Bayesian Inference of Accelerated Life Tests with Lognormal Distribution and Inverse Power Law Model

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Abstract:
A Bayesian method for obtaining predictive distribution for the Lognormal life distribution under normal operating stress level is proposed for using failure data from accelerated levels of stress. Failure times of experimental samples from higher than the nominal level of stress, under constant stress levels is considered. Estimation of the model parameters by means of posterior distributions and posterior predictive distribution of future failure times, under nominal use conditions is achieved through Gibbs Sampler Markov Chain Monte-Carlo (MCMC) approach. An example is used to demonstrate the method.

Key Words: Bayesian ALT, OpenBUGS, Lognormal

1. Introduction

Accelerated life tests (ALT) are commonly used to save time and reduce cost in product development in a highly competitive manufacturing environment. Accelerated test methods have been developed over the last 50 years mostly from the frequentist approach. There are more and more Bayesian ALT articles in literature in recent years. Achcar (1991), proposes a Bayesian analysis of the data set using Laplace method of approximation and Jeffreys priors but fits an Exponential model to the data and obtains a Pareto predictive distribution. Achcar and Louzado-Neto (1992), develop Bayesian analysis theory for the Weibull life model with the Eyring model, Inverse Power Law model, and Arrhenius model using the Laplace approximation and Jeffreys priors. Nelson (1972) discusses a data set of an accelerated life experiment for breakdown of insulating fluids subject to several constant elevated test voltages. Barbosa and Louzada-Neto (1994) assume a small proportion of Type II censoring and analyze the data via generalized linear models. Mattos and Migon (2001) uses the same censoring scheme proposed by Barbosa and Louzada-Neto (1994) and report a fully Bayesian analysis of the same data set under Type II censoring using Gibbs sampler and Inverse Power Law. They assume a Gamma prior for the shape parameter and locally uniform priors for the proportionality constant and the exponent of the Power Law Model. Leon et al. (2007) analyze a different set of accelerated life test data with random effects using Bayesian Weibull regression. They assume Gamma prior for the shape parameter and Normal priors for the proportionality constant and the exponent of the Power Law Model. They also report a detailed and complete WinBUGS code for the Bayesian analysis.

2. Problem

This research was motivated by an article written by Zhang et al. (2012) on an ALT study of white organic light-emitting diodes. They analyzed the data (Table 1) using the Lognormal distribution and assuming the Inverse Power law. They report that testing under nominal conditions will take more than18 months. The failure was measured as less than 50% brightness from the initial intensity. Nominal level of stress is reported to be 3.20 mA. Their test consists of constant stress ALT experiment at three levels of acceleration 9.64 mA, 17.09 mA, and 22.58 mA. In an earlier paper in 2012 Zhang et al. explain that 9.64 mA and 17.09 mA were constant stress test levels but data from a step-stress test with four levels were converted to pretend a constant stress at 22.58 mA. They converted the step-stress data into constant stress failure times using a method described in Nelson (1980). This conversion of data assumes that "the remaining life of specimens depends only on the current cumulative fraction failed and current stress regardless how the fraction accumulated." They used maximum likelihood methods to analyze the data. They concluded that the Lognormal distribution fit the ALT data well and the acceleration was consistent with the Inverse Power Law.

2.1 Model

Zhang et al. (2012) modeled the data using a Lognormal distribution with pdf

$f(t) = \frac{1}{\sqrt{2\pi}\sigma t} \exp\left[-\frac{(\ln t - \mu)^2}{2\sigma^2} \right]$ and σ is assumed to be a constant and the mean is modeled as

$\mu = \alpha + \beta \ln(I)$, where I is the current (stress) measured in mA.

2.2 Data

Table 1 displays the data from Zhang et al. (2012).

Table 1											
		Failure times in hours									
Current Stress	Stress Method	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
9.46 mA	Constant Stress	1691.5	2084.7	210.3	2374.5	2421.5	2586.0	2621.5	2680.5	2868.0	2879.5
17.09 mA	Constant Stress	601.5	689.7	697.3	716.5	785.5	854.5	889.5	1115.7	1131.3	1251.5
22.58 mA	Converted	529.67	543.24	592.63	615.33	629.80	741.30	781.30	816.30		

2.3 Estimates of Parameters

Parameter estimates calculated at each level of stress from Zhang et al. (2012) are displayed in Table 2.

Table 2			
Stress	9.64 mA	17.09 mA	22.58 mA
$\hat{\mu}_i$ (MLE)	7.7840	6.7445	6.3320
$\hat{\sigma}_i$ (MLE Corrected for Bias)	0.1669	0.2462	0.2229

They calculate the weighted average of the standard deviations using sample sizes as the weights and report it as $\sigma_0 = 0.21161$. Using the three points and plotting a straight line they derived a linear relationship between the mean μ and the current I, $\mu = 11.6735 - 1.7223\ln(I)$. Then for the nominal current $I_0 = 3.20$ mA, average log life is equal to $\mu_0 = 9.6702$ hours and therefore the average life $\bar{\mu} = \exp\left(\mu_0 + \frac{1}{2}\sigma^2\right) = 16{,}196.6$ hours and median life $t_{0.5} = \exp(\mu_0) = 15{,}838$ hours.

3. Bayesian Method

Our objective of this research is to use fully Bayesian analysis to analyze the data reported by Zhang et al. First we used the OpenBUGS program to find equivalent numbers for the maximum likelihood estimators of parameters at each level. OpenBUGS codes for this part is given in Appendix I. Then we used all the data to get a fully Bayesian estimation of parameters. OpenBUGS codes for this part is given in Appendix II.

3.1 Model Formulation

We also assume the Lognormal distribution and model the mean as $\mu_{LN} = \frac{\alpha}{V^{\beta}} = \exp\left(\mu_N + \frac{1}{2}\sigma^2\right)$ where α and β are parameters of the Inverse Power Law, V is the stress, and μ_N and σ are the parameters of the corresponding normal distribution. This equation can be rewritten as $\mu_N = \beta_0 + \beta_1 \ln(V)$ where $2\beta_0 = 2\ln(\alpha) - \sigma^2$ and $\beta_1 = \beta$. Parameter σ is assumed to be constant and the mean is modeled as $\mu_N = \beta_0 + \beta_1 \ln(V)$, where $V = I$ is the current (stress) measured in mA.

3.2 Comparison of MLE and Bayesian Estimates of the Parameters

Table 3

Stress Level	9.64 mA	17.09 mA	22.58 mA
$\hat{\mu}_l$ (MLE)	7.784	6.744	6.332
$\hat{\mu}_l$(Bayesian)	7.784	6.744	6.332
$\hat{\sigma}_l$ (MLE)	0.1669	0.2462	0.2229
$\hat{\sigma}_l$ (Bayesian)	0.2083	0.2299	0.2020

Maximum Likelihood Estimates and the Bayesian estimates of the mean are identical to 3 decimal places. Parameter σ is assumed to be a constant and the Bayesian method produces more consistent estimates than the MLE estimates.

4. Results

4.1 Overall Comparison

Table 4		
Parameter or Quantity	Estimated from Zhang et al. Paper	Bayesian Estimates
β_0	11.6735	11.66
β_1	-1.7223	-1.718
σ	0.2116	0.2118
μ_0	9.6702	9.6617
Average life under 3.2 mA	16,196.6 hours	16,160.7 hours
Median life under 3.2 mA	15,838 hours	15,704.5 hours

5. Conclusions

Parameter estimates of the Inverse Power Law are similar up to the first decimal place. Standard deviation parameter is similar up to the third decimal place. For all practical purposes, the estimate of average life are less than 50 hours away from each other and the estimates of the median life are less than 150 hours away from each other. Bayesian method performs as well as Maximum Likelihood methods.

OpenBUGS did not have any convergence issues and error estimates justified that the Lognormal and Inverse Power Model were appropriate ways to model the data.

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Appendix I

OpenBUGS Codes to Replicate Zhang et al. MLE Results at Each Level of Stress:

(Using one data line at a time)

```
Model {
        for (j in 1:N)
        {
            z[j]<-log(y[j])
            z[j]~dnorm(mu, tau)
            }
        mu~dunif(5,15)
        tau~dunif(10,30)
    }
#Data
list(N=10, y=c(1691.5, 2084.7, 2100.3, 2374.5, 2421.5,2586, 2621.5, 2680.5, 2868, 2879.5))
#list(N=10,y=c(601.5, 689.7, 697.3, 716.5, 785.5, 854.5, 889.5, 1115.7, 1131.3, 1251.5))
#list(N=9, y=c(406,440.50,463.50,532.50,555.50,643.67,651.33, 716.50, 762.50))
#Init
list( mu=10, tau=20)
```

Appendix II

OpenBUGS Codes to Estimate Parameters using all the Data:

```
model{
for (i in 1:N){
        mu[i]<- beta0+beta1*log(V[i])
                for (j in 1:r[i])
                {
                    y[i,j]~dnorm(mu[i],tau)
                }
          }
                beta1~dnorm(0,0.001)
                beta0~dnorm(0,0.001)
                tau~dgamma(0.001,0.001)
      }
#Data
list(N=3, r=c(10,10,10), V=c(9.64, 17.09, 22.58),
y=structure(
.Data=c(
7.4333, 7.6424, 7.6498, 7.7725, 7.7921, 7.8579, 7.8715, 7.8937, 7.9614, 7.9654,
6.3994, 6.5362, 6.5472, 6.5744, 6.6663, 6.7505, 6.7907, 7.0172, 7.0311, 7.1321,
6.0063, 6.0879, 6.1388, 6.2776, 6.3199, 6.4672, 6.4790, 6.5744, 6.6366,NA),
.Dim=c(3,10)
)
)
#Initial values
list( beta0=15, beta1=-2, tau=25)
```