt{U_i \exp(-\beta x_i)}$ and $b_i = \sqrt{U_i \exp(-\beta x_i)}$, $h(\beta;\boldsymbol{x},\boldsymbol{n},\boldsymbol{U})$ in (17) becomes

$$h(\beta;\boldsymbol{x},\boldsymbol{n},\boldsymbol{U}) = -\left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right) + \left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq 0$$

by the Cauchy-Schwarz inequality, which establishes the required property that $H(\beta;\boldsymbol{x},\boldsymbol{n},\boldsymbol{U})$ is indeed a monotone decreasing function of β. Let us now inspect the limit of $H(\beta;\boldsymbol{x},\boldsymbol{n},\boldsymbol{U})$. Note that under the condition $x_1 < x_2 < \cdots < x_k$, we have

$$\lim_{\beta\to-\infty} H(\beta;\boldsymbol{x},\boldsymbol{n},\boldsymbol{U}) = \lim_{\beta\to-\infty} \frac{\sum_{i=1}^{k} U_i x_i \exp(-\beta x_i)}{\sum_{i=1}^{k} U_i \exp(-\beta x_i)} = x_k \geq \frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i}$$

and

$$\lim_{\beta\to\infty} H(\beta;\boldsymbol{x},\boldsymbol{n},\boldsymbol{U}) = \lim_{\beta\to\infty} \frac{\sum_{i=1}^{k} U_i x_i \exp(-\beta x_i)}{\sum_{i=1}^{k} U_i \exp(-\beta x_i)} = x_1 \leq \frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i}.$$

Therefore, (9) has a unique solution $\hat{\beta}$, and by using $\hat{\beta}$, the unique value of $\hat{\alpha}$ can be computed from (8). Hence, there exist the unique MLEs of α and β. □

Proof of Theorem 3: Since $\Delta_1 + \Delta_2 = \tau_2$ with pre-fixed $\tau_2 > 0$, the objective function $\phi(\Delta_1, \Delta_2)$ in (11) can be written as

$$\phi(\Delta_1, \tau_2 - \Delta_1) = \phi(\Delta_1) = \frac{(1+\xi)^2}{A_1} + \frac{\xi^2}{A_2},$$

where $A_1 = 1 - \exp(-\Delta_1/\theta_1)$ and $A_2 = [1 - \exp(-(\tau_2 - \Delta_1)/\theta_2)]\exp(-\Delta_1/\theta_1)$. Taking the derivative of $\phi(\Delta_1)$ with respect to Δ_1, and setting it to zero, we obtain

$$\frac{\left(\frac{\theta_1}{\theta_2} - 1\right)\exp\left(-\frac{\tau_2-\Delta_1}{\theta_2}\right) + 1}{\left[1 - \exp\left(-\frac{\tau_2-\Delta_1}{\theta_2}\right)\right]^2} = \frac{c}{\left[\exp\left(\frac{\Delta_1}{\theta_1}\right) - 1\right]^2}, \tag{18}$$

where $c = (1+\xi)^2/\xi^2$ is a positive constant. Now, let $u_L(\Delta_1)$ and $u_R(\Delta_1)$ represent the LHS and RHS of (18). It is easy to check that $u_L(\Delta_1)$ is an increasing function and $u_R(\Delta_1)$ is a decreasing function in $\Delta_1 \in (0, \tau_2)$ upon $\theta_1 > \theta_2$. Also, since

$$\lim_{\Delta_1\to 0} u_L(\Delta_1) = \frac{\left(\frac{\theta_1}{\theta_2} - 1\right)\exp\left(-\frac{\tau_2}{\theta_2}\right) + 1}{\left[1 - \exp\left(-\frac{\tau_2}{\theta_2}\right)\right]^2} < \lim_{\Delta_1\to 0} u_R(\Delta_1) = \infty$$

and

$$\lim_{\Delta_1\to\tau_2} u_L(\Delta_1) = \infty > \lim_{\Delta_1\to\tau_2} u_R(\Delta_1) = \frac{c}{\left[\exp\left(\frac{\tau_2}{\theta_1}\right) - 1\right]^2},$$

$u_L(\Delta_1)$ and $u_R(\Delta_1)$ intersect in the range of $(0, \tau_2)$. Therefore, there exists the unique value $\Delta_1^* \in (0, \tau_2)$ that satisfies $\phi'(\Delta_1^*) = 0$.

Moreover, the second derivative of $\phi(\Delta_1)$ is obtained as

$$\begin{aligned}\frac{\partial^2\phi(\Delta_1)}{\partial\Delta_1^2} &= \frac{\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)\xi^2\exp\left(\frac{\Delta_1}{\theta_1}-\frac{\tau_2-\Delta_1}{\theta_2}\right)}{\theta_2\left[1-\exp\left(-\frac{\tau_2-\Delta_1}{\theta_2}\right)\right]^2}+\frac{\xi^2\exp\left(\frac{\Delta_1}{\theta_1}-\frac{\tau_2-\Delta_1}{\theta_2}\right)}{\theta_1\theta_2\left[1-\exp\left(-\frac{\tau_2-\Delta_1}{\theta_2}\right)\right]^2} \\ &\quad +\frac{2\xi^2\exp\left(\frac{\Delta_1}{\theta_1}-\frac{2(\tau_2-\Delta_1)}{\theta_2}\right)}{\theta_2^2\left[1-\exp\left(-\frac{\tau_2-\Delta_1}{\theta_2}\right)\right]^3}+\frac{\xi^2\exp\left(\frac{\Delta_1}{\theta_1}\right)}{\theta_1^2\left[1-\exp\left(-\frac{\tau_2-\Delta_1}{\theta_2}\right)\right]} \\ &\quad +\frac{(1+\xi)^2\exp\left(-\frac{\Delta_1}{\theta_1}\right)}{\theta_1^2\left[1-\exp\left(-\frac{\Delta_1}{\theta_1}\right)\right]}+\frac{2(1+\xi)^2\exp\left(-\frac{2\Delta_1}{\theta_1}\right)}{\theta_1^2\left[1-\exp\left(-\frac{\Delta_1}{\theta_1}\right)\right]^3},\end{aligned}$$

where each term is positive when $\Delta_1 \in (0, \tau_2)$. This indicates that $\phi(\Delta_1)$ forms a convex function of Δ_1, and it is minimized by Δ_1^* that satisfies $\phi'(\Delta_1^*) = 0$. □

Proof of Theorem 4: Since $\Delta_1+\Delta_2 = \tau_2$ with pre-fixed $\tau_2 > 0$, the objective function $\delta(\Delta_1, \Delta_2)$ in (13) can be written as

$$\delta(\Delta_1, \tau_2-\Delta_1) = \delta(\Delta_1) = A_1A_2(x_2-x_1)^2,$$

where $A_1 = 1-\exp(-\Delta_1/\theta_1)$ and $A_2 = [1-\exp(-(\tau_2-\Delta_1)/\theta_2)]\exp(-\Delta_1/\theta_1)$. Taking the derivative of $\log\delta(\Delta_1)$ with respect to Δ_1, and setting it to zero, we obtain

$$\frac{\partial\log\delta(\Delta_1)}{\partial\Delta_1} = \frac{1/\theta_1}{\exp\left(\frac{\Delta_1}{\theta_1}\right)-1}-\frac{1}{\theta_1}-\frac{1/\theta_2}{\exp\left(\frac{\tau_2-\Delta_1}{\theta_2}\right)-1} = 0,$$

from which we can drive

$$2\exp\left(\frac{\tau_2-\Delta_1}{\theta_2}\right)+\left(1-\frac{\theta_1}{\theta_2}\right)\exp\left(\frac{\Delta_1}{\theta_1}\right)+\left(\frac{\theta_1}{\theta_2}-2\right) = \exp\left(\frac{\tau_2}{\theta_2}-\left(\frac{1}{\theta_2}-\frac{1}{\theta_1}\right)\Delta_1\right). \tag{19}$$

Let $u_L(\Delta_1)$ and $u_R(\Delta_1)$ represent the LHS and RHS of (19), respectively. With $\theta_1 \geq \theta_2$, it is easy to show that both $u_L(\Delta_1)$ and $u_R(\Delta_1)$ are decreasing in $\Delta_1 \in (0, \tau_2)$. Also, since

$$u_L(0) = 2\exp\left(\frac{\tau_2}{\theta_2}\right)-1 > u_R(0) = \exp\left(\frac{\tau_2}{\theta_2}\right)$$

and

$$u_L(\tau_2) = \left(1-\frac{\theta_1}{\theta_2}\right)\exp\left(\frac{\tau_2}{\theta_1}\right)+\frac{\theta_1}{\theta_2} < u_R(\tau_2) = \exp\left(\frac{\tau_2}{\theta_1}\right),$$

$u_L(\Delta_1)$ and $u_R(\Delta_1)$ intersect in the range of $(0, \tau_2)$. Therefore, there exists the unique value $\Delta_1^* \in (0, \tau_2)$ that satisfies $\delta'(\Delta_1^*) = 0$.

Moreover, the second derivative of $\log\delta(\Delta_1)$ is obtained as

$$\frac{\partial^2\log\delta(\Delta_1)}{\partial\Delta_1^2} = -\frac{Q}{\theta_1^2\theta_2^2\left[\exp\left(\frac{\tau_2-\Delta_1}{\theta_2}\right)-1\right]^2\left[\exp\left(\frac{\Delta_1}{\theta_1}\right)-1\right]^2},$$

where

$$
\begin{aligned}
Q &= \theta_1^2 \exp\left(\frac{2\Delta_1}{\theta_1} + \frac{\tau_2 - \Delta_1}{\theta_2}\right) + \theta_1^2 \exp\left(\frac{\tau_2 - \Delta_1}{\theta_2}\right) \\
&\qquad + \left[\theta_2^2 \exp\left(\frac{2(\tau_2 - \Delta_1)}{\theta_2}\right) - 2(\theta_1^2 + \theta_2^2)\exp\left(\frac{\tau_2 - \Delta_1}{\theta_2}\right) + \theta_2^2\right] \exp\left(\frac{\Delta_1}{\theta_1}\right) \\
&= \theta_1^2 \exp\left(\frac{\tau_2 - \Delta_1}{\theta_2}\right)\left[\exp\left(\frac{\Delta_1}{\theta_1}\right) - 1\right]^2 \\
&\qquad + \theta_2^2 \exp\left(\frac{\Delta_1}{\theta_1}\right)\left[\exp\left(\frac{\tau_2 - \Delta_1}{\theta_2}\right) - 1\right]^2 > 0.
\end{aligned}
$$

Since $\frac{\partial^2 \log \delta(\Delta_1)}{\partial \Delta_1^2} < 0$, $\delta(\Delta_1)$ forms a concave function of Δ_1, and it is maximized by Δ_1^* that satisfies $\delta'(\Delta_1^*) = 0$. □

Proof of Theorem 5: Since $\Delta_1 + \Delta_2 = \tau_2$ with pre-fixed $\tau_2 > 0$, the objective function $a(\Delta_1, \Delta_2)$ in (15) can be written as

$$
a(\Delta_1, \tau_2 - \Delta_1) = a(\Delta_1) = \frac{\xi_2^2}{A_1} + \frac{\xi_1^2}{A_2},
$$

where $A_1 = 1 - \exp(-\Delta_1/\theta_1)$ and $A_2 = [1 - \exp(-(\tau_2 - \Delta_1)/\theta_2)]\exp(-\Delta_1/\theta_1)$. Since $a(\Delta_1)$ is similar to $\phi(\Delta_1) = \phi(\Delta_1, \Delta_2)$ in (11), a similar argument can be used to prove the unique existence of Δ_1^* that minimizes $a(\Delta_1)$. □

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