# Conditional Median Run Length as Performance Measure of Shewhart $S^2$ Control Chart with Estimated Parameter

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**Abstract**
Several recent works, in Statistical Monitoring Process, have addressed the effect of parameter estimation on the performance of control charts. Although the measures of this performance are generally expressed based on particular associated characteristics of the unconditional run length ($RL$) distribution, recently, some researchers have proposed the use of the conditional $RL$ distribution that is considered more meaningful in real applications. The most traditional performance indicator is the average $RL$ ($ARL$). Nevertheless, this expected value is not a good representative measure for the center because the $RL$ distribution is usually right-skewed. Therefore, we study the $q$-quantiles of the conditional in-control run length ($CICRL_q$) of the $S^2$ and $S$ control charts. It is worth to note that these quantiles, the distribution of which is obtained analytically, are random variables when the parameter of the $IC$ process dispersion is estimated. One specific quantile is proposed: the conditional in-control median run length $CICMRL$ ($CICRL_{0.50}$). We analyze and compare the entire distribution of the $CICMRL$ and the conditional $IC$ $ARL$ ($CICARL$). The results show that the variability of the $CICMRL$ is smaller than the $CICARL$. Moreover, the $CICMRL$ and $CICARL$ requires similar minimum values of number of Phase I samples ($m$), to guarantee a specified minimum limit (a percentage of its nominal value) with a high probability. For that reason, the proposed robust $CICMRL$ may be a good alternative performance measure.

**Key Words:** $S^2$ and $S$ Control Chart, Conditional Control Chart Performance, Run Length Distribution, Run Length Quantiles, Median Run Length ($MRL$), Average Run Length ($ARL$)

## 1. Introduction

The $S^2$ control chart is a well-known and widely used tool to monitor the variability of a quality characteristic of interest in manufacturing and service processes. The run length ($RL$) of a control chart is a discrete random variable that can be defined as the number of collected samples (subgroups) until the occurrence of the first out-of-control ($OOC$) signal. The control chart performance is usually measured in terms of its run length ($RL$) distribution. A customary metric of the control charts performance has been the average run length ($ARL$) (see, for example, Montgomery (2012)). It represents “the mean value” of the $RL$ distribution. However, since the $RL$ distribution is not symmetrical, but rather is generally right -skewed (see, for instance, Teoh et al. (2016)), the $ARL$ is not anymore a reasonable summary measure of centrality. For that reason, several researchers (e.g., see Chakraborti (2007), Mei (2008), Graham et al. (2014), Teoh et al. (2016)) have pointed out some disadvantages of the use of the $ARL$ as a unique performance metric, so the use of some representative $RL$ quantiles (or percentiles), such as the median, have been proposed. It is well known that in the case of a right-skewed distribution, the extreme values can impact on the corresponding mean because it is pulled in the direction of the longest distribution tail and therefore it could end up excessively extended. Hence, as the $RL$ distribution is generally right-skewed, the study of the complete $RL$ distribution is fundamental. For one thing, the median run length ($MRL$) is preferred since it is more robust central measure than the $ARL$. For another, using $RL$ quantiles, summary measures for the variation or spread of the $RL$ distribution, such as the interquartile range and the difference between two opposed and extreme percentiles (e.g., $95th$ and $5th$) can be obtained.

Barnard (1959) and Bissell (1969) started to work on the quantiles of the $RL$ distribution as performance measures of control charts. The study of the main percentiles of the $RL$ distribution for the Shewhart $\bar{X}$ chart, when the process parameters are specified or known, was dealt by Khoo (2004). He also proposed the percentiles of the time to signal and the percentiles of the number of individual units sampled in place of the traditional $ARL$. Radson and Boyd (2005) presented a graphic display of the $RL$ distribution that shows the main representative percentiles like a boxplot. Chakraborti (2007), contrary to Khoo (2004), examined the $RL$ distribution when the process parameters are unknown. He put forward a design of control charts based on the $MRL$. In the same direction, others authors have advocated this $MRL$ criterion for applying on different types of control charts, for instance, Lee and Khoo (2006), Khoo et al. (2011), Teoh et al. (2014, 2016), among others. These researches just considered the unconditional (or marginal) $RL$ distribution.

The $RL$ of Shewhart chart, when the process parameters are known or specified, follows a geometric distribution with parameter equal to the probability of a signal. For the case of $IC$ process, this parameter is the constant $\alpha$ (the probability of a false alarm or also called the false alarm rate ($FAR$)). Nevertheless, this is an unrealistic situation because the $IC$ process parameters are generally unknown in practical application. So these process parameters have to be estimated from the Phase I reference, and then the (unconditional) $IC$ $RL$ no longer follows a geometric distribution and its corresponding parameter ($FAR$) changes into a random variable. This reference sample is collected over time in Phase I analysis (e.g., see Chakraborti et al. (2009) and Jones-Farmer et al. (2014)) in order to estimate the $IC$ process parameters, and subsequently, to estimate the control limits and monitoring the process in Phase II analysis. Since parameters are estimated, the $S^2$ chart performance may be extremely different from the parameters known case one (see, for instance, Chen (1998), Castagliola (2009), Epprecht et al. (2015)).

Most of the works about the effect of parameter estimation on the performance of control charts focus on the unconditional $RL$ distribution (an overview of the related literature can be found in Jensen et al. (2006) and Psarakis et al. (2014)). However, the use of this unconditional distribution can be considered as conceptual and not practical. Indeed, theoretically, the unconditional $RL$ distribution and its associated measures can be determinate using expectation of their conditional counterpart, for instance, the unconditional $ARL$ of the control chart for dispersion can be thought of as the value obtained by averaging $ARL$ given the standard deviation estimator over its distribution. In this way, the practitioner-to-practitioner effect (i.e., the variability among different reference samples of practitioners even when coming from the same $IC$ process) is averaged and is not considered in this unconditional distribution. It is important to remark that it is contrary to the real applications because the user will have a single reference sample to estimate the process parameter(s), the control limits, and the performance of the resulting chart. In this context, Epprecht et al. (2015) and Faraz et al. (2015) for the case of the $S^2$ control chart, have claimed and argued that the use of the conditional $RL$ distribution (i.e., the $RL$ distribution given an certain reference sample, or given the process parameter(s) estimated with that sample, or given the chart limit(s) estimated using that (or those) process parameter(s) estimated) is more proper and more real to the practitioner since a specific single reference sample is used and practitioner-to-practitioner variability is considered.

Since the $IC$ chart performance depends on its right-skewed $IC\ RL$ distribution, it can be better described regarding the use of the $IC\ RL$ quantiles. Then, the study of them in this paper can enables us to obtain valuable information and an in-depth understanding of the Phase II chart performance. Hence, from a practical point of view, this work is looking for a more insight into the $IC$ performance of the Shewhart $S^2$ chart when the process variance is estimated, seeking to take advantage of the characteristics of the conditional $IC\ RL$ distribution and quantiles that account the effects of the amount of $IC$ Phase I reference sample and the practitioner-to-practitioner variability. We propose two performance measures related to the conditional $IC\ RL$ quantiles, namely, the $MRL$ and the $RL$ $0.05$-quantile, and we argue that the use of them may be more proper and more robust than the $CARL$. The scope of this paper is limited to the case of the one-sided $S^2$ chart with a probability upper control limit ($UCL_{S^2}$) that is calculated regarding a specified probability of type I error or probability of a false alarm ($\alpha$).

This work is organized as follows: Section 2 presents the $RL$ of Shewhart $S^2$ chart and its properties in both the known and unknown $IC$ process variance cases. In Section 3, based on analytical derivations, the general expression of the conditional $IC\ MRL$ ($CICMRL$ or $CICRL_{0.5}$) and its corresponding distribution are provided. In Section 4, we obtain the required minimum $m$ to guarantee a specific minimum $CICMRL$ with a high probability. The general conclusions are given in Section 6.

## 2. The Run Length of Shewhart $\mathbf{S^2}$ Control Chart

### 2.1 The Run Length of Shewhart $\mathbf{S^2}$ Control Chart when the IC process variance is known or specified

Once the $UCL_{S^2}$ is calculated using the known $IC$ process variation $\sigma_0^2$, the process variability of the future production can be examined in the Phase II monitoring process. We suppose that $n$ ($n > 1$) is the size of each sample (subgroup) taken from Phase II process. These $n$ observations of each sample are independent and identically distributed (iid) and follow a normal distribution with mean $\mu$ and variance $\sigma^2$ ($X_{ij} \sim N(\mu, \sigma^2)$, where $X_{ij}$ is the jth observation of the measure quality characteristic ($j = 1, 2, \dots, n$) in the ith sampling time). A standard deviation shift may occur during a given moment in Phase II monitoring process. Then, in order to figure out the effect of this shift on the Phase II performance, we define the standard deviation ratio ($\gamma$):

$$\gamma = \sigma/\sigma_0 \tag{1}$$

When the Phase II process variability is IC (a state of statistical control just related to the process variability), then $\sigma = \sigma_0$ and $\gamma = 1$. On the other hand, if the process standard deviation increases ($\sigma > \sigma_0$) at a given moment during Phase II monitoring process because of the existence of any special or assignable cause, then $\gamma > 1$ and the process is OOC. Likewise, if the process standard deviation decreases ($\sigma < \sigma_0$), then $\gamma < 1$. The ith chart point (plotting statistic obtained using the ith sampling time) in the Phase II monitoring process represents the corresponding ith sample variance $S_i^2$. So, given a specified value of the nominal $FAR\ (\alpha)$, each $S_i^2$ value can be compared with the $UCL_{S^2}$ value in order to detect an OOC signal. $UCL_{S^2}$ is obtained from Equation (2) as follow (see, for example, Montgomery (2012)):

$$UCL_{S^2} = \frac{\chi^2_{n-1,1-\alpha}}{n-1}\sigma_0^2 \tag{2}$$

Where $\chi^2_{n-1,1-\alpha}$ denotes the$1-\alpha$ () quantile of the distribution of a chi-squared random variable with $n-1$ degrees of freedom (df), that is, $F^{-1}_{\chi^2_{n-1}}(1-\alpha)$. Let $A_i$ be the ith signalling event, namely, the event that $S_i^2$ exceeds the $UCL_{S^2}$, whose probability of occurrence is called the probability of a signal (an alarm) $p_s$, i.e., the probability that any chart point fall outside the upper control limit ($P(A_i) = P(S_i^2 > UCL_{S^2}) = p_s$). Then, using Equations (1) and (2), we have $p_s = P\left(S_i^2 > \frac{\chi^2_{n-1,1-\alpha}}{n-1}\sigma_0^2\right) = P\left(\frac{(n-1)S_i^2}{\sigma^2} > \chi^2_{n-1,1-\alpha}\frac{\sigma_0^2}{\sigma^2}\right)$. It is known that $(n-1)\,S_i^2/\sigma^2$ follows a chi-square distribution with $n-1$ df. So, it can be reduced as

$$p_s = P\left(S_i^2 > UCL_{S^2}\right) = 1 - F_{\chi^2_{n-1}}\left(\frac{\chi^2_{n-1,1-\alpha}}{\gamma^2}\right) \tag{3}$$

Where $F_{\chi^2_{n-1}}$ is the cumulative distribution function (cdf) of the chi-square random variable with $n-1$ df. The RL is a discrete random variable that is defined as the number of collected samples from the Phase II process until the occurrence of the first signaling event (including the sample that causes this event), and it is considered as a paramount performance measure of the control charts. It is widely known that the RL of the $S^2$ control chart follows a geometric distribution with parameter $p_s$ (see, for example, Montgomery (2012)) since, in the first place, the samples of the Phase II monitoring process are independent (then, the samples variances $\{S_i^2\}$ are also independent random variables and the corresponding signalling events $\{A_i\}$ are independent) and, in the second, $n, \gamma, \alpha$ and $\sigma_0$ are known or specified values (thus, from Equations (2) and (3), the control limits and $p_s$ are constant known values). Hence, we can figure out the performance of control charts for dispersion through the properties of the geometric distribution. Then, the probability mass function (pmf) and the cdf of the $RL$ ($f_{RL}(l)$ and $F_{RL}(l)$, respectively) may be obtained as:

$$f_{RL}(l) = P(RL = l) = p_s(1-p_s)^{l-1} \tag{4}$$

$$F_{RL}(l) = P(RL \le l) = 1 - (1-p_s)^l \tag{5}$$

Where $l = \{1,2,3, \dots\}$. If the process is $IC$ ($\gamma = 1$), $p_s$ is the probability of a false alarm ($p_s = nominal\,FAR = \alpha$), otherwise ($OOC$ process with $\gamma > 1$) $p_s$ is called the probability of a true alarm or the power of the chart $p_s = 1-\beta$ (, where $\beta$ is the type II error probability). Thus, in both situation (IC and OOC Phase II process), given specified values of $(\alpha, n, \gamma)$, the parameter of the RL distribution ($p_s$) is a constant known value that can be obtained from Equation (3). Typically, the mean and the standard deviation of the RL (ARL and SDRL) are examined to characterize the performance of the chart. Thus,

$$E(RL) = ARL = \frac{1}{p_s} \tag{6}$$

$$SD(RL) = SDRL = \sqrt{\frac{(1-p_s)}{p_s^2}} \tag{7}$$

As was explained earlier, both measures should not be used as the singles indicators of the control chart because of the shape of the RL distribution. Then, the RL quantiles provide relevant and meaningful additional information to clarify the description of the RL through summary measures for the center and for the variation of its distribution. The $q$-quantile (or the $100$qth percentile, where $0 < q < 1$) of the run length ($RL_q$) is defined as the smallest integer so that the cdf at $RL_q$ is at least equal to q (i.e., $F_{RL}(RL_q) = P(RL \leq RL_q) \geq q$). Next, using Equation (5), $1-(1-p_s)^{RL_q} \geq q$. Thus,

$$RL_q = \left\lceil \frac{ln(1-q)}{ln(1-p_s)} \right\rceil \tag{8}$$

Where $\lceil a \rceil$ denotes the smallest integer greater or equal to $a$, then $RL_q = \{1,2,3,\ldots\}$. For example, when the process is $IC$ and $\alpha = 0.0027$, the 0.5-quantile of the in-control run length ($ICRL_{0.5}$), or also called the in-control median run length ($ICMRL$), is obtained from Equation (8), using $q = 0.5$ and $p_s = \alpha = 0.0027$. Thus, the $ICMRL = 257$. In addition, using Equation (6), the in-control average run length ($ICARL$) is equal to $370.4$. Both performance measures are quite different and the practitioners should avoid misleading and confusing interpretations. The $ICARL$ just points out that a false alarm will be detected, on average approximately, by the $370$th sample, but it does not provide any likelihood of this occurring. Conversely, the $ICMRL$ indicates that, at least 50% of the time, a false alarm will be detected within the first 257 samples.

In this work, as well as in several other researches (see, e.g., Khoo (2004), Chakraborti (2007) and Teoh et al. (2016)), the $RL_q$ is defined in such a way that the cdf at this q-quantile is at least (and not just exactly) equal to q. It is due to the RL is a discrete random variable (as was defined above). Thus, $P(RL \leq RL_q - 1) < q \leq P(RL \leq RL_q)$,.

### 2.2 The Run Length of Shewhart $S^2$ Control Chart when the $IC$ process variance is unknown

The IC process parameters are unknown in most often real control chart applications. As a result, the control limit (see Equation (2)) is no longer a known constant value and need to be estimated based on the estimator of the IC process variance $\hat{\sigma}_0^2$, which, in turn, is obtained using a specific reference sample taken from the IC Phase I process. We consider mn as the number of Phase I reference sample observations $\{X_{Iij}\}$ such that m is the number of independent random samples each of size $n$ (these observations are iid, $X_{Iij} \sim N(\mu_0, \sigma_0^2)$, where $i = \{1, 2, \ldots, m\}$ and $j = \{1, 2, \ldots, n\}$).

After the control limit of the $S^2$ Control Chart is estimated (using Equation (10)), the Phase II monitoring process can be started (prospective) using independent samples (plotting statistics) of the same size of the Phase I samples (n). We assume that the observations of each sample are iid in Phase II process, and follow a normal distribution with an unknown standard deviation σ. Moreover, as mentioned before, the performance of the $S^2$ Control Chart is affected by the variance estimation. Next, we need to measure this estimation effect. So k is defined as the error factor of the estimate $\hat{\sigma}_0^2$:

$$k = \hat{\sigma}_0^2 / \sigma_0^2 \tag{9}$$

Given a control chart application, we have

$$\widehat{UCL}_{S^2} = \frac{\chi^2_{n-1,1-\alpha}}{n-1} \hat{\sigma}_0^2 \tag{10}$$

$$P(B_i) = P\left(S_i^2 > \widehat{UCL}_{S^2}\right) = p_s \tag{11}$$

Because the random variables $(S_i^2 - \widehat{UCL}_{S^2})$ and $(S_j^2 - \widehat{UCL}_{S^2})$ have $\hat{\sigma}_0^2$ in common, the signaling events $B_i$ and $B_j$ are dependent for $i \neq j$. Hence, the distribution of the (unconditional) RL is no longer geometric. Next, using Equations (1), (9) and (10), the $p_s$ from Equation (11) can be expressed as $p_s = P\left(S_i^2 > \frac{\chi^2_{n-1,1-\alpha}}{n-1} \hat{\sigma}_0^2\right) = P\left(\frac{(n-1)S_i^2}{\sigma^2} > \chi^2_{n-1,1-\alpha} \frac{\hat{\sigma}_0^2}{\sigma_0^2} \frac{\sigma_0^2}{\sigma^2}\right)$, which can be reduced as follow

$$p_s = P\left(S_i^2 > \widehat{UCL}_{S^2}\right) = 1 - F_{\chi^2_{n-1}}\left(\frac{k}{\gamma^2} \chi^2_{n-1,1-\alpha}\right) \tag{12}$$

The value of $p_s$ can be obtained unconditionally, i.e., averaging $p_s$ (given $k$) over the distribution of $k$ ($p_s = E_{k^2}[p_s \mid k]$), or equivalently, by averaging over all possible values of the estimation of $\sigma_0^2$ ($\hat{\sigma}_0^2$) that are calculated using a very large amount of reference samples collected from the same IC Phase I process. In a similar way, we can determinate the unconditional RL distribution and its associated measures for the estimated process variance case (see, for example, Chen (1998), Maravelakis et al. (2002) and Castagliola et al. (2009)). However, because the users just have a single IC Phase I reference sample to estimate the process parameter(s) in real applications and given the practitioner-to-practitioner variability, the value of $p_s$ obtained conditionally on $k$ (or $\hat{\sigma}_0^2$) is more proper and more practical to the control chart users. So, from Equation (12), $p_s$ is a function of the random variable k, then the $p_s$ given a specific Phase I estimator of $\sigma_0$ ($\hat{\sigma}_0$) is called the conditional probability of a signal (CPS)

$$CPS(\hat{\sigma}_0^2, \gamma) = P\left(S_i^2 > \widehat{UCL}_{S^2} \mid \hat{\sigma}_0^2\right) = 1 - F_{\chi^2_{n-1}}\left(\frac{k}{\gamma^2} \chi^2_{n-1,1-\alpha} \mid k\right) \tag{13}$$

The random variable CPS can assume a particular realization given a value of $\hat{\sigma}_0^2$ which, in turn, is computed using a specific IC Phase I reference sample. Hence, this particular value of CPS is the geometric parameter (i.e., the probability of a success) of the conditional run length (CRL) distribution. Thus, the cdf of the CRL is

$$F_{CRL}(t) = P(CRL \le t) = P(RL \le t \mid \hat{\sigma}_0^2) = 1 - \left(1 - CPS(\hat{\sigma}_0^2, \gamma)\right)^t$$
$$= 1 - \left(F_{\chi^2_{n-1}}\left(\frac{k}{\gamma^2}\chi^2_{n-1,1-\alpha} \mid k\right)\right)^t$$

We considered the traditional unbiased estimator of the $\sigma_0^2$, i.e., the pooled sample variance ($S_p^2$):

$$\hat{\sigma}_0^2 = S_p^2 = \frac{\sum_{i=1}^m S_{Ii}^2}{m} \qquad (14)$$

Where $S_{Ii}^2 = \frac{1}{n-1}\sum_{j=1}^n \left(X_{Iij} - \bar{X}_{Ii}\right)^2$ and $\bar{X}_{Ii} = \frac{1}{n}\sum_{j=1}^n X_{Iij}$ are the sample variance and the sample mean that are obtained in the IC Phase I process from a specific reference sample. Then, we can take advantage of the fact that when $\hat{\sigma}_0^2 = S_p^2$, the random variable Y ($Y = m(n-1)\, S_p^2/\sigma_0^2$) follows a chi-square distribution with $m(n-1)$ df. In addition, we focus on CRL when the process is IC, it means that the standard deviation is the same in Phase II and Phase I processes ($\sigma = \sigma_0$ and $\gamma = 1$). Then, we address the conditional in-control run length ($CICRL$) of the $S^2$ chart that follows a geometric distribution and whose parameter is one realization of the conditional false alarm rate ($CFAR$), that is, from Equation (13), $CFAR(\hat{\sigma}_0) = CPS(\hat{\sigma}_0^2 = S_p^2, \gamma = 1)$. Thus, the random variable CFAR can be rewritten as

$$CFAR(Y) = 1 - F_{\chi^2_{n-1}}\left(\frac{S_p^2}{\sigma_0^2}\chi^2_{n-1,1-\alpha}\right) = 1 - F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1,1-\alpha}\right) \qquad (15)$$

Epprecht et al. (2015) presented a thorough examination of the behavior of $CFAR$ (by a particular $S^2$ control chart with a specific estimator $\hat{\sigma}_0^2$ or Y) and its distribution and quantiles, while we study the quantiles of the CICRL as a performance measures, which also represent the characteristics of the control charts. The $CICRL$ and its associated measures can be derived through the known properties of the geometric distribution. Then, the $CICRL$ distribution is obtained as the cdf of the ICRL conditioned on (or given) $\hat{\sigma}_0^2$ (or equivalently, conditionally on the corresponding $Y$). Thus, similar to Equation (5), the cdf of the $CICRL$ can be expressed as

$$F_{CICRL}(t) = P(CICRL \le t) = P(ICRL \le t \mid \hat{\sigma}_0^2) = P(ICRL \le t \mid Y)$$
$$= 1 - \left(1 - CFAR(Y)\right)^t$$

Where $t = \{1,2,3, \dots\}$. Then, substituting Equation (15) in the latter expression, the cdf of the CICRL ($F_{CICRL}$) can be rewritten as

$$F_{CICRL}(t) = P(CICRL \le t) = 1 - \left(F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1,1-\alpha}\right)\right)^t \qquad (16)$$

Note that, from Equation (16), the cdf of $CICRL$, given a specified value of α, depends on the random variable Y which, in turn, depends on the error factor of the estimate $\hat{\sigma}_0^2$ ($k = S_p^2/\sigma_0^2$) and its distribution depends on the number ($m$) and size (n) of the Phase I reference sample, namely, $Y \sim \chi^2_{m(n-1)}$. The pmf, the conditional in-control average run length ($CICARL$) and the conditional in-control standard deviation run length ($CICSDRL$) may be obtained straightaway by substituting $CFAR$ (Equation (15)) for $p_s$ in Equations (4), (6) and (7), respectively. Next, in Section 3, we will evaluate the quantiles of the $CICRL$.

For the case of the $RL$ properties of the Shewhart $\bar{X}$ Chart (pmf, cdf, $ARL$, $SDRL$ and quantiles) when the process parameters are known were examined and studied, for example, by Khoo (2004). On the other hand, the case when parameters are unknown was addressed, for example, based on the unconditional RL distribution, by Chakraborti (2006, 2007).

## 3. Conditional In-Control Run Length q-Quantile ($\mathrm{CICRL_q}$) of the Shewhart $\mathbf{S^2}$ and S Control Charts

### 3.1 The expression of $CICRL_q$

The study of the $RL$ distribution and its quantiles provide more information and a better understanding of $S^2$ and $S$ charts performance. Since the $CICRL$, by definition, is a discrete random variable as the $CRL$, its $q$-quantile ($CICRL_q$), where $0 < q < 1$, is defined as the smallest positive integer $CICRL_q$ so that the cdf at $CICRL_q$ is greater or equal to $q$ ($F_{CICRL}\left(CICRL_q\right) \geq q$). Therefore, similar to the procedure for obtaining the expression of Equation (8) in the known process variance case, the $CICRL_q$ can be calculated using Equation (16), as being the smallest integer that satisfies: $1 - \left(F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1,\ 1-\alpha}\right)\right)^{CICRL_q} \geq q$. Thus,

$$CICRL_q = \left\lceil \frac{ln(1-q)}{ln\left(F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1,\ 1-\alpha}\right)\right)} \right\rceil \tag{17}$$

Where $\lceil b \rceil$ denotes the smallest integer greater or equal to $b$, then $CICRL_q = \{1, 2, 3, \ldots\}$. Thus, from Equation (17), we can obtain and examine any quantile of the $CICRL$. Our work proposes the use of the $CICRL_{0.50}$ (i.e., the conditional in-control median run length ($CICMRL$)). As was mentioned earlier, since the $IC\ RL$ is right-skewed, the $ICMRL$ is more proper and more useful summary measure of centrality than the $ICARL$. Then, we consider the $CICMRL$. It indicates that the first false alarm is detected after the $CICMRLth$ sample with approximately (a little less than) 50% probability. Put another way, it means that approximately 50% of the conditional $IC\ RL's$ will be larger than the $CICMRL$ (for a very large number of reference sample taken from the same $IC$ process or, equivalently, for a very large amount of control chart constructed with the corresponding reference samples).

### 3.2 The $CICRL_q$ distribution

The probability mass function (pmf) and the cumulative distribution function (cdf) of the Conditional $IC\ RL$ $q$-Quantile ($CICRL_q$), that is, $f_{CICRL_q}(\tau)$ and $F_{CICRL_q}(t)$, are presented in this section. We can find the pmf of $CICRL_q$ as

$$f_{CICRL_q}(\tau) = P\left(CICRL_q = \tau\right)$$

$$= F_{\chi^2_{m(n-1)}}\left(\frac{m(n-1)\chi^2_{n-1,(1-q)^{1/\tau}}}{\chi^2_{n-1,\,1-\alpha}}\right) - F_{\chi^2_{m(n-1)}}\left(\frac{m(n-1)\chi^2_{n-1,\,(1-q)^{1/(\tau-1)}}}{\chi^2_{n-1,\,1-\alpha}}\right), \tau = \{1,2,3,\dots\} \quad (18)$$

Note that, from Equation (18), the pmf of $CICRL_q$ depends on $m$, $n$ and $\alpha$ values. We can obtain the pmf of the $CICMRL$, substituting $q = 0.50$ ($f_{CICMRL}(\tau)$) into Equation (18).

The pmfs of the proposed performance measures is displayed in Figures 1 (for $m = \{25, 50, 100, 200, 1000\}$, $n = 5$, $\alpha = 0.0027$). We can note that, the right-skewed shape of the $CICMRL$ distribution is more pronounced for small values of $m$. For another side, when the value of $m$ increases, the pmf curve tends to be more peaked and its modal value (the $CICMRL$ value at which its pmf takes its maximum value) tends to the corresponding nominal value, i.e., $ICMRL = 257$. For instance, the $f_{CICMRL}(\tau)$ (see Figure 1), the modal values are $91,\ 147,\ 192,\ 222,\ 249$ for the five considered values of $m$.

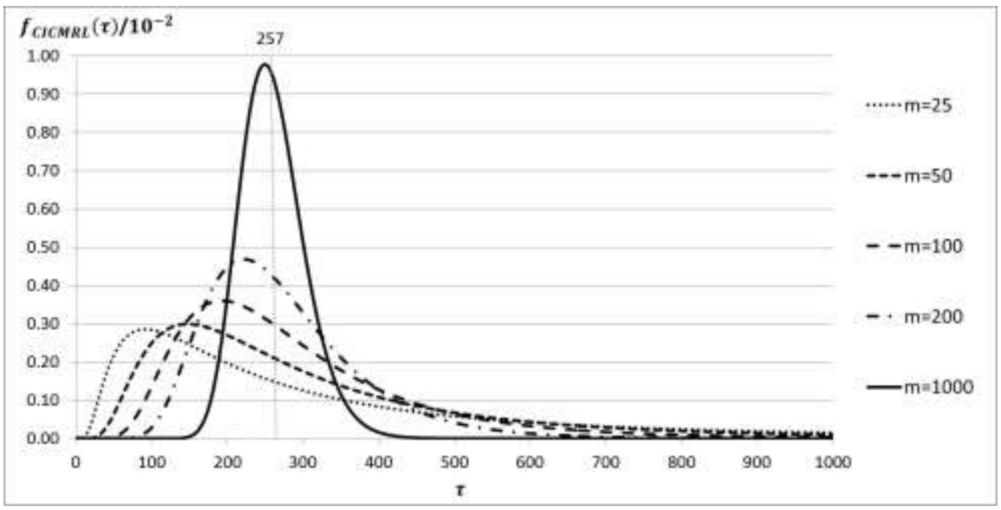


**Figure 1**: $f_{CICMRL}(\tau)$: Probability mass function (pmf) envelope of the Conditional $IC$ $MRL$ ($CICMRL$) for $m = \{25, 50, 100, 200, 1000\}$, $n = 5$ and $\alpha = 0.0027$ (nominal $ICMRL\ = 257$).

Note that since the $CICRL_q$ is a discrete random variable (positive integer values), its pmf is zero for all non-integer values of $CICRL_q$. Thus, the pmf is a non-continuous function, but we decided to draw the curve connecting the points (pmf envelope, see Figure 1) in order to make easier the visualization of important patterns of pmf.

Next, we can find the cdf of $CICRL_q$ as

$$F_{CICRL_q}(t) = P\left(CICRL_q \le t\right) = \begin{cases} 0,\ t < 1 \\ F_{\chi^2_{m(n-1)}}\left(\frac{m(n-1)\chi^2_{n-1,\,(1-q)^{1/\lfloor t \rfloor}}}{\chi^2_{n-1,\,1-\alpha}}\right),\ t \ge 1 \end{cases} \quad (19)$$

Where $t$ is a real value and $\lfloor t \rfloor$ denotes the largest integer less or equal to $t$. Note that, from Equation (19), the cdf of $CICRL_q$, which is a non-decreasing step function, depends on $m$, $n$ and $\alpha$ values. We obtained the cdf of the $CICMRL$, substituting $q = 0.50$ ($F_{CICMRL}(t)$) into Equation (19).

**3.3 $CICMRL$ assessment**

In order to evaluate the proposed performance measure $CICMRL$, we compare it to the $CICARL$ (the conditional value of the most traditional performance, the $ARL$). Then, as an initial assessment, the expected value ($E$), standard deviation ($SD$), coefficient of variation ($CV$) and skewness (measured with Pearson's moment coefficient) of these three performance measures are shown in Table 1 for $m = \{25,50,100,200,1000,\infty\}$, $n = 5$ and $\alpha = 0.0027$ (nominal $ICARL = 370.4$ and nominal $ICMRL = 257$). These theoretical values were obtained using the formula of the $r$th moment of a random variable (continuous and discrete) as a function of its cdf (see Chakraborti et al. (2017)):

$$E(CICARL^r) = 1 + r\int_1^{\infty} t^{r-1}(1 - F_{CICARL}(t))dt$$

$$E((CICMRL)^r) = 1 + \sum_{i=1}^{\infty}((i+1)^r - i^r)(1 - F_{CICMRL}(i))$$

The cdf of the $CICARL$ ($F_{CICARL}(t)$) is related to the cdf of the $CFAR$, that is, $F_{CICARL}(t) = P(CICARL \leq t) = P(CFAR \geq t^{-1}) = 1 - F_{CFAR}(t^{-1})$). An expression for the $F_{CFAR}(t^{-1})$ was obtained by Epprecht et al. (2015). We also used the cdf of the $CICMRL$ ($F_{CICMRL}(i)$) for $q = 0.5$ given by Equation 19. The expected values and standard deviation values from Table 1 were verified using computer simulations. The values of the corresponding nominal performance measures are given in the last column from Table 1, i.e., when the process variance is known ($m = \infty$). We can note that, for $m = 1000$, the expected values of both performance measures are close to those when process variance is known (nominal values); conversely, when $m$ is small, these expected values and standard deviation values are significantly large, for instance, for $m = 25$, the expected values (i.e., $E(CICARL)$ and $E(CICMRL)$) are roughly 80% larger than their corresponding nominal values and the standard deviation values of these performance measures (i.e., $SD(CICARL)$ and $SD(CICMRL)$ are roughly 90% larger than their corresponding expected values. A comparison of the variability (or spread), based on $SD$ and $CV$ values, across these three performance measures, from Table 1, displays that the variability of the $CICMRL$ is smaller than the $CICARL$ variability. When $m$ becomes smaller, the differences in variability among these measures increase. The numerical skewness results given in Table 1 support the highly skewed to the right shape of the $CICMRL$ distribution (when $m$ is small) shown in Figure 1. On the other side, when $m$ becomes larger, these distributions tend to be symmetric. For each value of $m$, the three performance measures distributions have similar skewness values.

**Table 1**: The expected values ($E$), standard deviation ($SD$), coefficient of variation ($CV$) and skewness of the $CICARL$ and $CICMRL$ and for $m = \{25, 50, 100, 200, 1000\}$, $n = 5$ and $\alpha = 0.0027$ (nominal $ICARL = 370.4$, and nominal $ICMRL = 257$).

| | | $m$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 25 | 50 | 100 | 200 | 1000 | ∞ |
| $CICARL$ | $E(CICARL)$ | 674.15 | 490.76 | 424.61 | 396.18 | 375.34 | 370.40 |
| | $SD(CICARL)$ | 1292.88 | 458.14 | 244.10 | 151.57 | 61.39 | 0 |
| | $CV(CICARL)$ | 1.9178 | 0.9335 | 0.5749 | 0.3826 | 0.1636 | 0 |
| | $Skewness$ | 33.60 | 4.76 | 2.25 | 1.37 | 0.55 | 0 |
| $CICMRL$ | $E(CICMRL)$ | 467.44 | 340.33 | 294.47 | 274.76 | 260.32 | 257 |
| | $SD(CICMRL)$ | 896.15 | 317.56 | 169.20 | 105.06 | 42.55 | 0 |
| | $CV(CICMRL)$ | 1.9171 | 0.9331 | 0.5746 | 0.3824 | 0.1635 | 0 |
| | $Skewness$ | 33.00 | 4.76 | 2.25 | 1.37 | 0.55 | 0 |

Given the number and size of a specific Phase I reference sample ($m$ and $n$), Probability-Integral Transformation can be used, i.e., the cdf of $Y$ ($F_{\chi^2_{m(n-1)}}(Y)$) can be expressed as a uniform random variable $U$ ($0 < u < 1$), then one particular realization of $Y$ ($Y = y$) is the $u$-quantile of the chi-square distribution with $m(n-1)$ df ($F_{\chi^2_{m(n-1)}}(y) = u$) so that $y = F^{-1}_{\chi^2_{m(n-1)}}(u)$. Therefore, from Equation (17), the $CICRL_q$ can be expressed as a function of $Y$ (say, a monotonically non-decreasing function $g$), then we have:

$$CICRL_q = \lceil g(Y) \rceil = \left\lceil g\left(F^{-1}_{\chi^2_{m(n-1)}}(U)\right)\right\rceil = h(U)$$

It can be rewritten as a function of $U$ (we use the notation $\left(CICRL_q\right)_U$):

$$\left(CICRL_q\right)_U = h(U) = \left\lceil \frac{ln(1-q)}{ln\left(F_{\chi^2_{n-1}}\left(\frac{F_Y^{-1}(U)}{m(n-1)}\chi^2_{n-1,\ 1-\alpha}\right)\right)} \right\rceil \tag{20}$$

Where $h$ is a positive integer-valued function. Moreover, from Equation (15), the conditional in-control average run length ($CICARL$) is obtained as the reciprocal of $CFAR(Y)$, i.e., $CICARL = 1/1 - F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1,\ 1-\alpha}\right)$. Next, $CICARL$ can also be expressed as a function $v$ of $U$:

$$(CICARL)_U = v(U) = \frac{1}{1-F_{\chi^2_{n-1}}\left(\frac{F_Y^{-1}(U)}{m(n-1)}\chi^2_{n-1,\ 1-\alpha}\right)} \tag{21}$$

Note that, from Equations (20) and (21), the $CICRL_q$ and $CICARL$ are expressed as monotonically non-decreasing functions ($h$ is discrete and $v$ is continuous) of a uniform random variable $U$. Then, we can take advantage of the fact that the cdf of $CICRL_q$ at

$h(u)$ is greater or equal to the cdf of $U$ at $u$, i.e., $F_U(u) = u$. Thus, we have the cdf of $CICRL_q$ at $h(u)$:

$$F_{CICRL_q}(h(u)) = P\left(CICRL_q \leq h(u)\right) \geq F_U(u) = P(U \leq u) = u \qquad (22)$$

Note that $h(u)$ (namely, from Equation (20), $h(u) = \left(CICRL_q\right)_u$) is the $u$-quantile of the $CICRL_q$. In the case of the $CICARL$, its cdf at $v(u)$ is equal to the cdf of $U$ at $u$ (i.e., $F_{CICARL}(v(u)) = P(CICARL \leq v(u)) = F_U(u) = P(U \leq u) = u$). Likewise, $v(u)$ ($(CICARL)_u$) is the $u$-quantile of the $CICARL$. In this way, these performance measures can be calculated and plotted as functions of $U$ that only take values in the $\langle 0, 1\rangle$ interval (while $Y$ and $k$ take values in $\langle 0, +\infty\rangle$) in order to figure out the distributions, the quantiles and the asymptotic behavior of the $CICRL_q$ and $CICARL$. Hence, this described transformation enables us to obtain the quantile functions of the $CICRL_q$ and $CICARL$ (Equations (20) and (21), respectively), which are the inverse of their corresponding cdfs.

Despite that the plots shown in Figure 1enable us to see the $CICMRL$ distribution shape for different values of $m$, we want to compare the entire distribution and quantiles values of the $CICMRL$ and $CICARL$ for each value of $m$. Hence, from Figure 2, a graphical representation that is similar to the one used by Radson and Boyd (2005) depicts the comparison of the distributions and quantiles (using Equations (20) and (21)) of the $CICARL$ and $CICMRL$ for $m = \{25, 50,100,1000\}$, $n = 5$, and $\alpha = 0.0027$ (nominal $ICARL = 370.4$ and nominal $ICMRL = 257$ are indicated with dashed vertical lines). This graph, like the standard boxplots, displays the extremes percentiles: 1th and 99th (large vertical lines), 5th and 95th (added short vertical lines); and the three quartiles: 25th, 50th and 75th (vertical lines in the box). Thus, using the plots from Figure 2, we can easily distinguish the right tails of the three analyzed performance measures. In addition to Figure 2, the quantile values of the two analyzed performance measures are given and compared in Tables 2 and 3.

Furthermore, using Equations (21) and (20), the $CICARL$ is compared with the proposed performance measure $CICMRL$. Table 2 shows and compares some $u$-quantiles values of both random variables ($(CICARL)_u$ and $(CICMRL)_u$)) for five different numbers of Phase I samples ($m = \{25,50,100,200,1000\}$) each of size $n = 5$ and $\alpha = 0.0027$. Additionally, some summary measures of these performance measures are provided, complementing information given in Table 1. Figure 3 is the graphical representation of quantiles values from Tables 2. It is worth to note that, although the plot of $CICMRL$ is expressed as a function of $U$ (Figures 3) is non-decreasing step function, which have infinite number of $U$-intervals, it was drawn as a continuous curve (similar to Figure 1).

In relation to the first comparison, Figure 3 shows the plots of the $u$-quantiles of $CICARL$ and $CICMRL$ and some of their values are show in Table 2 considering $u = \{0.01, 0.05, 0.1, 0.25, 0.50, 0.75, 0.90, 0.95 \text{ and } 0.99\}$ and five additional values of $u$ ($u^*$, where $u^*$ varies between 0.5033 and 0.5188). $u^*$ is a value so that, for each one of the five values of $m$, the mean of $Y$ is the $u^*$-quantile of $Y$ ($F^{-1}_{\chi^2_{m(n-1)}}(u^*) = Y = E(Y) = m(n-1)$), then each $u^*$ satisfies (from Equations (20) and (21)): $u^* = h^{-1}(257) = v^{-1}(370.4)$, i.e., both conditional $IC\ RL$ measures equal the corresponding nominal values ($CICMRL = 257$ and $CICARL = 370.4$, see the diagonal values in the middle of Table 2). The estimate of the $IC$ process standard deviation has a strongly influence in determining the $CICARL$ and $CICMRL$, thus we can quantify this influence using the $\langle 0, 1\rangle$ $U$-interval. Note that when $Y < E(Y)$ ($U < u^*$), the process standard deviation is underestimated ($S_p < \sigma_0$, then $k < 1$), and on the other hand, when $Y > E(Y)$ ($U > u^*$), the process standard deviation is overestimated ($S_p > \sigma_0$, then $k > 1$). From Table 2 and Figure 5, we can deduce that the intersections between curves of the $CICMRL$ (or the $CICARL$) plots and one horizontal straight line equal to $257$ (or $370.4$) correspond to values of $u^*$ between 0.5033 and 0.5188 (see the diagonal values in the middle of Table 2). The curves displayed in Figure 3 show that as the value of $m$ increases, each conditional $IC\ RL$ measure tends to a horizontal straight line, the equation of which equals the corresponding nominal value. Next, the asymptotic behavior of the $CICMRL$ and $CICARL$ is examined. In relation to the left tail, if $u \to 0$, then $Y \to 0$ and $CICMRL, CICARL \to 1$. On the other hand, in relation to the right tail, if $u \to 1$, then $Y \to \infty$ and $CICMRL, CICARL \to \infty$.

From Figure 3 (both plots with the same scale) and Table 2, we can note that even though the $CICARL$ and $CICMRL$ curves have similar shape, the variability of them are different. The skewness of the distribution of both performance measures (see the values of the Pearson's moment coefficient of skewness from Table 1) makes the curves much steeper for the upper than for the lower $u$-quantiles. Therefore, from Figure 3 and Table 2, it is easier to note that, unless $m$ is large (say, 1000 or more), standard deviation overestimation has a very larger impact on these performance measures values than the underestimation case. It is due to the longer right tails. For example, for $m = 25$, when $u = 0.01$ ($\sigma_0$ underestimation) $CICARL = 44.3$ (11.96% of its nominal value) and $CICMRL = 31$ (12.06% of its nominal value), on the other hand, when $u = 0.99$ ($\sigma_0$ overestimation) $CICARL = 5152.6$ (1391% of its nominal value) and $CICMRL = 3572$ (1390% of its nominal value). Moreover, as was indicated before, we can note that the variability (or spread) of the $CICMRL$ is smaller than the variability of the $CICARL$ and it can be verified from Table 2 using the values of summary measures for dispersion based on percentiles, such as the difference between 0.95-quantile and 0.05-quantile ($95th - 5th$) or the interquartile range ($IQR$). For instance, from Table 1, for $m = 25$, the extreme quantiles difference ($95th - 5th$) for $CICARL$ is $2123.4$ and for $CICMRL$ is $1472$; for $m = 1000$, this difference for $CICARL$ is $199.9$ and for $CICMRL$ is $139$.

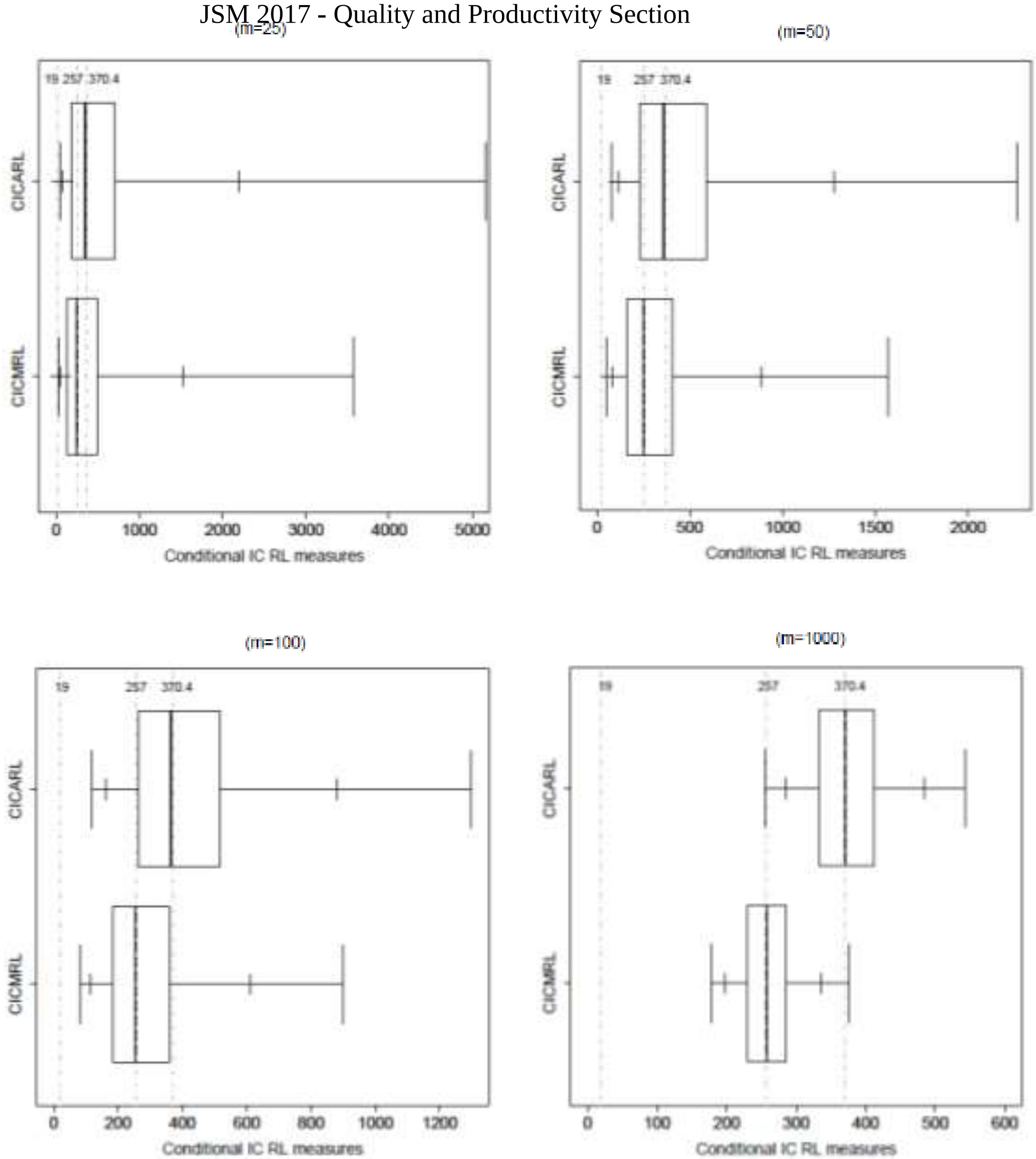


**Figure 2:** Comparison of the $CICARL$ and $CICMRL$ distributions and quantiles for $m = \{25, 50,100,1000\}$, $n = 5$, and $\alpha = 0.0027$ (nominal $ICARL = 370.4$ and nominal $ICMRL = 257$ are indicated with dashed vertical lines). These plots (like boxplots) show 1th, 5th, 25th, 50th, 75th, 95th and 99th percentiles.

**Table 2**: The $u$-quantiles of $CICARL$ ($(CICARL)_u$) and the $u$-quantiles of $CICMRL$ ($(CICMRL)_u$) for some values of $U$ $(0 < u < 1)$. Comparison between some summary measures ($SM$) of the $CICARL$ and $CICMRL$: the expected value ($E$), standard deviation ($SD$), coefficient of variation ($CV$), interquartile range ($IQR$) and difference between 0.95-quantile and 0.05-quantile ($95th - 5th$) (for $m = \{25, 50, 100, 200, 1000\}$, $n = 5$ and $\alpha = 0.0027$ (nominal $ICARL = 370.4$ and nominal $ICMRL = 257$)).

| | $m$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 25 | | 50 | | 100 | | 200 | | 1000 | |
| $u, SM$ | $CICARL$ | $CICMRL$ | $CICARL$ | $CICMRL$ | $CICARL$ | $CICMRL$ | $CICARL$ | $CICMRL$ | $CICARL$ | $CICMRL$ |
| 0.01 | 44.3 | 31 | 78.3 | 54 | 120.1 | 83 | 164.8 | 114 | 255.8 | 177 |
| 0.05 | 76.7 | 53 | 118.9 | 83 | 163.9 | 114 | 207.0 | 144 | 284.6 | 197 |
| 0.1 | 104.8 | 73 | 150.0 | 104 | 194.5 | 135 | 234.3 | 163 | 301.4 | 209 |
| 0.25 | 182.1 | 126 | 225.0 | 156 | 260.9 | 181 | 289.4 | 201 | 331.9 | 230 |
| 0.50 | 353.0 | 245 | 361.6 | 251 | 365.9 | 254 | 368.1 | 255 | 369.9 | 257 |
| 0.5033 | | | | | | | | | 370.4 | 257 |
| 0.5066 | | | | | | | 370.4 | 257 | | |
| 0.5094 | | | | | 370.4 | 257 | | | | |
| 0.5133 | | | 370.4 | 257 | | | | | | |
| 0.5188 | 370.4 | 257 | | | | | | | | |
| 0.75 | 719.8 | 499 | 595.9 | 413 | 519.8 | 360 | 471.3 | 327 | 412.8 | 286 |
| 0.90 | 1429.5 | 991 | 955.3 | 662 | 720.9 | 500 | 591.9 | 410 | 456.2 | 316 |
| 0.95 | 2200.1 | 1525 | 1280.1 | 887 | 881.2 | 611 | 680.1 | 472 | 484.5 | 336 |
| 0.99 | 5152.6 | 3572 | 2262.9 | 1569 | 1297.5 | 900 | 887.1 | 615 | 543.1 | 377 |
| $E$ | 674.15 | 467.44 | 490.76 | 340.33 | 424.61 | 294.47 | 396.18 | 274.76 | 375.34 | 260.32 |
| $SD$ | 1292.88 | 896.15 | 458.14 | 317.56 | 244.10 | 169.20 | 151.57 | 105.06 | 61.39 | 42.55 |
| $CV$ | 1.9178 | 1.9171 | 0.9335 | 0.9331 | 0.5749 | 0.5746 | 0.3826 | 0.3824 | 0.1636 | 0.1635 |
| $IQR$ | 537.7 | 373 | 370.9 | 257 | 258.9 | 179 | 181.9 | 126 | 80.9 | 56 |
| $95th - 5th$ | 2123.4 | 1472 | 1161.2 | 804 | 717.3 | 497 | 473.1 | 328 | 199.9 | 139 |

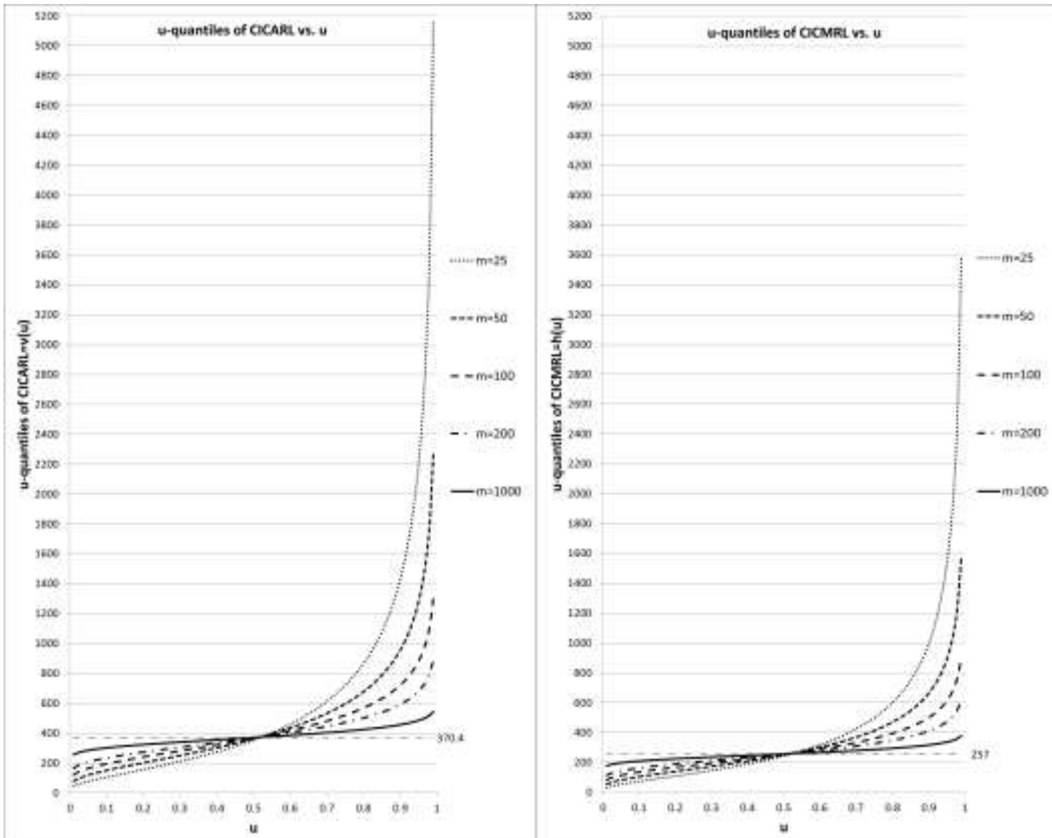


**Figure 3**: Comparison between the $u$-quantiles of $CICARL$ (($v(u)$)) and the $u$-quantiles of $CICMRL$ ($h(u)$) ($0 < u < 1$) for different values of $m = \{25, 50, 100, 200, 1000\}$, $n = 5$ and $\alpha = 0.0027$ (nominal $ICARL = 370.4$ and nominal $ICMRL = 257$).

## 4. Minimum number of Phase I samples ($m$) for a guaranteed in-control Phase II performance

In this section, the Phase II $IC$ performance is measured using the proposed $CICMRL$. It may be useful information for the users to know the minimum number $m$ of Phase I reference samples that guarantees that $CICRL_q$ is greater or equal to a tolerated lower bound $tlb_p$ (a given percentage of the nominal $ICRL_q$, i.e., $tlb_p = (1-\varepsilon) nominal\, ICRL_q$) with a specified high probability $1-p$ (e.g., 90%) such that $P\big(CICRL_q \geq tlb\big) = 1-p$. The value of $tlb_p$ considers the effect of parameter estimation and practitioner-to-practitioner variability and the users can choose the value of $\varepsilon$ ($0 \leq \varepsilon < 1$, e.g., $\varepsilon = 10\%$).

$$P\big(CICRL_q \geq (1-\varepsilon) nominal\, ICRL_q\big) \geq 1-p$$

Then, $P\big(CICRL_q \leq \lceil tlb \rceil - 1\big) \leq p$, where $\lceil tlb \rceil$ denotes the smallest integer greater or equal to $tlb$. Next, using using Equation (19), we obtain

$$m(n-1)\chi^2_{n-1,\,(1-q)^{1/(\lceil(1-\varepsilon)nominal\ ICRL_q\rceil-1)}} \le \chi^2_{m(n-1),\,p}(\chi^2_{n-1,\ 1-\alpha}) \quad (23)$$

Note that, from Equation (23), the solution of $m$ needs a search, since the number of degrees of freedom of $\chi^2_{m(n-1),\,p}$ (chi-squared quantile) is a function of $m$. Then, $m$ is the smallest integer value that satisfies Equation (23).

For given values of $n$, $\alpha$, $\varepsilon$ and $p$, we are also interested in finding the minimum value of $m$ for a guaranteed $IC$ performance that is based on the traditional $CICARL$ (i.e., $P(CICARL \ge (1-\varepsilon)nominal\ ICARL) \ge 1-p$) in order to compare this $m$ value to the ones based on the $CICMRL$ in Table 3. Next, in a similar way that a search is needed to find the minimum value of $m$ that satisfies Equation (23), for a guaranteed $CICARL$ value, a search is needed to find the minimum value of $m$ that satisfies

$$m(n-1)\chi^2_{n-1,1-(\alpha/(1-\varepsilon))} \le \chi^2_{m(n-1),\,p}(\chi^2_{n-1,\ 1-\alpha}) \quad (24)$$

Epprecht et al. (2015) dealt with the minimum $m$ for a guaranteed $CFAR$ value (when the process is $IC$), using an equivalent expression to Equation (24).

Table 3 shows, for different values of $n$, the minimum number of Phase I samples required ($m$) for guaranteeing a desired $IC$ performance in terms of the $CICMRL$ and in terms of the $CICARL$ (namely, as a percentage of their nominal value), for same values of $\alpha$, $\varepsilon$ and $p$. We consider $\varepsilon = \{\,20\%\}$, $p = \{0.05, 0.10\}$ and $\alpha = 0.0027$ (nominal $ICARL = 370.4$ and nominal $ICMRL = 257$). For instance, when $\varepsilon = 20\%$ and $p = 0.10$, we have, in Table 3, the minimum values of $m$ that satisfy: $P(CICMRL \ge 205.6) \ge 0.9$ and $P(CICARL\ \ge 296.3) \ge 0.9$. Then, these values of $m$, as was indicated by Epprecht et al. (2015) for the case of maximum tolerated values of $CFAR$, are much larger than the customary and recommended values of $m = 25$ or 30 and $n = 4$ or 5 (see, e.g., Montgomery (2012)) and also larger than the recommended values (with values of $m$ of up to 200 and values of $n$ of up to 10 units) by Chen (1998), Maravelakis et al. (2002) and Castagliola et al. (2009), who based their recommendations on the analysis of the unconditional $RL$ distribution.

From Table 3 when $\varepsilon = 20\%$, the values of $m$ for guaranteeing a minimum limit of the $CICMRL$ are slightly smaller than the corresponding values of $m$ for guaranteeing a minimum limit of the $CICARL$, for same values of $p$. These differences of $m$ values vary between 0 and 3.

**Table 3.** Minimum number of Phase I samples required ($m$) as a function of $n$ for a guaranteed specified minimum limit of the $CICMRL$ ($CICARL$ in parentheses) value, with $\varepsilon = \{20\%\}$, $p = \{0.05, 0.10\}$ and $\alpha = 0.0027$ (nominal $ICMRL = 257$ and nominal $ICARL = 370.4$)

| | $\varepsilon = 20\%$ | |
|---|---|---|
| | $p$ | |
| $n$ | 0.05 | 0.10 |
| 2 | 2591 (2594) | 1584 (1586) |
| 3 | 1868 (1871) | 1141 (1143) |
| 4 | 1568 (1570) | 957 (958) |
| 5 | 1397 (1399) | 853 (854) |
| 6 | 1285 (1287) | 784 (785) |
| 7 | 1204 (1206) | 734 (735) |
| 8 | 1143 (1144) | 697 (698) |
| 9 | 1094 (1096) | 667 (668) |
| 10 | 1055 (1056) | 643 (644) |
| 15 | 930 (932) | 567 (567) |
| 20 | 862 (863) | 525 (526) |
| 25 | 818 (819) | 498 (499) |
| 30 | 786 (787) | 479 (479) |
| 35 | 763 (764) | 464 (465) |
| 40 | 744 (745) | 453 (453) |
| 45 | 728 (729) | 443 (444) |
| 50 | 716 (717) | 435 (436) |

## 5. Conclusions

It is known that, when the process parameters are unknown, the control charts performance may be substantially affected by the effect of parameter estimation. This study was addressed by several authors, including Chen (1998), Maravelakis et al. (2002), Jensen et al. (2006), Castagliola (2009) and Psarakis et al. (2014), among others. However, these researches are based on the unconditional $RL$ distribution. Recent papers (e.g., Saleh et al. (2015)) have pointed out the practitioner-to-practitioner variability (not explicitly considered by previous research works). Indeed, given the fact that the practitioner just has a single Phase I reference sample ($m$ subgroups of size $n$) for estimating the $IC$ process parameters, the use of the conditional $RL$ distribution (conditioned on the Phase I standard deviation estimator) is more proper and more meaningful. Previous papers (e.g., Epprecht et al. (2015), Faraz et al. (2015), Goedhart et al. (2017)) have considered the distribution of the conditional in-control $ARL$ ($CICARL$). Since the conditional $RL$ follows a geometric distribution, which is generally right-skewed, the study of the complete distribution and quantiles of the $RL$, as its median or some of its extreme quantiles, should be recommended to the practitioner and can be preferred to the traditional $ARL$, not just in analyzing the chart performance, but also in chart design. This study was addressed by several authors, including Khoo (2004), Radson and Boyd (2005), Lee and Khoo (2006), Chakraborti (2007), Khoo et al. (2011) and Teoh et al. (2014, 2016), among others. Nevertheless, these works consider the unconditional $RL$ distribution, i.e., when the quantiles of the $RL$ distribution are not random variables (constant values). Thus, we consider here the conditional $MRL$. We argue that it can be preferred to the traditional $ARL$ (or, in this case, the $CICARL$), when analyzing the effect of parameter estimation on the chart performance. To the best of our knowledge, there is no study in the literature, focusing on the quantiles of the conditional in-control $RL$ ($CICRL_q$) of the $S^2$ and $S$ control charts. Note that such quantiles, in the typical case of estimated parameters, are random variables. Here, the pmf and cdf of the $CICRL_q$ are obtained analytically. The results show that the variability (spread) of the proposed $CICMRL$ is smaller than the variability of the $CICARL$.

Since these conditional performance measures ($CICMRL$ and $CICARL$) are random variables, we think that, in control chart performance analysis, it is useful to assess the probability of a guaranteed minimum value of each one of these measures. This minimum limit of each measure is a specified percentage ($(1-\varepsilon)100\%$) of its nominal value. Then, given specified values of $n$ and $\alpha$, and for a same value of $\varepsilon$, the $CICMRL$ requires similar values of $m$ for guaranteeing a minimum limit with a specified high probability $(1-p)$ to the $CICARL$ require to guarantee their minimum limit with the same probability. Finally, it is clear that these required numbers of $m$ obtained in our results are impractically large. For this reason, a follow-up of this work will focus on the use of an adjusted control limit, given realistic and practical values of $m$ and $n$, for guaranteeing a specified minimum value of $CICMRL$ with a specified high probability. In this way, we advocate the use of the $CICMRL$ as a performance measure of $S^2$ charts, and we also propose a future work using them as a criterion in chart design.

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