Use of Markov Chains in Process Control with the Incorporation of Possible Inspection Errors

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Abstract

A process is periodically sampled, however there might be errors in assessing whether an item meets specifications. This, in turn, has an effect on correctly classifying whether the process is in or out of control. Markov chain techniques prove useful in the analysis of various aspects.

Key Words: Acceptance Sampling, Attribute, Markov Chains, Misclassification, Process Control, Runs

1. Introduction

In Taguchi, Elsayed, and Hsiang(1989) and Taguchi, Chowdhury, and Wu (2004) on-line process control by attributes involves inspecting every h^{th} item produced. When the process is in control, it is assumed to have some high fraction of conforming items, close to 100 %. That is, an item conforms to specifications with probability p_1 very close to 1 when the process is in control. When the process goes out of control, there is a shift to p_2 ($< p_1$) for the fraction conforming, i.e., the probability that the selected item is really conforming. When an inspected item is considered nonconforming, the process is stopped for adjustment.

Several authors have studied this scenario with various assumptions. Nayebpour and Woodall (1993) assume the time until the shift from p_1 to p_2 follows a geometric distribution. That is, the items produced are modeled as independent and identically distributed trials with a constant probability π for each item to be the first item produced with the process out of control. Since only every h^{th} item is inspected, the first item produced with the process out of control may not be inspected and thus there may be some initial number of items produced before the possibility of the detection of this shift even exists.

Borges, Ho, and Turnes (2001) note that the inspection process itself may be subject to possible diagnostic errors, meaning that in a single classification a conforming item might be mistakenly classified as nonconforming. We let p_{CN} denote the probability of this type of misclassification. In addition, a nonconforming item might mistakenly be judged as conforming and we let p_{NC} be the probability of this misclassification. We will also define probability p_{CC} (p_{NN}) of the correct classification that a conforming (nonconforming) item is classified as conforming (nonconforming). This suggests making repeated classifications of each inspected item before making the final decision as to whether to judge the item as conforming or nonconforming. If the item has been judged in this final decision to be nonconforming, the process is judged out of control and is stopped for adjustment. Otherwise, the process is considered in control and is not stopped for adjustment. Because of the possibility of misclassification errors during the repeated classifications, it is possible that an item can be judged to be nonconforming and thus that the process is judged out of control, when it actually is not. Still it is stopped for adjustment. However, in that case, no cause can be found and the process then is restarted and we assume has not somehow been put out of control by the stopping and searching for a cause. On the other hand, it is also possible that the process goes out of control, but is not detected. In that case, it remains out of control until this is detected at a later time, when it will be adjusted and be put back in control.

In Trindade, Ho, and Quinino (2007), the rule for the final decision of whether the inspected item is conforming, and thus whether the process is in control, is based on a pre-specified number of repeated classifications and using majority rule. Quinino, Colin, and Ho (2009) consider a rule in which the item is judged to be conforming and the process to be in control if and only if there are k classifications as conforming before f classifications as nonconforming, where k and f are some pre-specified positive integers. We will use the acronym TCTN since the decision is based on the total number of classifications as conforming and nonconforming. Smith and Griffith (2009) and Griffith and Smith (2011) have further studied this rule and another rule called CCTN.

In this paper, we continue the study of the alternative rule CCTN in which the final determination that an item is conforming, and thus the process is in control, if and only if k consecutive classifications as conforming occur before a total of f classifications as nonconforming.

2. Probabilistic Analysis

Proposition 1: If the item being inspected is conforming (nonconforming), the probability that it is judged to be conforming is

$$P(judged\ conforming|\ actually\ conforming) = CCTN(p_{CC})$$
$$= 1 - \left(1 - p_{CC}^{k}\right)^{f}$$

$$P(judged\ conforming|\ actually\ nonconforming) = CCTN(p_{NC})$$

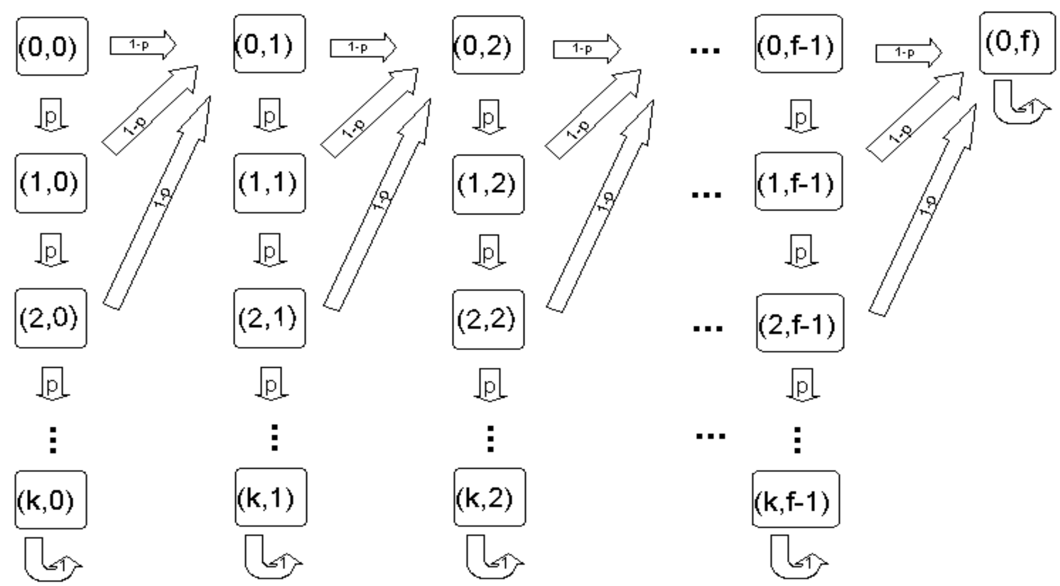
$$= 1 - \left(1 - p_{NC}^{k}\right)^{f}$$

PROOF:
Consider the following Markov chain $\{X_n\}$ with state space

$$\{(r,s)\!: 0 \le r \le k, 0 \le s < f\} \cup \{(0,f)\}$$

where $X_n = (r,s)$ means that after the n^{th} test there are r consecutive successes and a total of s failures. Let p_{CC} (p_{NC}) be the probability that a conforming (nonconforming) item is classified as conforming. In the analysis below, we let p will be equal to p_{CC} or p_{NC} depending on the true nature of the item. The transition probabilities are of the form

$$P(X_n = (r + 1,s)|\ X_{(n-1)} = (r,s)) = p \text{ and } P(X_n = (0,s + 1)|\ X_{(n-1)} = (r,s)) = q.$$



The process begins in state *(0,0)*, and absorbed in state *(k,0)* with probability p^k, i.e.

P(absorbed in state *(k,0)*)=p^k

or moves to the next column with probability $1\text{-}p^k$. Thus, to be absorbed in state *(k,1)*, the chain must first move out of column 1 (not be absorbed in state *(k,0)*) and must go from the top to the bottom of the column 2. The probability of doing this is $p^k(1\text{-}p^k)$. Hence,

P(absorbed in state *(k,1)*)=$p^k(1\text{-}\ p^k)$.

In general for $i<f$, reaching state *(k,i)* requires failure to reach absorbing state in the first i columns and to reach state *(k,i)* of the $(i+1)^{st}$ column. This occurs with probability $p^k(1-p^k)^i$, thus

P(absorbed in state *(k,i)*)= $p^k(1\text{-}\ p^k)^i$ *for* $i < f$.

On the other hand, the chain is absorbed into state *(0,f)* with probability $(1\text{-}p^k)^f$, hence

P(absorbed in state *(0,f)*)= $(1\text{-}p^k)^f$.

From these observations we can conclude that the probability of judging it nonconforming is $(1\text{-}p^k)^f$ and the probability of judging conforming is $1\text{-}(1\text{-}p^k)^f$.

Proposition 2: If the process is in control, the probability that it is judged to be in control is

$$\begin{aligned} P_{II} &= P(judged\ in\ control|\ actually\ control) \\ &= p_1 CCTN(p_{CC}) + \ (1-p_1)CCTN(p_{NC}) \end{aligned}$$

Proof: If it is in control, then the inspected item is conforming with probability p_1 and nonconforming with probability 1-p_1. In light of proposition 1 and using the law of total probability the result follows.

Proposition 3: If the process is out of control, the probability that is judged to be in control is

$$P_{OI} = P(judged\ in\ control |\ out\ of\ control)$$
$$= p_2 CCTN(p_{CC}) + \ (1 - p_2)CCTN(p_{NC})$$

Proof: If out of control, then inspected item conforms with probability p_2 and fails to conform with probability $1 - p_2$. In light of proposition 1 and using the law of total probability the result follows.

Proposition 4: When the process is out of control, the average run length is $\frac{1}{1-P_{OI}}$.

Proof: This is geometric distribution with parameter $1 - P_{OI}$.

Proposition 5: When the process is in control, the average run length is $\frac{1}{1-P_{II}}$.

Proof: This is geometric distribution with parameter $1 - P_{II}$.

3. Short Term Analysis Using Markov Chains

In this section we will use Markov Chains to study the probability of judging the process to be out of control when it is in control as well as judging it to be out of control when it is out of control. We will also look at the distribution of the time until the process is declared out of control using first passage probabilities. To this end, we create a Markov Chain whose state space contains four ordered-pairs whose elements are one or zeros. A one stands for in control and a zero stands for out of control. The first coordinate is the actually state of the process and second coordinate is the judgement. For example, (1,1) means that at a decision point the process in in control and judged to be in control. Whereas, (0,1) means that the process is actually out of control but judged to be in control. Let $\theta = 1 - (1 - \pi)^h$. So, $1 - \theta = (1 - \pi)^h$ is the probability that the process has remained in control while those h items have been produced. The one-step probability matrix for the transitions of this Markov Chain are given in the following transition matrix.

$$\begin{array}{c} \\ \mathbf{(1,1)} \\ \mathbf{(0,1)} \\ \mathbf{(1,0)} \\ \mathbf{(0,0)} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{(1,1)} & \mathbf{(0,1)} & \mathbf{(1,0)} & \mathbf{(0,0)} \end{array} \\ \begin{bmatrix} (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ 0 & P_{OI} & 0 & 1 - P_{OI} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Using standard Markov Chain theory one can use first-passage probabilities to find the probability distribution of the time until the process is declared out of control. One can also use first-step analysis to find the probability of absorption into (1,0) and into (0,1). Note: P_{IO} = 1 - P_{II} and P_{OO} = 1 - P_{OI}.

3. Long Term Analysis Using Markov Chains

We can modify the analysis of the proceeding section in order to study the long term behavior of this decision process. Whenever we reach state (1,0) or state (0,0) the process is judged out of control. When the cause is found and corrected or when it is determined that the process is in control and there is no cause the process is put back online and the transitions are like the transition from state (1,1). Thus in analyzing the long term behavior of the decision process the rows in the new matrix that correspond to transitions out of (1,0) and (0,0) are like the transitions out of state (1,1). Therefore, the one-step transition probability matrix useful for long term analysis is given below.

$$\begin{array}{c} \\ (\mathbf{1},\mathbf{1}) \\ (\mathbf{0},\mathbf{1}) \\ (\mathbf{1},\mathbf{0}) \\ (\mathbf{0},\mathbf{0}) \end{array} \begin{array}{c} \begin{array}{cccc} \quad(\mathbf{1},\mathbf{1}) & \quad(\mathbf{0},\mathbf{1}) & \quad(\mathbf{1},\mathbf{0}) & \quad(\mathbf{0},\mathbf{0}) \end{array} \\ \begin{bmatrix} (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ 0 & P_{OI} & 0 & 1-P_{OI} \\ (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \\ (1-\theta)P_{II} & \theta P_{OI} & (1-\theta)P_{IO} & \theta P_{OO} \end{bmatrix} \end{array}$$

This matrix corresponds to an irreducible, aperiodic, positive recurrent Markov Chain and the limiting probabilities exist and are independent of the starting state. These probabilities can be interpreted as the long term proportion of time spent in each state and can be found by solving a system of linear equations.

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