An Efficient Method for Quantile Regression

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Abstract

Quantile regression has wide applications in many fields. Studies of heavy-tailed distributions have also rapidly developed. Estimation of high conditional quantiles for multivariate heavy-tailed distributions is an important and interesting problem. This paper proposes a new weighted quantile regression method in high quantile regression. The Monte Carlo simulations of the bivariate Pareto distribution show good efficiency of the proposed weighted estimator relative to the regular quantile regression estimator. The paper also investigates a real-world example by using the proposed weighted method. Comparisons of the proposed method with existing methods are given.

Keywords: Bivariate Pareto distribution, conditional quantile, extreme value distributions, linear programing, Monte Carlo simulation, weighted loss function.

AMS 2010 Subject Classifications: primary: 62G32; secondary: 62J05

1. Objective and Motivation

Extreme value analysis deals with extreme events in various worldwide disciplines including finance, earth sciences, and biological sciences. This is an important field in probability and statistics for studying extreme events. Estimating the high conditional quantiles of extreme values of related variables is thus a very important task.

The well-known mean regression model assumes that the conditional mean of y given $\mathbf{x}$ is

$$\mu_{y|\mathbf{x}} = E(y|x_1, x_2, ..., x_k) = \mathbf{x}^T\boldsymbol{\beta} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k,$$

where $\mathbf{x} = (1, x_1, x_2, ..., x_k)^T$ contains given variables and $\boldsymbol{\beta} = (\beta_0, \beta_1, ..., \beta_k)^T$ where $\boldsymbol{\beta} \in R^p$ $(p = k + 1)$. By solving via minimizing the L_2-squared distance, one obtains a least squares

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estimator for $\boldsymbol{\beta}$ from a random sample $(y_i, x_{i1}, x_{i2}, ..., x_{ik})$ where $i = 1, 2, ..., n$, namely

$$\widehat{\boldsymbol{\beta}}_{LS} = \arg \min_{\boldsymbol{\beta} \in R^p} \sum_{i=1}^{n} (y_i - \mathbf{x}_i^T \boldsymbol{\beta})^2 \quad \text{and} \quad \widehat{\mu}_{LS} = \widehat{\mu}_{y|\mathbf{x}} = \mathbf{x}^T \widehat{\boldsymbol{\beta}}_{LS}. \qquad (1)$$

The mean linear regression provides the mean relationship between a response variable and explanatory variables. There are limitations present in the conditional mean models. When analyzing extreme value events, where the response variable y is heavy-tailed distributed, the mean linear regression cannot be extended to non-central locations. Quantile regression estimates conditional quantiles and it will be used to estimate values of extreme events (Yu *et al.*, 2003; Hao and Naiman, 2007). We will study a real world example: Apple Stock Prices Example.

Table 1. Top 10 data of Apple closing stock prices in USD (1990 - 2015).

$Date$	$CSP - Apple$	x_1: $CSP - IBM$	x_2: $CSP - EMC$
29/8/2012	673.47	195.08	26.89
11/10/2012	628.10	205.76	25.68
18/7/2012	606.26	188.25	25.08
5/5/2014	600.96	191.26	25.78
27/11/2012	584.78	191.23	24.57
23/4/2012	571.70	198.62	27.66
23/12/2013	570.09	182.23	25.07
5/6/2012	562.83	189.20	23.38
9/3/2012	545.17	200.62	29.01
21/3/2014	532.87	186.67	27.98

The closing stock price (CSP) is the final price at which a security is traded on a given trading day. The closing price represents the most up-to-date valuation of a security until trading begins again on the following trading day. Investors are extremely interested in predicting when particular stock prices will be high in order to maximize their rate of return. The daily Apple closing stock prices from January 1, 1990 until December 31, 2015 have been collected from Yahoo Canada Finance with only every 30^{th} data point being retained (https://ca.finance.yahoo.com/). A closing stock price of 30 US dollars is the selected threshold due to the fact that a stock with a closing stock price of over 30 US dollars is considered to have much potential to grow, which leads to 163 records from the original 218 remaining in the data set (Chaturvedi, 2009). The top 10 Apple closing stock price (CSP-Apple) data is in Table 1.

One can see linear behavior in Figure 1 along with both the least squares lines and the least squares surface. Since Apple Inc., IBM Corporation, and EMC Corporation are all popular technology companies with excellent reputations, their closing stock prices are well-known to be very predictive of one another. Utilizing the traditional linear regression model with two regressors (x_1 and x_2), namely

$$E(y|x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2, \qquad (2)$$

where x_1 is (CSP-IBM (USD))$^{0.5}$, x_2 is (CSP-EMC (USD))$^{0.5}$, and y is (CSP-Apple (USD))$^{0.5}$, $\widehat{\boldsymbol{\beta}}_{LS}$ is computed by *(1)* where $n = 163$. However, the $\widehat{\mu}_{LS}$ surface merely estimates the average of (CSP-Apple)$^{0.5}$ for a given level of both (CSP-IBM)$^{0.5}$ and (CSP-EMC)$^{0.5}$. Recalling that one is interested in high CSP-Apple which can lead to noteworthy return rates, the quantile regression method will be used to estimate the desired high (i.e. 95%) conditional quantiles of CSP-Apple. We will continue to study this example in greater detail in Section 4.

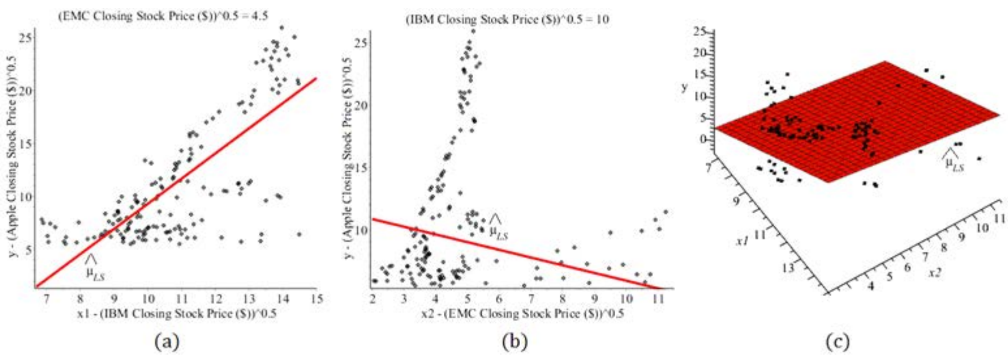

Figure 1. Scatter diagrams ($n = 163$) of y - (CSP-Apple)$^{0.5}$ related to x_1 - (CSP-IBM)$^{0.5}$ and x_2 - (CSP-EMC)$^{0.5}$ with the mean regression lines in (a) and (b) and surface in (c) $\widehat{\mu}_{LS}$.

Main Methods and Results:

In this paper, we propose a new weighted quantile regression method in order to improve the regular quantile regression method. In this paper, we will do three studies:

1. A weight as a function of local conditional density is proposed. An estimate of this weight is also given.

2. Monte Carlo simulations will be performed to show the efficiency of the new weighted quantile regression estimator relative to the regular quantile regression estimator.

3. The new proposed method will be applied to a real-world example of extreme events and compared to mean regression and regular quantile regression.

In Section 2, we review some notation and propose a weighted quantile regression method with conditional density as the weight. In Section 3, the results of Monte Carlo simulations generated from the bivariate Pareto Type II distribution show that the proposed weighted method produces high efficiencies relative to existing methods. In Section 4, the three regression methods: mean regression, regular quantile regression, and the proposed weighted quantile regression, are applied to the Apple Stock Prices example. Three goodness-of-fit tests are used to assess the distributions of the data. Studies of the example illustrate that the proposed weighted quantile regression model fits data better than the existing quantile regression method.

2. Proposed Weighted Quantile Regression

Pickands (1975) first introduced the Generalized Pareto Distribution (GPD).

Definition 1. The cumulative distribution function (c.d.f.) of the two-parameter GPD(γ, σ) with the shape parameter $\gamma > 0$ and scale parameter $\sigma > 0$ of a random variable X is given by

$$F(x) = 1 - \left(1 + \gamma \frac{x}{\sigma}\right)^{1/\gamma}, \quad \gamma > 0, \quad \sigma > 0, \quad x > 0; \qquad (3)$$

Definition 2. The τth conditional quantile of a continuous random variable y with the c.d.f. $F(y)$ for given $\mathbf{x}$ is defined as

$$Q_y(\tau|\mathbf{x}) = Q_\tau(y|x_1, x_2, ..., x_k) = F^{-1}(\tau|\mathbf{x}).\ 0 < \tau < 1.$$

In this paper, we assume that

$$Q_y(\tau|\mathbf{x}) = \mathbf{x}^T\boldsymbol{\beta}(\tau) = \beta_0(\tau) + \beta_1(\tau)x_1 + \cdots + \beta_k(\tau)x_k,\ 0 < \tau < 1,$$

where $\boldsymbol{\beta}(\tau)= (\beta_0(\tau), \beta_1(\tau), \beta_2(\tau), ..., \beta_k(\tau))^T$.

Koenker and Bassett (1978) proposed an L_1-loss function to obtain estimator $\widehat{\boldsymbol{\beta}}(\tau)$ by solving

$$\widehat{\boldsymbol{\beta}}(\tau) = \arg \min_{\boldsymbol{\beta}(\tau)\in R^p} \sum_{i=1}^{n} \rho_\tau(y_i - \mathbf{x}_i^T\boldsymbol{\beta}(\tau)),\ 0 < \tau < 1, \tag{4}$$

where ρ_τ is a loss function, namely

$$\rho_\tau(u) = u(\tau - I(u < 0)) = \begin{cases} u(\tau - 1), & u < 0; \\ u\tau, & u \geq 0. \end{cases}$$

Huang *et al.* (2015) proposed a weighted quantile regression method

$$\widehat{\boldsymbol{\beta}}_w(\tau) = \arg \min_{\boldsymbol{\beta}(\tau)\in R^p} \sum_{i=1}^{n} w_i(\mathbf{x}_i, \tau)\rho_\tau(y_i - \mathbf{x}_i^T\boldsymbol{\beta}(\tau)),\ 0 < \tau < 1, \tag{5}$$

where $w_i(\mathbf{x}_i, \tau)$ is any uniformly bounded positive weight function independent of $y_i,\ i = 1, ..., n,$ for $\mathbf{x}_i = (1, x_{i1}, x_{i2}, ..., x_{ik})^T$.

In this paper, we propose weights $w_i(\mathbf{x}_i, \tau)$ in *(6)* as the local conditional density of y for given $\mathbf{x}_i$, $f(y|\mathbf{x}_i)$, at the τth quantile point $\xi_i(\tau|\mathbf{x}_i)$ of y given $\mathbf{x}_i$, which is

$$w_i(\mathbf{x}_i, \tau) = (f_i(\xi_i(\tau|\mathbf{x}_i)))^\theta,\ 0 < \theta < M,\ 0 < \tau < 1,\ i = 1, 2, ..., n, \tag{6}$$

where $f_i(\xi_i(\tau|\mathbf{x}_i))$ is uniformly bounded at the quantile points $\xi_i(\tau|\mathbf{x}_i)$ and M is a finite real number.

As Koenker (2005) suggested, when the conditional densities of the response are heterogeneous, it is natural to consider whether weighted quantile regression might lead to efficiency improvements. Next, we discuss properties of the proposed weighted estimators. In this paper, we are looking for improvement of efficiency by using weight *(6)* in Section 3 simulations, and by using *(4)* and *(5)* in Section 4 for the Apple Stock Price example.

In order to compare the regular and weighted quantile regression models in *(4)* and *(5),* we extend the idea of measuring goodness of fit by Koenker and Machado (1999) and suggest to use a Relative $R(\tau)$ which is defined as

$$\textit{Relative } R(\tau) = 1 - \frac{V_{weighted}(\tau)}{V_{regular}(\tau)}, \quad -1 \leq R(\tau) \leq 1, \tag{7}$$

where

$$V_{regular}(\tau) = \sum_{y_i \geq \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}(\tau)} \frac{\tau}{n} \left| y_i - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}(\tau) \right| + \sum_{y_i < \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}(\tau)} \frac{(1-\tau)}{n} \left| y_i - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}(\tau) \right|,$$

where $\widehat{\boldsymbol{\beta}}(\tau)$ is given by *(4)*, and

$$V_{weighted}(\tau) = \sum_{y_i \geq \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}_w(\tau)} w_i \tau \left| y_i - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}_w(\tau) \right| + \sum_{y_i < \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}_w(\tau)} w_i (1-\tau) \left| y_i - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}_w(\tau) \right|,$$

where $w_i = w_i(\mathbf{x}_i, \tau)$ and $\widehat{\boldsymbol{\beta}}_w(\tau)$ are given by *(5)* with weight in *(6).*

3. Simulations

In this section, Monte Carlo simulations are performed. We generate m random samples of size n each from the bivariate Pareto distribution Type II (Arnold, 2015) for random vector $(\mathbf{X},\mathbf{Y})$ with a joint c.d.f.

$$F(x,y) = 1 - \frac{1}{(1+x)^\alpha} - \frac{1}{(1+y)^\alpha} + \frac{1}{(1+x+y)^\alpha},\ x>0,\ y>0,\ \alpha>0, \tag{8}$$

and the conditional quantile function of y given x with distribution in *(8)* is

$$\xi(\tau|x) = Q_y(\tau|x) = (1+x)\left(\frac{1}{(1-\tau)^{1/(\alpha+1)}} - 1\right),\ x>0,\ \alpha>0,\ 0<\tau<1. \tag{9}$$

The conditional density of y for given x is

$$f(y|x) = \frac{(\alpha+1)x^{(\alpha+1)}}{(1+x+y)^{(\alpha+2)}},\ x>0,\ y>0,\ \alpha>0,$$

and the τth conditional density of y for given x at the τth quantile is

$$f(\xi(\tau|x)) = \frac{(\alpha+1)(1-\tau)^{(\alpha+2)/(\alpha+1)}}{1+x},\ x>0,\ \alpha>0,\ 0<\tau<1.$$

Assume that the true conditional quantile is $Q_y(\tau|x) = \beta_0(\tau) + \beta_1(\tau)x$. We use two quantile regression methods:

1. The regular quantile regression $Q_R(\tau|x)$ estimation based on *(4),* namely

$$Q_R(\tau|x) = \widehat{\beta}_0(\tau) + \widehat{\beta}_1(\tau)x \tag{10}$$

2. The weighted quantile regression $Q_W(\tau|x)$ estimation based on *(5),* namely

$$Q_W(\tau|x) = \widehat{\beta}_{w0}(\tau) + \widehat{\beta}_{w1}(\tau)x. \tag{11}$$

For each method, we generate size $n = 300$, $m = 1000$ samples. $Q_{R,i}(\tau|x)$ or $Q_{W,i}(\tau|x),\ i = 1,...,m,$ are estimated for the ith sample. Let $\alpha = 3,\ \theta = 1$ in *(6)*, then the weights are

$$w_{ij}(x_{ij},\tau) = f_{ij}(\xi_{ij}(\tau)) = \frac{4(1-\tau)^{(5/4)}}{1+x_{ij}},\ x_{ij}>0,\ 0<\tau<1,\ j=1,2,...,n. \tag{12}$$

The simulation mean squared errors (SMSE) of the estimators *(10)* and *(11)* are:

$$SMSE(Q_R(\tau)) = \frac{1}{m}\sum_{i=1}^{m}\int_0^N (Q_{R,i}(\tau|x) - Q_y(\tau|x))^2 dx; \quad (13)$$

$$SMSE(Q_W(\tau)) = \frac{1}{m}\sum_{i=1}^{m}\int_0^N (Q_{W,i}(\tau|x) - Q_y(\tau|x))^2 dx, \quad (14)$$

where the true τth conditional quantile $Q_y(\tau|x)$ is defined in *(9)*. N is a finite x value such that the c.d.f. in *(8)* $F(N, N) \approx 1$. We let $N = 1000$ and the simulation efficiencies (SEFF) are given by

$$SEFF(Q_W(\tau)) = \frac{SMSE(Q_R(\tau))}{SMSE(Q_W(\tau))}, \quad (15)$$

where $SMSE(Q_R(\tau))$ and $SMSE(Q_W(\tau))$ are defined in *(13)* and *(14)*, respectively.

Table 2 displays the $SEFF(Q_W(\tau))$ for varying τ values by using the weight in *(12)*. It shows that all of the $SEFF(Q_W(\tau))$ are larger than 1 when $\tau = 0.95, ..., 0.99$. Figure 2 compares the $SMSE(Q_R(\tau))$ with the $SMSE(Q_W(\tau))$ for $\tau = 0.95, ..., 0.99$. It demonstrates that all $SMSE(Q_W(\tau))$ for our proposed weight in *(12)* have smaller values than $SMSE(Q_R(\tau))$.

Table 2. Simulation Mean Square Errors (SMSEs) and Efficiencies (SEFFs) of Estimating $Q_y(\tau|x), m = 1000, n = 300, N = 1000$.

τ	0.95	0.96	0.97	0.98	0.99
$SMSE(Q_R(\tau))$	1.6340x10^8	2.0268x10^8	2.6606x10^8	4.9618x10^8	1.8385x10^9
$SMSE(Q_W(\tau))$	0.9508x10^8	1.4324x10^8	2.3373x10^8	4.8441x10^8	1.7994x10^9
$SEFF(Q_W(\tau))$	1.7186	1.4149	1.1383	1.0243	1.0217

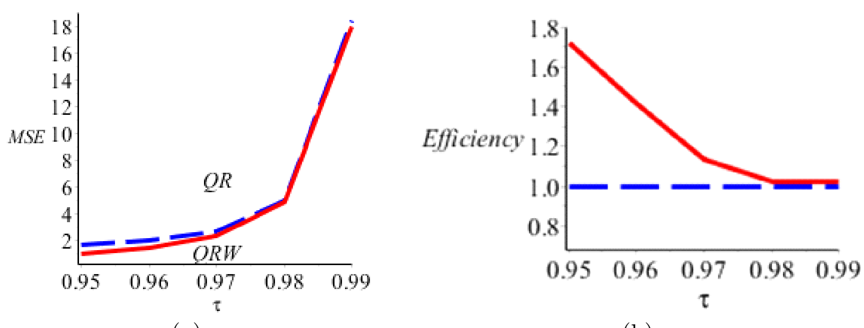

Figure 2. (a) $SMSE(Q_R(\tau))$ is the blue line, $SMSE(Q_W(\tau))$ is the red line; (b) $SEFF(Q_W(\tau))$ is the red line.

From the results of the simulation, we can conclude that Table 2 and Figure 2 show that for $\tau = 0.95, ..., 0.99$, the proposed weighted regression $Q_W(\tau|x)$ with the weight in *(12)* is more efficient relative to the regular quantile regression $Q_R(\tau|x)$.

4. Real-World Examples of Applications

In this section, we applied the following three regression models to the Apple Stock Prices example in Section 1:

1. The traditional mean linear regression (LS) estimator $\widehat{\boldsymbol{\beta}}_{LS}$ in *(1)*;
2. The regular quantile regression Q_R estimator $\widehat{\boldsymbol{\beta}}(\tau)$ in *(4)*;
3. The proposed weighted quantile regression Q_W estimator $\widehat{\boldsymbol{\beta}}_w(\tau)$ in *(5)* with weight $w_i(\mathbf{x}_i, \tau)$ in *(6)*.

Remark. To estimate the proposed local conditional density in weight $w_i(\mathbf{x}_i, \tau) = (f_i(\xi_i(\tau|\mathbf{x}_i)))^\theta$ in *(6)*, we use kernel density estimation (Silverman, 1986; Scott, 1992).

$$\widehat{w}_i(\mathbf{x}_i, \tau) = \widehat{f}_i(\widehat{\xi}_i(\tau|\mathbf{x}_i)), \qquad \text{where} \quad \widehat{f}(y|\mathbf{x}) = \frac{\widehat{f}(y, \mathbf{x})}{\widehat{\mu}(\mathbf{x})}, \tag{16}$$

where $\widehat{f}(y, \mathbf{x})$ is an estimator of the joint density of y and $\mathbf{x}$ and $\widehat{\mu}(\mathbf{x})$ is an estimator of marginal density of $\mathbf{x}$. We estimate the conditional quantile function $\xi(\tau|\mathbf{x})$ of y given $\mathbf{x}$ by inverting an estimated conditional c.d.f. $\widehat{F}(y|\mathbf{x})$

$$\widehat{\xi}(\tau|\mathbf{x}) = \widehat{Q}_y(\tau|\mathbf{x}) = \inf\{y : \widehat{F}(y|\mathbf{x}) \geq \tau\} = \widehat{F}^{-1}(\tau|\mathbf{x}),$$

where $\widehat{F}(y|\mathbf{x})$ is the estimated conditional c.d.f. $F(y|\mathbf{x})$.

Now we are back to the Apple Stock Prices example in Section 1. The mean regression model in *(2)* where y - (CSP-Apple)$^{0.5}$, x_1 - (CSP-IBM)$^{0.5}$, and x_2 - (CSP-EMC)$^{0.5}$ with $\widehat{\boldsymbol{\beta}}_{LS}$ in *(1)* is applied along with the regular quantile regression model with $\widehat{\boldsymbol{\beta}}(\tau)$ in *(4)* and the weighted quantile regression model with $\widehat{\boldsymbol{\beta}}_w(\tau)$ in *(5)* with the proposed weight $w_i(\mathbf{x}_i, \tau) = f_i(\xi_i(\tau|\mathbf{x}_i))$ in *(6)* with $\theta = 1$. We use the quantile regression model

$$Q_y(\tau|x) = \beta_0(\tau) + \beta_1(\tau)x_1 + \beta_2(\tau)x_2. \tag{17}$$

The data $y_1, y_2, ..., y_n$ is transformed to $y_i^* = \frac{y_i - \mu}{\sigma}$ to fit the GPD model *(3)* with $\mu = 30^{0.5}$, MLE $\widehat{\sigma}_{MLE} = 5.5388$, and MLE $\widehat{\gamma}_{MLE} = 0.0178$. The $\log - \log$ plot labeled Figure 3(a) indicates that the data fits the distribution well. Table 3 indicates that the Kolmogorov-Smirnov (K-S) test (Kolmogorov, 1933) allows one to conclude that the transformed data fits the GPD model with a probability of 21.50%, and, comparably, the Anderson-Darling (A-D) and the Cramer-von-Mises (C-v-M) tests (Anderson and Darling, 1952) imply a similar conclusion with a 11.35% probability and a 15.45% probability, respectively. Figure 3(b) shows that the transformed data fits the GPD model well.

Figure 4 further indicates that there is a linear relationship between (Apple closing stock price (USD))$^{0.5}$ (y) and both (IBM closing stock price (USD))$^{0.5}$ (x_1) and (EMC closing stock price (USD))$^{0.5}$ (x_2). The 2-regressor (x_1 and x_2) model in *(17)* was used. It can be seen that, when the quantile is high, Q_W, in general, has a higher Apple closing stock price than Q_R.

Table 3. The test statistics of three goodness-of-fit tests for the Apple Stock Prices example.

K-S		**A-D**		**C-v-M**	
Test Statistic	p-value	Test Statistic	p-value	Test Statistic	p-value
0.0806	0.2150	1.8345	0.1135	0.2796	0.1545

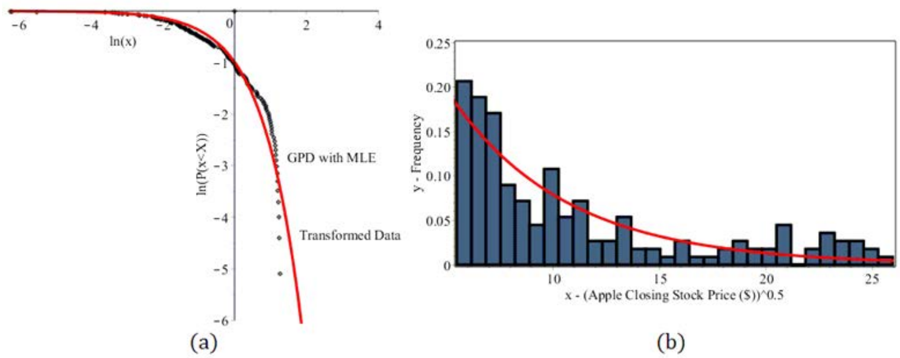

Figure 3. (a) log − log plot for the Apple stock prices example; data–dots; estimated GPD–red curve; (b) Histogram of the transformed Apple closing stock price data ($n = 163$); GPD–red curve.

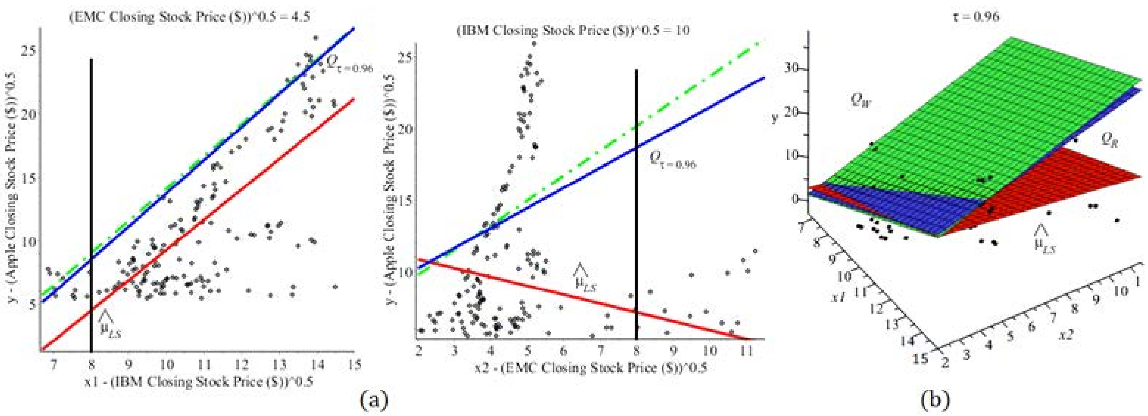

Figure 4. (a) Two 2D plots of the quantile regressions of the Apple closing stock price data at $\tau = 0.96$, Q_R–blue solid; Q_W–green dash; $\widehat{\mu}_{LS}$–red solid; (b) A 3D plot of the quantile regressions of the Apple closing stock price data at $\tau = 0.96$; Q_R–blue; Q_W–green; $\widehat{\mu}_{LS}$–red; where $w_i(\mathbf{x}_i, \tau) = f_i(\xi_i(\tau))$; x_1 - (CSP-IBM $)^{0.5}$, x_2 - (CSP-EMC $)^{0.5}$; y - (CSP-Apple$)^{0.5}$.

It is interesting to see the difference in the relationship between the transformed Apple closing stock price and the transformed EMC closing stock price when using the mean regression method in comparison to the quantile regression methods. The mean regression method allows one to conclude that there seems to be a negative relationship between the two whereas the quantile regression methods suggest a positive relationship. The mean regression method appears to put more weight on the low transformed Apple closing stock prices whereas the quantile regression method appears to put more weight on the high transformed Apple closing stock prices. The stock market is very complicated and this is therefore worth investigating further to find out which relationship is more plausible in the real world.

Figure 5 and Table 4 show that $R(\tau) > 0$ when $\tau \geq 0.95$ where $R(\tau)$ is defined in formula *(7)*, which means that $V_{weighted}(\tau) < V_{regular}(\tau)$, further allowing one to conclude that the Q_W curves fit the data better than the Q_R curve in these cases.

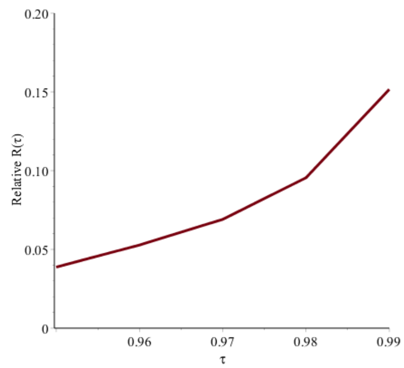

Figure 5. Relative $R(\tau)$ by using weighted quantile regression for Apple Stock Price example.

Table 4. Relative $R(\tau)$ values for Apple Stock Price example.

	$\tau = 0.95$	$\tau = 0.96$	$\tau = 0.97$	$\tau = 0.98$	$\tau = 0.99$
Relative $R(\tau)$	0.0388	0.0528	0.0691	0.0955	0.1516

The quantile regression model of the transformed closing stock price, and moreover the weighted quantile regression models of the transformed closing stock price, has proven to be very useful as a guide as well as an assistive tool in proper stock price forecasting and, hence, providing correct purchasing decisions for those interested in investing in Apple stocks since these models appear to fit the data significantly better than the alternative mean regression model as shown by several visual representations and numerical computations.

5. Overall Conclusions and Suggestions

In this paper, we proposed a new weighted quantile regression method. The main contributions are:

1. Quantile regression has an efficient way to estimate high conditional quantiles with an L_1-loss function which overcomes the limitation of the traditional mean regression, particularly in the analysis of extreme events.

2. The proposed weighted quantile regression method with weight as a function of local conditional density at the quantile point in *(6)* performed better than the regular quantile regression method in the computational simulations. The simulation results show the higher efficiencies of the proposed weighted quantile regression estimator relative to the regular quantile regression estimator.

3. The proposed weighted quantile regression method behaved better via goodness-of-fit than the regular quantile regression in the Apple Stock Price example discussed. Additionally, the high quantile values of the response variable related to the explanatory variable(s) can be predicted. The proposed method gives an alternate way to study extreme events.

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