Approximating the finite-time t probability distributions in an extreme renewal process

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Abstract

In an extreme renewal process with Pareto(II) interarrival times with shape parameter $\alpha \in (0,1]$, we approximate the time-t pdfs for *large* t by the related *limiting* pdfs of a renewal process with *right-truncated* Pareto(II) interarrival times, which have very simple formulas. The distance between the approximating pdfs and time-t pdfs is measured by an L1 metric with values in $(0,1)$.

Key words: extreme renewal process; no-mean interarrival time; time-t excess; age and total life; regenerative process; level crossing method; integral equations; distance between pdfs.

1. Introduction

Recent work in statistics and stochastic modelling, has generated interest in the finite-time t pdfs of renewal processes with *no-mean*, heavy-tailed Pareto interarrivals (e.g., Huang et al., 2013; Harris et al., 2000). Here, we approximate the three finite-time t pdfs of excess, age and total life for *large* t, in a renewal process with Pareto(II) interarrivals and shape parameter $\alpha \in (0,1]$, which we call an *'extreme renewal process'*.

All three time-t pdfs are expressed in terms of the solution of a key integral equation, which may be tedious to compute. To approximate the time-t pdfs in the extreme renewal process, we use the corresponding *limiting* pdfs of a related renewal process where interarrivals have a right-truncated Pareto(II) distribution with the same shape parameter α, which *does have a finite mean*. When the truncation point is $K \geq t$, we call the latter a Pareto(II)-trun(K) (briefly trun(K)) renewal process , whose limiting pdfs of excess, age and total life *exist*, having very simple formulas. This simplicity motivates us to approximate the *finite-time* t pdfs in the extreme renewal process, by using the *limiting* pdfs in the corresponding Pareto(II)-trun(K) renewal process. We give formulas for the time-t pdfs *in terms of the key limiting pdf of a basic regenerative process*. This key pdf is the solution an integral equation derived via a level crossing technique (Brill[3]. (In *real-world* extreme value problems, Huang et al. (2013) uses a truncated Pareto(II) distribution to approximate the Pareto(II) distribution.)

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2. Preliminaries

2.1. Notation for extreme and trun(K) renewal processes

Denote the extreme renewal process by $\{Z_n\}_{n=1,2,..}$, where $Z_n \underset{dis}{:\equiv}$ *no-mean* Pareto(II)(x, α), $0 < x < \infty$. In $\{Z_n\}_{n=1,2,...}$the finite time-t pdfs of excess, age, and total life exist.

Denote the Pareto(II)-trun(K) renewal process by $\{Z_{K,n}\}_{n=1,2,..}$ where $Z_{K,n}$:= $Z_K \underset{dis}{:\equiv}$ *right-truncated* $Pareto(II)(x, \alpha)$, $x \in (0, K)$, $K \geq t$. In $\{Z_{K,n}\}_{n=1,2,...}$ the limiting pdfs of excess, age, and total life, as *time* $\rightarrow \infty$, exist because $E(Z_K) < \infty$.

To approximate the time-t pdfs in $\{Z_n\}_{n=1,2,..}$for a large time t, we use a right truncation point $K \geq t$ for the interarrival times in $\{Z_{K,n}\}_{n=1,2,...}$. The selected truncation point K for the interarrival times may be different in the analysis of excess, age or total life.

The approximations of the *pdfs of the* ζ_ts, viz., the $f_{\zeta_t(\cdot)}$s, are measured by an integral of an L1 measure of distance between the approximating limiting pdfs and the time-t pdfs, ζ = γ, δ, β.

3. Brief outline of the analysis

1. Fix a large time $t > 0$. Compute the *key* limiting pdf of a *basic regenerative process*, which is unique for each fixed t in $\{Z_n\}_{n=1,2,...}$, denoted by $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{x\in(0,t)}$ (see details in Brill 2014.) We utilize the key pdf in the form of the numerical solution of a single integral equation.

2. Obtain formulas for the pdfs of the ζ_ts, $\zeta = \gamma$, δ, β in $\{Z_n\}_{n=1,2,...}$, *in terms of the key pdf* $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$. The formulas for the pdfs of γ_t and β_t use the key pdf in the form $\frac{f^{(t)}(x)}{\pi_0^{(t)},}$ in the integrands of integrals. The formula for the pdf of δ_t uses the key pdf directly in the form $\frac{f^{(t)}(x)}{\pi_0^{(t)},}$ in a linear term.

3. Approximate the *finite-time* t pdfs in $\{Z_n\}_{n=1,2,...}$ by using the corresponding *limiting* pdfs in $\{Z_{K,n}\}_{n=1,2,...}$. The truncation point point K of Z_K is: t when approximating δ_t; it is much greater than t when approximating the pdf of γ_t or β_t (Section 8.3 below).

4. Give a measure of distance between pdfs of ζ_t and the pdfs of ζ_K, by using an integral of an L1 metric. The distance measure has a value in $(0, 1)$ (Section 8.3).

4. Extreme renewal process $\{Z_n\}_{n=1,2,...}$

Denote the cdf, ccdf (:= 1- cdf), and pdf of each interarrival in $\{Z_n\}_{n=1,2,...}$ by $B(\cdot)$, $\bar{B}(\cdot)$:= $1 - B(\cdot)$, and $b(\cdot)$, respectively. Thus

$$\left.\begin{aligned} B(x) \ &= \ 1 - (1+x)^{-\alpha}\,,\ x \in (0,\infty)\,, \\ \bar{B}(x) \ &= 1 - B(x) = (1+x)^{-\alpha}\,,\ x \in [0,\infty), \\ b(x) &= \tfrac{d}{dx}B(x) = \alpha\,(1+x)^{-\alpha-1}\,,\ x \in (0,\infty), \end{aligned}\right\} \tag{1}$$

where $B(x) = P(Z \leq x)$. If $\alpha \in (0, 1]$ then Z is heavy-tailed (Sigman, 1999) and has no mean, implying that the limiting pdfs of γ_t, δ_t, β_t $(t \to \infty)$ do not exist.

5. Basic regenerative process $\{X(s)\}_{s \geq 0}$ and its key limiting pdf

Denote the basic regenerative process by $\{X(s)\}_{s \geq 0}$ (see Brill, 2014). It has a key limiting pdf denoted by $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$. Equations (2) and (3) below give an integral equation and initial condition for the key pdf $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$, where $\pi_0^{(t)}$ = $\lim_{s \to \infty} P(X(s) = 0)$. *All three formulas for the pdfs of the time-t r.v.s are uniquely expressed in terms of* $\frac{f^{(t)}(x)}{\pi_0^{(t)}}$.

5.1. Integral equation for key pdf $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$

Applying the method in Sections 3.2.1-3.2.2, p. 198, in Brill (2014), leads to the integral equation and normalizing condition for $f^{(t)}(x), x \in (0, t)$

$$f^{(t)}(x) = \pi_0^{(t)} \alpha (1 + x)^{-\alpha - 1} + \alpha \int_{y=0}^{x} (1 + x - y)^{-\alpha - 1} f^{(t)}(y) dy, \ 0 < x < t. \quad (2)$$

$$\pi_0^{(t)} + \int_{x=0}^{\infty} f^{(t)}(x) dx = 1. \quad (3)$$

Formulas (3.13) and (3.14) in Brill (2014) yield the solution of (2) and (3) as

$$\pi^{(t)} = \frac{1}{1 + M(t)}, \qquad f^{(t)}(x) = \frac{M'(x)}{1 + M(t)}, 0 < x < t, \quad (4)$$

where $M(x)$ is the renewal function (e.g., p. 169 in Karlin and Taylor, 1975). $M(t)$ is equal to a series of self-convolutions of the interarrival cdf $B(x)$, which may be tedious to compute. Therefore we use a simple numerical procedure (Section 5.1.1 below) to compute the solution of (3) for the pdf $f^{(t)}(x)$, $x \in (0, t)$ in terms of $\pi_0^{(t)}$ Then we apply (3) to compute the probability $\pi_0^{(t)}$. It can be shown that $f^{(t)}(x)$, $x \in (0, t)$ is bounded.

5.1.1. Computation of the key mixed pdf $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}, x \in (0, t)$

The computation used for solving equations (2) and 3 for $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{x \in (0,t)}$, is based on the definition of an integral on a finite interval using a Riemann-Stieltjes sum (e.g., p. 141 in Apostol, 1974). The resulting numerical solution is a step function on a preassigned partition of $(0, t)$ with a norm h. To get a useful solution for $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{x \in (0,t)}$, we choose a 'small' $h > 0$ such that t = Nh where N is a large positive integer.

6. Formulas for the finite-time t pdfs of the extreme renewal process $\{Z_n\}_{n=1,2,...}$ in terms of the key pdf $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$

Denote the pdfs of γ_t, δ_t and β_t by: $f_{\gamma_t}(x)$, $x \in (0,\infty)$; $\{\pi_{\delta_t}, f_{\delta_t}(x)\}_{0<x<t}$ where $\pi_{\delta_t} = P(\delta_t = t)$; $f_{\beta_t}(x)$, $x \in (0,\infty)$. Formula (4.4a) in Brill (2014) gives

$$f_{\gamma_t}(x) = b(t+x) + \int_{y=0}^{t} b(t+x-y)\frac{f^t(y)}{\pi_0^{(t)}}dy,\ 0 < x < \infty; \tag{5}$$

Formula (4.11) (ibid) gives

$$f_{\delta_t}(x) = \bar{B}(x)\frac{f^{(t)}(t-x)}{\pi_0^{(t)}},\ 0 < x < t; \qquad \pi_{\delta_t} = \bar{B}(t). \tag{6}$$

Formulas (4.13) and (4.14) (ibid) give

$$f_{\beta_t}(x) = b(x)\int_{y=0}^{x} \frac{f^{(t)}(t-y)}{\pi^{(t)}}dy,\ 0 < x < t, \tag{7}$$

$$f_{\beta_t}(x) = b(x)\left(1 + \int_{y=0}^{t} \frac{f^{(t)}(t-y)}{\pi^{(t)}}dy\right),\ t \le x < \infty. \tag{8}$$

where $\bar{B}(\cdot)$ and $b(\cdot)$ are given in (1).

Remark 1. δ_t t π_{δ_t} $(1+t)^{-\alpha}$ $f_{\beta_t}(x)$ $x = t$

$$f_{\beta_t}(t^+) - f_{\beta_t}(t^-) = b(t) = \alpha(1+t)^{-\alpha-1}$$

We get the time-t pdfs in (5)-(8) by substituting the computed $\left\{\widehat{\pi_0^{(t)}}, \widehat{f^{(t)}}(x)\right\}_{0<x<t}$ for $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$ in formulas (5)-(8), respectively.

7. The Pareto(II)-trun(K) renewal process $\{Z_{K,n}\}_{n=1,2,...}$

The renewal process $\{Z_{K,n}\}_{n=1,2,...}$ has right-truncated Pareto(II) interarrivals $\underset{dis}{=} Z_K$ with support $(0,K)$, $t \le K < \infty$, and the same shape parameter α as in the extreme renewal process (see formula (1)). Substituting from formula (1), the cdf, ccdf and pdf of Z_K, are respectively

$$\left.\begin{aligned} B_K(x) &= \frac{B(x)}{B(K)} = \frac{1-(1+x)^{-\alpha}}{1-(1+K)^{-\alpha}},\ x \in (0,K), \\ \bar{B}_K(x) &= 1 - B_K(x) = 1 - \frac{1-(1+x)^{-\alpha}}{1-(1+K)^{-\alpha}},\ x \in (0,K), \\ b_K(x) &= \tfrac{d}{dx}B_K(x) = \frac{\alpha\,(1+x)^{-\alpha-1}}{1-(1+K)^{-\alpha}},\ x \in (0,K). \end{aligned}\right\} \tag{9}$$

7.1. Expected value of Z_K

For all $\alpha > 0$, Z_K has a finite mean which, using $\bar{B}_K(x)$ in (9), is

$$\left.\begin{aligned} &E(Z_K) = \int_{x=0}^{K} \bar{B}_K(x)dx = \int_{x=0}^{K} \left(1 - \frac{1-(1+x)^{-\alpha}}{1-(1+K)^{-\alpha}}\right) dx \\ &= \int_{x=0}^{K} \left(\frac{(1+x)^{-\alpha} - (1+K)^{-\alpha}}{1-(1+K)^{-\alpha}}\right) dx \\ &= K - \frac{(-\alpha+1)\,K - (1+K)^{-\alpha+1} + 1}{(-\alpha+1)\left(1-(1+K)^{-\alpha}\right)},\ \text{if } 0 < \alpha < 1; \\ &E(Z_K) = (1+\frac{1}{K})\ln(1+K) - 1,\ \text{if } \alpha = 1. \end{aligned}\right\} \quad (10)$$

Since $E(Z_K) < \infty$, the limiting pdfs of excess, age and total life in $\{Z_{K,n}\}_{n=1,2,\ldots}$exist as *time* $\to \infty$(see Section 7.2 below).

7.2. Limiting pdfs of excess, age and total life in trun(K) renewal process $\{Z_{K,n}\}_{n=1,2,\ldots}$

Denote the *limiting* excess, age and total life in $\{Z_{K,n}\}_{n=1,2,\ldots}$ by γ_K, δ_K and β_K respectively; with corresponding pdfs $f_{\gamma_K}(x)$, $x \in (0,K)$; $f_{\delta_K}(x)$, $x \in (0,K)$; $f_{\beta_K}(x)$, $x \in (0,K)$. Using well known formulas for the limiting pdfs of an ordinary renewal process where the interarrivals have a finite mean (e.g., formulas (6.2), (6.5) and (6.6), pp. 193-194, in Karlin and Taylor, 1975), and substituting from (9), we obtain for the trun(K) renewal process:

$$f_{\gamma_K}(x) = \frac{1}{E(Z_{K\gamma})}\bar{B}_{K_\gamma}(x) = \frac{1}{E(Z_{K_\gamma})}\left(1 - \frac{1-(1+x)^{-\alpha}}{1-(1+K_\gamma)^{-\alpha}}\right), x \in (0, K_\gamma), \quad (11)$$

$$f_{\delta_K}(x) = \frac{1}{E(Z_{K_\delta})}\bar{B}_{K_\delta}(x) = \frac{1}{E(Z_{K_\delta})}\left(1 - \frac{1-(1+x)^{-\alpha}}{1-(1+K_\delta)^{-\alpha}}\right), x \in (0, K_\delta), \quad (12)$$

$$f_{\beta_K}(x) = \frac{1}{E(Z_{K_\beta})}x b_{K_\beta}(x) = \frac{1}{E(Z_{K_\beta})}x\left(\frac{\alpha\,(1+x)^{-\alpha-1}}{1-(1+K_\beta)^{-\alpha}}\right), x \in (0, K_\beta), \quad (13)$$

where $E(Z_{K_\zeta})$ is given in (10) upon replacing K by K_ζ (ζ = γ, δ, or β).

Remark 2.
$(0,\infty)$ $f_{\zeta_K}(x)$ 0 $x \in (K_\zeta, \infty)$ ζ γ δ β

8. Using limiting pdfs of $\{Z_{K,n}\}_{n=1,2,\ldots}$ to approximate time-t pdfs of $\{Z_n\}_{n=1,2,\ldots}$

We use pdfs $f_{\zeta_K}(x)$ (ζ = γ, δ, β) given by (11)-(13) to approximate the time-t pdfs $f_{\zeta_t}(x)$, $x > 0$ (ζ = γ, δ, β), in $\{Z_n\}_{n=1,2,\ldots}$. The pdfs $f_{\zeta_t}(x)$, are relatively tedious to compute, requiring a numerical solution of an integral equation for the key pdf $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{x\in(0,t)}$ in (2)–(3), which is then used as part of the integrand in other formulas (Section 6 above). This suggests using $f_{\zeta_K}(x)$, $x > 0$, to approximate $f_{\zeta_t}(x)$, ζ= γ, δ β.

8.1. The truncation points K_ζ for $f_{\zeta_t}(x)$, ζ= γ, δ, β

We select right truncation points of the pdf $b(x)$, denoted by K_γ, $K_{\delta,}$, K_β. The resulting interarrival times, Z_{K_γ}, $Z_{K_{\delta,}}$, Z_{K_β}, have pdfs given in (9), with K replaced by K_γ, $K_{\delta,}$, and K_β, respectively, i.e., b_{K_ζ}, $x \in (0, K_\zeta)$, ζ = γ, δ, β. Thus $\{Z_{K_\varsigma,n}\}_{n=1,2,...}$ will be a trun(K_ζ) renewal process with pdfs of *interarrival times* $\underset{dis}{=} b_{K_\varsigma}(x)$, $x \in \left(0, \zeta_{K_\varsigma}\right)$, ζ= γ, δ, β.

8.2. Choice of truncation points K_γ, $K_{\delta,}$, K_β depending on t

The truncation points depend on the fixed time t in $\{Z_n\}_{n=1,2,...}$. We assume, for example, that t is "large" if $t \geq u$ such that $\overline{B}(u) \leq 0.05$. Then K_γ and K_β are selected as arbitrary large numbers much greater than t, because the support of $f_{\varsigma_t}(x)$, $\xi = \gamma, \beta$ is $(0, \infty)$. However, we select K_δ = t, since the support of $f_{\varsigma t}(x)$ is $(0, t)$, with an atom at t having probability π_{δ_t}.

8.3. Distance between time-t pdfs and limiting pdfs

By Remark 2 we assume, without loss of generality, that pdfs $f_{\zeta_t}(x)$ and $f_{\zeta_K}(x)$, ($\zeta = \gamma, \delta, \beta$) have the same support, i.e., $(0, \infty)$. Note that the Stieltjes integral $\int_0^{K_\zeta} dF_{\zeta_K(x)} = 1$, and the Riemann integral $\int_{K_\zeta}^\infty f_{\zeta_K}(x)dx = 0$, $\zeta = \gamma, \delta, \beta$. Also, $\int_0^\infty f_{\zeta_t}(x) = 1$, ζ = γ, β, and $\int_0^{K_\delta} dF_{\delta_t}(x) = 1$ $(K_\delta = t)$. In order to quantify the notion "$f_{\zeta_K}(\cdot)$ approximates $f_{\zeta_t}(\cdot)$" for ζ = γ, δ, β, we use an L1 measure based on the metric $\left|f_{\zeta_t}(x) - f_{\zeta_K}(x)\right|$, $x \in (0, \infty)$. This leads to an integral measure for the distance between the pdfs $f_{\zeta_t}(x)$, $\zeta = \gamma$, β and $\left\{\pi_\delta, f_{\zeta_t}(x)\right\}_{x\in(0,t)}$, and the corresponding approximating pdfs $f_{\zeta_K}(x)$, $x \in (0, \infty)$, ζ = γ, δ, β. It can be shown that $f_{\zeta_K}(x)$ and $f_{\zeta_t}(x)$, $x \in (0, \infty)$, (ζ = γ, δ, β) are bounded. It follows that a measure of distance between $f_{\zeta_t}(x)$ and $f_{\zeta_K}(x)$ is

$$\rho(f_{\zeta_t}, f_{\zeta_K}) = \frac{1}{2} \int_{x=0}^{\infty} \left|f_{\zeta_t}(x) - f_{\zeta_K}(x)\right| dx, \zeta = \gamma, \beta, \delta, \tag{14}$$

and

$$0 < \rho(f_{\zeta_t}, f_{\zeta_K}) < 1. \tag{15}$$

8.4. Range of distance measure $\rho(f_{\zeta_t}, f_{\zeta_K})$

Note that $\rho(f_{\zeta_t}, f_{\zeta_K}) > 0$ because $f_{\zeta_t}(x) > 0$, $x \in [t, \infty)$ due to: (i) the tail probabilities of $f_{\zeta_t}(x), x \in (K_\zeta, \infty)$ $(\zeta = \gamma, \beta)$, (ii) the atom π_{δ_t} in the mixed pdf $\{\pi_{\delta_t}, f_{\delta_t}(x)\}_{x\in(0,t)}$ (recalling that $K_\delta = t$), and (iii) $f_{\zeta_K}(x) = 0, x \in (K_\zeta, \infty)$. Moreover $\rho(f_{\zeta_t}, f_{\zeta_K}) < 1$, since both $f_{\zeta_t}(x) > 0$ and $f_{\zeta K}(x) > 0$, $x \in (0, K_\zeta), \zeta = \gamma, \beta, \delta$. The closer the distance $\rho(f_{\zeta_t}, f_{\zeta_K})$ is to 0, the better is the approximation; the closer the distance $\rho(f_{\zeta_t}, f_{\zeta_K})$ is to 1, the worse is the approximation. For other measures of discrepancy between pdfs, see p. 35 and Section 3.7 in Silverman (1986).

8.5. Example: Using $\rho(f_{\beta_t}, f_{\beta_K})$ when approximating the pdf of total life

As an example, we look at the approximation of $f_{\beta_t}(x)$. First assume t= 400 and K_β = 800. The pdfs $f_{\beta_t}(\cdot)$ and $f_{\beta_K}(\cdot)$ are plotted in Fig. 1. The function $\left|f_{\beta_t}(x) - f_{\beta_K}(x)\right|$, $x \in (0, K_\beta)$ is plotted in Fig. 2. Note that $\left|f_{\beta_t}(x) - f_{\beta_K}(x)\right| \neq 0$, $x \in (0, 800)$. The distance measure turns out to be $\rho(f_{\zeta_t}, f_{\zeta_K}) = 0.393383 + 0.484346 =$ **0.877729** where "0.484346" is the *tail probability* $\int_{K_\beta}^{\infty} f_{\beta_t}(x)dx = \int_{x=800}^{\infty} f_{\beta_t}(x)dx$. This indicates that the approximation is poor.

Second, assume $t = 400$ and $K_\beta = 1600$. The pdfs of $f_{\beta_t}(\cdot)$ and $f_{\beta_K}(\cdot)$ are plotted in Fig. 3. The function $\left|f_{\beta_t}(\cdot) - f_{\beta_K}(\cdot)\right|$ is plotted in Fig. 4. In this case $\left|f_{\beta_t}(x) - f_{\beta_K}(x)\right| = 0$, $x \in (0, 1600)$, at the two points where $f_{\beta_t}(x) = f_{\beta_K}(x)$. The distance measure $\rho(f_{\beta_t}, f_{\beta_K})$ is $0.177457 + 0.342485 =$ **0.519942** where "0.342485" is the *tali probability* $\int_{x=1600}^{\infty} f_{\beta_t}(x)dx$. This approximation is somewhat better. These two cases indicate that the distance measure decreases in value as K_β increases. If we choose t larger, and choose $K_\beta >> t$, both $\left|f_{\beta_t}(x) - f_{\beta_K}(x)\right|$ and the tail probability $\int_{x=K_\beta}^{\infty} f_{\beta_t}(x)dx$will decrease and $\rho(f_{\beta_t}, f_{\beta_K})$ will be much closer to 0, thus giving a very good approximation. Similar results occur for approximating the pdf of excess $f_{\gamma_t}(x), x \in (0, \infty)$. In the case of the pdf of age $\left\{\pi_\delta^{(t)} f_{\delta_t}(x)\right\}_{x \in (0, K_\delta)}$, the distance measure will decrease as the fixed time t $(= K_\delta)$ increases.

References

[1] Apostol, T.M. (1974). *Mathematical Analysis*, *Second Edition*, Addison Wesley Publishing Company, Reading, Massachusetts.

[2] Arnold, B.C. (2015).*Pareto Distributions*, *Second Edition*, CRC Press, Taylor & Francis Group, Boca Raton.

[3] Brill, P.H. (2014). Alternative Analysis of Finite-time Probability Distributions of Renewal Theory, *Probability in the Engineering and Informational Sciences*, **28**(2), 183-201.

[4] Brill, P.H. (2017). *Level Crossing Methods in Stochastic Models, Second Edition*, Springer, New York.

[5] Cox, D.R. (1962). *Renewal Theory (reprinted 1970)*, Methuen, London..

[6] Harris, C.M., Brill, P.H., Fischer, M.J. (2000). Internet-Type Queues with Power-Tailed Interarrival Times and Computational Methods for Their analysis, INFORMS J. on Computing, 12(4), 261-271.

[7] Huang, M.L., Coia, V., Brill, P.H. (2013). A Cluster Truncated Pareto Distribution and Its Applications, ISRN Probability and Statistics, Volume 2013, Article ID 265373, 10 pages, http://dx.doi.org/10.1155/2013/265373.

[8] Karlin, S., Taylor, H.M. (1975). *A First Course in Stochastic Processes, 2nd ed.*, Academic Press, New York.

[9] Kleiber, C., Kotz, S. (2003), Statistical Size Distribution in Economics and Actuarial Sciences, John Wiley, New York.

[10] Ross, S.M. (1970). *Applied Probability with Optimization Applications*, Holden Day, San Francisco.

[11] Sigman, K. (1999). A Primer on Heavy-tailed Distributions, *Queueing Systems*, **33**, 261-275**.**

[12] Sigman, K., Wolff, R.W. (1993). A Review of Regenerative Processes, *SIAM Review*, 35(2), 269-286.

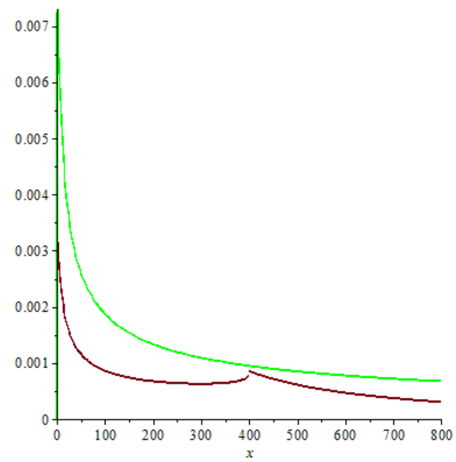

Figure 1: pdfs $f_{\beta_t}(x)$ (Red) and $f_{K_\beta}(x)$ (Green), $t = 400$, $K_\beta = 800$

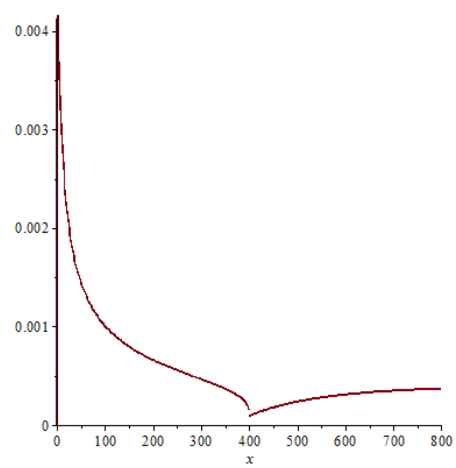

Figure 2: $\left|f_{\gamma_t}(x) - f_{K\gamma}(x)\right|, x \in (0, 1600)$ $t = 400$, $K_\beta = 800$. Distance measure = 0.393383 + 0.484346 = **0.877729** (high)

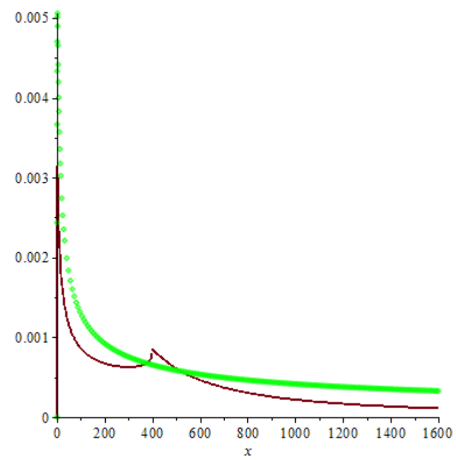

Figure 3: pdfs $f_{\gamma_t}(x)$ (Red) and $f_{K\gamma}(x)$ (Green), $t = 400$, $K_\beta = 1600$

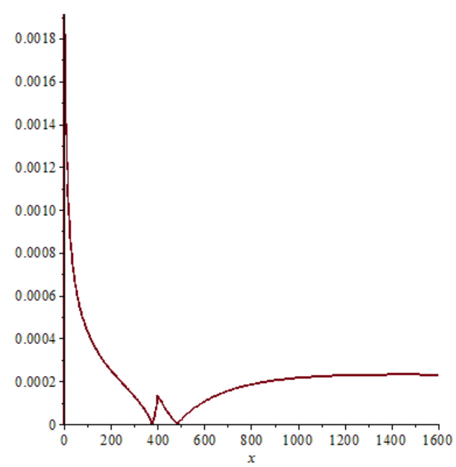

Figure 4: $\left|f_{\gamma_t}(x) - f_{K\gamma}(x)\right|, x \in (0, 1600)$ $t = 400$, $K_\beta = 1600$. Distance = 0.177457 + 0.342485 = **0.519942** (better but still high)