On an Efficient Estimator of Exponential Parameter and Its Distributional Fit

Khairul Islam[1], Tanweer Shapla[2]

[1,2] Department of Mathematics and Statistics, Eastern Michigan University, Ypsilanti, Michigan 48197

Abstract

Exponential distribution plays an important role in modeling real-life data relating to the continuous waiting time. In this presentation, a new estimator of the Exponential parameter has been proposed. Some important characteristics of the estimator have been studied. The performance of the new estimator has been compared theoretically, and empirically with the one using maximum likelihood estimator. Empirical study with simulation, and examples to real-life data reveal that the new estimator is more efficient than the maximum likelihood estimator.

Keywords: Moment generating function, method of moments, exponential parameter, relative efficiency, simulation.

1. Introduction

Exponential distribution has widely been used in modeling distributions in areas ranging from studies on the lifetimes of manufactured items [1-3] to research involving survival or remission times in chronic diseases [4-5]. The wide applicability of exponential distribution in lifetime modeling is due to the availability of simple statistical methods for it [2] and it represents the lifetimes of many things such as various types of manufactured items [1]. Exponential distribution and of its parameter estimation appear in any standard book of statistics [6-8]. We say that a continuous random variable X follows an exponential distribution with parameter θ (mean) if the probability density function is given by

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}; x > 0;\ \theta > 0$$

In general, θ is unknown and estimated using sample data. Let $x_1, x_2, \cdots, x_n$ be a random sample of size n. Then, the maximum likelihood function of $f(x)$ is given by

$$L(\theta) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}$$

Taking logarithm on both sides of the likelihood function, we get

$$l(\theta) = lnL(\theta) = -nln\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

---

[1] kislam@emich.edu

Taking derivative of l with respect to θ, and setting equal to zero, a maximum likelihood estimate (MLE) of θ, $\hat{\theta}$ is given by

$$\frac{\partial l}{\partial \theta} = -n\frac{1}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = 0$$

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

It is easy to see that $\hat{\theta}$ is an unbiased estimate of θ, i.e.,

$$E(\hat{\theta}) = \theta$$

with variance of $\hat{\theta}$ given by

$$V(\hat{\theta}) = \frac{\theta^2}{n}$$

In the next section, we propose a new estimator of the exponential parameter θ and study some statistical properties of the proposed estimator.

2. Proposed Estimator

The moment generating function of $X \sim Exp(\theta)$ is

$$M_X(t) = E(e^{Xt}) = (1 - \theta t)^{-1}$$

Given a random sample $X_1, X_2, \cdots, X_n$ of size n, the moment generating function of $\sum_{i=1}^{n} X_i$ is given by

$$\begin{aligned} M_{\sum_{i=1}^{n} X_i}(t) &= E\left(e^{\sum_{i=1}^{n} X_i t}\right) \\ &= \prod_{i=1}^{n} E(e^{X_i t}) \\ &= \prod_{i=1}^{n} M_{X_i}(t) \\ &= \prod_{i=1}^{n} (1 - \theta t)^{-1} \\ &= (1 - \theta t)^{-n} \end{aligned}$$

Then, by the method of moments, a new estimator of θ, $\tilde{\theta}$ follows from solving the equation

$$e^{\sum_{i=1}^{n} X_i t} = (1 - \tilde{\theta} t)^{-n}$$

After an algebraic manipulation, we have the following new estimator of θ

$$\tilde{\theta} = \frac{1 - e^{-t\bar{x}}}{t}; t \neq 0$$

3. Properties of New Estimator

In this section, we study some properties of the proposed estimator, which we state in terms of the following theorems:

Theorem 3.1 The expected value of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is $E(\tilde{\theta}) = \frac{1}{t}\left[1 - \left(1 + \frac{t\theta}{n}\right)^{-n}\right]$ and if $t \to 0$, then $\tilde{\theta}$ is an unbiased estimate of θ.

Proof: The expected value of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$E(\tilde{\theta}) = \frac{1 - E(e^{-t\bar{x}})}{t} = \frac{1}{t}\left[1 - \left(1 + \frac{t}{n}\theta\right)^{-n}\right]; t \neq 0$$

Taking limit as $t \to 0$ and applying the L' Hospital Rule, we have

$$\lim_{t\to 0} E(\tilde{\theta}) = \lim_{t\to 0} \frac{(-)(-n)\left(1 + \frac{t}{n}\theta\right)^{-n-1}\left(\frac{1}{n}\theta\right)}{1} = \theta$$

Theorem 3.2 The bias of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$B(\tilde{\theta}) = \frac{1}{t}\left[1 - \left(1 + \frac{t\theta}{n}\right)^{-n}\right] - \theta$$

and if $t \to 0$, then bias of $\tilde{\theta}$ is 0.

Proof: The bias of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$B(\tilde{\theta}) = E(\tilde{\theta}) - \theta = \frac{1 - E(e^{-t\bar{x}})}{t} - \theta = \frac{\left[1 - \left(1 + \frac{t}{n}\theta\right)^{-n}\right]}{t} - \theta; t \neq 0$$

Taking limit as $t \to 0$ and applying the L' Hospital Rule, we have

$$\lim_{t\to 0} B(\tilde{\theta}) = \lim_{t\to 0} \frac{n\left(1 + \frac{t}{n}\theta\right)^{-n-1}\left(\frac{1}{n}\theta\right)}{1} - \theta = \theta - \theta = 0$$

Theorem 3.3 The variance of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$V(\tilde{\theta}) = \frac{1}{t^2}\left[\left(1 + \frac{2t\theta}{n}\right)^{-n} - \left(1 + \frac{t\theta}{n}\right)^{-2n}\right]$$

and if $t \to 0$, then variance of $\tilde{\theta}$ is same as the variance of $\hat{\theta}$.

Proof: Note that

$$M_{\sum_{i=1}^{n} X_i}(t) = E\left(e^{\sum_{i=1}^{n} X_i t}\right) = (1-\theta t)^{-n}$$

It also follows that

$$M_{\bar{x}}(t) = M_{\sum_{i=1}^{n} X_i}(t/n) = \left(1-\frac{\theta t}{n}\right)^{-n}$$

$$M_{\bar{x}}(-t) = \left(1+\frac{\theta t}{n}\right)^{-n}$$

$$M_{\bar{x}}(-kt) = \left(1+\frac{\theta k t}{n}\right)^{-n}$$

Now, the variance of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$V(\tilde{\theta}) = V\left(\frac{1-e^{-t\bar{x}}}{t}\right) = \frac{1}{t^2} V(e^{-t\bar{x}})$$

$$\begin{aligned} V(e^{-t\bar{x}}) &= E(e^{-t\bar{x}})^2 - [E(e^{-t\bar{x}})]^2 \\ &= E(e^{-2t\bar{x}}) - [E(e^{-t\bar{x}})]^2 \\ &= M_{\bar{x}}(-2t) - [M_{\bar{x}}(-t)\,]^2 \\ &= \left(1+\frac{2t\theta}{n}\right)^{-n} - \left[\left(1+\frac{t\theta}{n}\right)^{-n}\right]^2 \\ &= \left(1+\frac{2t\theta}{n}\right)^{-n} - \left(1+\frac{t\theta}{n}\right)^{-2n} \end{aligned}$$

Then,

$$V(\tilde{\theta}) = \frac{1}{t^2}\left[\left(1+\frac{2t\theta}{n}\right)^{-n} - \left(1+\frac{t\theta}{n}\right)^{-2n}\right]; t \neq 0$$

Taking limit as $t \to 0$ and applying the L' Hospital Rule, we have

$$\begin{aligned} \lim_{t\to 0} V(\tilde{\theta}) &= \lim_{t\to 0} \frac{(-n)\left(1+\frac{2t\theta}{n}\right)^{-n-1}\left(\frac{2\theta}{n}\right) - (-2n)\left(1+\frac{t\theta}{n}\right)^{-2n-1}\left(\frac{\theta}{n}\right)}{2t} \\ &= \lim_{t\to 0} \frac{-\theta\left(1+\frac{2t\theta}{n}\right)^{-n-1} + \theta\left(1+\frac{t\theta}{n}\right)^{-2n-1}}{t} \\ &= \lim_{t\to 0} \frac{\theta(n+1)\left(1+\frac{2t\theta}{n}\right)^{-n-2}\left(\frac{2\theta}{n}\right) - (2n+1)\theta\left(1+\frac{t\theta}{n}\right)^{-2n-2}\left(\frac{\theta}{n}\right)}{1} \\ &= \theta(n+1)(1+0)\left(\frac{2\theta}{n}\right) - (2n+1)\theta(1+0)\left(\frac{\theta}{n}\right) \end{aligned}$$

$$= \frac{2\theta^2(n+1)}{n} - \frac{\theta^2(2n+1)}{n}$$

$$= \frac{2\theta^2 - \theta^2}{n} = \frac{\theta^2}{n} = V(\hat{\theta})$$

Theorem 3.4 The mean square error (MSE) of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$MSE(\tilde{\theta}) = \frac{1}{t^2}\left[\left(1+\frac{2t\theta}{n}\right)^{-n} - \left(1+\frac{t\theta}{n}\right)^{-2n}\right] + \left[\frac{1}{t}\left\{1-\left(1+\frac{t}{n}\theta\right)^{-n}\right\} - \theta\right]^2$$

and if $t \to 0$, then MSE of $\tilde{\theta}$ is the same as the variance of $\hat{\theta}$.

Proof: The MSE of $\tilde{\theta} = \frac{t\bar{x}}{e^t - 1}$ is

$$MSE(\tilde{\theta}) = V(\tilde{\theta}) + [B(\tilde{\theta})]^2$$

$$= \frac{1}{t^2}\left[\left(1+\frac{2t\theta}{n}\right)^{-n} - \left(1+\frac{t\theta}{n}\right)^{-2n}\right] + \left[\frac{1}{t}\left\{1-\left(1+\frac{t}{n}\theta\right)^{-n}\right\} - \theta\right]^2$$

Taking limit as $t \to 0$ and applying the L' Hospital Rule, we have

$$\lim_{t\to 0} MSE(\tilde{\theta}) = \lim_{t\to 0} V(\tilde{\theta}) + \left[\lim_{t\to 0} B(\tilde{\theta})\right]^2$$

$$= \frac{\theta^2}{n} + 0$$

$$= \frac{\theta^2}{n}$$

$$= V(\hat{\theta})$$

Theorem 3.5 The relative efficiency (RE) of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ with respect to $\hat{\theta}$ is

$$RE = \frac{\theta^2/n}{\frac{1}{t^2}\left[\left(1+\frac{2t\theta}{n}\right)^{-n} - \left(1+\frac{t\theta}{n}\right)^{-2n}\right] + \left[\frac{1}{t}\left\{1-\left(1+\frac{t}{n}\theta\right)^{-n}\right\} - \theta\right]^2} \times 100\%$$

Proof: The relative efficiency of $\tilde{\theta} = \frac{t\bar{x}}{e^t - 1}$ with respect to $\hat{\theta}$ is given by

$$RE = \frac{V(\hat{\theta})}{MSE(\tilde{\theta})} \times 100$$

$$= \frac{\theta^2/n}{\frac{1}{t^2}\left[\left(1+\frac{2t\theta}{n}\right)^{-n} - \left(1+\frac{t\theta}{n}\right)^{-2n}\right] + \left[\frac{1}{t}\left\{1-\left(1+\frac{t}{n}\theta\right)^{-n}\right\} - \theta\right]^2} \times 100$$

It is easy to see that as $t \to 0$, $\tilde{\theta}$ and $\hat{\theta}$ are same. If $t \neq 0$, then there may exist a non-zero t such that

$$MSE(\tilde{\theta}) < V(\hat{\theta})$$

or,

$$\frac{1}{t^2}\left[\left(1+\frac{2t\theta}{n}\right)^{-n}-\left(1+\frac{t\theta}{n}\right)^{-2n}\right]+\left[\frac{1}{t}\left\{1-\left(1+\frac{t}{n}\theta\right)^{-n}\right\}-\theta\right]^2<\frac{\theta^2}{n} \qquad (1)$$

In section 5 below, we search for values of t, for selected values of θ and n, for which the relation (1) holds true and hence find the percent relative efficiency of the proposed estimator $\tilde{\theta}$ with respect to $\hat{\theta}$ using R code.

4. Fitting of an Exponential Distribution

In this section, we discuss how we can employ the two underlying estimates for fitting of exponential distributions to real life data using the chi-squared goodness of fit, and AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) criteria.

4.1 Chi-squared Goodness of Fit

For chi-squared goodness of fit [8-9], we compare observed and expected frequency using a chi-squared distribution. The algorithm for the goodness of fit is as follows:

Given an observed sample, we divide the range of the observed values into k equal intervals and evaluate the observed and expected frequency in each of the k intervals using the following procedure.

Let the k intervals be designated by $[u_0, u_1]$, $(u_1, u_2]$, …, $(u_{i-1}, u_i]$, …, $(u_{k-1}, u_k]$, where u_i is the upper end-point of ith interval, $i = 1, 2, \dots, k$. Note that an observation of the given sample can be observed in any of the intervals with probability $p = 1/k$. Then, it follows that

$$F(u_i) = ip$$

Also, by the property of the exponential distribution,

$$F(u_i) = \int_0^{u_i} \frac{1}{\theta} e^{-\frac{t}{\theta}}\, dt = 1 - e^{-\frac{u_i}{\theta}}$$

We solve the two expressions of $F(u_i)$ to find the upper end-point of the ith interval, u_i:

$$ip = 1 - e^{-\frac{u_i}{\theta}}$$

By algebraic manipulation, it follows that

$$u_i = \theta \ln\left(\frac{1}{(1-ip)}\right) \qquad (2)$$

Let o_i be the observed frequency in the ith interval. Then,

$$o_i = \#(u_{i-1} < x \le u_i) \qquad (3)$$

The expected frequency for the ith interval is

$$e_i = n \times p_i;\ p_i = \frac{1}{k} = 0.125, if\ k = 8$$

The goodness of fit statistic is then given by

$$\chi^2 = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}$$

which follows a chi-squared distribution with $(k - 1 - \nu)$ degrees of freedom, where ν is the number of parameters estimated. In our example, we only estimate one parameter θ. Therefore, $\nu = 1$.

4.2 Goodness of Fit Using AIC and BIC Criteria

The likelihood function of exponential distribution is given by:

$$L = \prod_{j=1}^{n} \frac{1}{\theta} e^{-\frac{x_j}{\theta}} = \left(\frac{1}{\theta}\right)^n e^{-\frac{\sum_{j=1}^{n} x_j}{\theta}}$$

In order to apply AIC and BIC [10-12], we find the estimated likelihood functions given by two estimators $\hat{\theta}$ and $\tilde{\theta}$ of θ.

Then, using the likelihood estimator $\hat{\theta}$,

$$\hat{L} = \left(\frac{1}{\hat{\theta}}\right)^n e^{-\frac{\sum_{j=1}^{n} x_j}{\hat{\theta}}} = \left(\frac{1}{\bar{x}}\right)^n e^{-n}$$

Using the new estimator $\tilde{\theta}$,

$$\tilde{L} = \left(\frac{1}{\tilde{\theta}}\right)^n e^{-\frac{\sum_{j=1}^{n} x_j}{\tilde{\theta}}} = \left(\frac{1}{\frac{1-e^{-t\bar{x}}}{t}}\right)^n exp\left(-\frac{n\bar{x}}{\frac{1-e^{-t\bar{x}}}{t}}\right)$$

Then, AIC and BIC due to the estimator $\hat{\theta}$ are

$$\widehat{AIC} = (-2)\left(ln\hat{L}\right) + 2\nu$$

$$\widehat{BIC} = (-2)\left(ln\hat{L}\right) + \nu\left(ln(n)\right)$$

Similarly, AIC and BIC due to the estimator $\tilde{\theta}$ are

$$\widetilde{AIC} = (-2)\left(ln\tilde{L}\right) + 2\nu$$

$$\widetilde{BIC} = (-2)\left(ln\tilde{L}\right) + \nu\left(ln(n)\right)$$

The method with the lower of values of AIC and BIC provides the better fit.

5. Real Life Applications with Examples

In order to assess the goodness of fit and compare performance of the two underlying estimates, we consider two examples (Examples 1 and 2) of real-life data. We evaluate chi-square goodness of fit and AIC-BIC for the two underlying estimates.

For both examples we consider 8 intervals (i.e., $k = 8$) designated by $[u_0, u_1]$, $(u_1, u_2]$, …, $(u_6, u_7]$, …, $(u_7, u_8]$, where u_i is the upper end-point of ith interval, $i = 1, 2, \ldots, 8$. Then, an observation of a given sample of size $n = 72$ can be observed in any of the intervals with probability $p = \frac{1}{8} = 0.125$, with an expected frequency of each interval $n \times 0.125 = 72 \times 0.125 = 9$. Tables (1) and (2) provide observed and expected frequencies for each examples using equation (2) and (3).

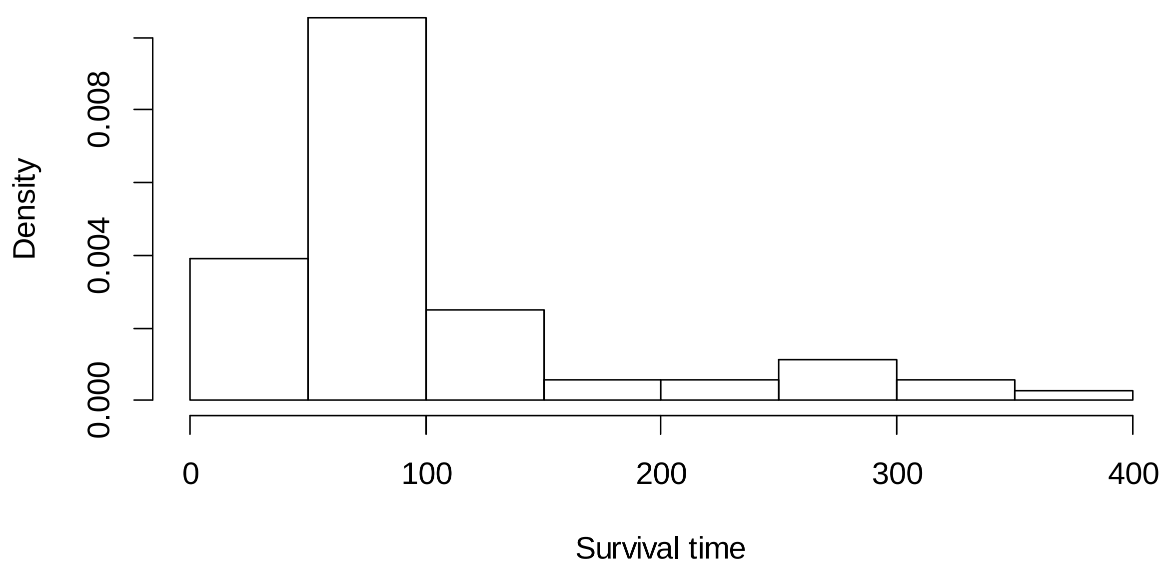
Example 1

This data represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [13], and appeared in [5].

12	15	22	24	24	32	32	33	34	38	38	43
44	48	52	53	54	54	55	56	57	58	58	59
60	60	60	60	61	62	63	65	65	67	68	70
70	72	73	75	76	76	81	83	84	85	87	91
95	96	98	99	109	110	121	127	129	131	143	146
146	175	175	211	233	258	258	263	297	341	341	376

The density histogram in Figure 1 demonstrates that the shape of the distribution of the data is positive skewed.

Figure 1: Density of survival time of guinea pigs for data in Example 1



The mean survival time is 99.82 days. We test the null hypothesis that the data come from an exponential distribution with mean 100.

Table 1 below provides observed and expected frequencies evaluated using equations (2) and (3) using the MLE estimate $\hat{\theta} = \bar{x}$.

Table 1: Observed and expected frequencies of survival times of guinea pigs infected with virulent tubercle bacilli for Example 1 using the MLE $\hat{\theta} = \bar{x}$ of θ.

Interval	o_i	e_i	$(o_i - e_i)^2/e_i$
[0,13.329)	1	9	7.111
[13.329, 28.716)	4	9	2.778
[28.716, 46.915)	8	9	0.111
[46.915, 69.189)	22	9	18.778
[69.189, 97.905)	15	9	4.000
[97.905, 138.379)	8	9	0.111
[138.379, 207.568)	5	9	1.778
[207.568, ∞)	9	9	0.000
Total	72	72	34.667

Then, the observed value of the Chi-square test statistic under the MLE estimate $\hat{\theta} = \bar{x}$ is

$$\chi^2_{\hat{\theta}} = \sum_{i=1}^{8} \frac{(o_i - e_i)^2}{e_i} = 34.667$$

and the p-value is 0.000005 with $d.f. = (8 - 1 - 1) = 6$.

It also follows that $AIC = 808.8836$ and $BIC = 811.1603$.

For the new estimate $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ with $t = 0.00051$, the observed and expected frequencies evaluated using equations (2) and (3) appear in Table 2 below.

Table 2: Observed and expected frequencies of survival times of guinea pigs infected with virulent tubercle bacilli for Example 1 using the new estimate $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ of θ.

Interval	o_i	e_i	$(o_i - e_i)^2/e_i$
[0, 12.995)	1	9	7.111
[12.995, 27.998)	4	9	2.778
[27.998, 45.741)	8	9	0.111
[45.741, 67.458)	21	9	16.000
[67.458, 95.455)	15	9	4.000
[95.455, 134.916)	9	9	0.000
[134.916, 202.373)	5	9	1.778
[202.373, ∞)	9	9	0.000
Total	72	72	31.778

Then, the observed value of the Chi-square test statistic under the new estimate $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$\chi^2_{\hat{\theta}} = \sum_{i=1}^{8} \frac{(o_i - e_i)^2}{e_i} = 31.778$$

and the p-value is 0.00002 with $d.f. = (8 - 1 - 1) = 6$.

It also follows that $AIC = 808.9303$ and $BIC = 811.2069$.

Given above analyses, at 5% level of significance we reject the null hypothesis that the data come from an exponential distribution with mean 100 using the both estimates $\hat{\theta}$ and $\tilde{\theta}$. However, the relative efficiency of the proposed estimate $\tilde{\theta}$ compared to $\hat{\theta}$ is 105.49%.

It is to be noted that we can search for values of t for selected values of θ and n, satisfying the relation in equation (1) in order to compute the percent relative efficiency of the proposed estimator $\tilde{\theta}$ with respect to $\hat{\theta}$.

Below we provide R code that was used to search for values of t and computing relative efficiency of $\tilde{\theta}$ with respect to $\hat{\theta}$, along with other computations aspects for data in Example 1.

```
n=length(x);
mean=mean(x)#observed value of mean=99.82
theta=100;
nu=1;
for (t in seq(0.00001,.001,.0001)){
a<-(1/t^2)*((1+(2*t*theta/n))^(-n)-(1+(t*theta/n))^(-2*n))+((1/t)*(1-(1+t*theta/n)^(-n))-
theta)^2
b<-theta^2/n;
ifelse (a<b,{print(t);print(b/a*100)},print(0))
}
est1=round(mean(x),digits=3);# MLE estimate
est2=round((1-exp(-est1*0.00051))/0.00051,digits=3);#New estimate

#Assessing GOF for using est1;
u=c();
for (i in 1: 7){u[i]=round(-est1*log(1-i*.125),digits=3)}
print(u)
o=c();
o[1]=sum(x<u[1]);
o[2]=sum(x>=u[1] & x<u[2])
o[3]=sum(x>=u[2] & x<u[3])
o[4]=sum(x>=u[3] & x<u[4])
o[5]=sum(x>=u[4] & x<u[5])
o[6]=sum(x>=u[5] & x<u[6])
o[7]=sum(x>=u[6] & x<u[7])
o[8]=sum(x>=u[7])
print(o)
e=rep(1,8)*.125*n;
chi2.1=round(sum((o-e)^2/e), digits=3);
pval.1=pchisq(chi2.1, df=6,nc=0,lower.tail=F)
like1=((1/est1)^n)*(exp(-n))
aic1=-(2*log(like1))+2*nu
```

```
bic1=-2*log(like1)+nu*log(n)

# Assessing GOF for using est2;
u=c();
for (i in 1: 7){u[i]=round(-est2*log(1-i*.125),digits=3)}
print(u)
o=c();
o[1]=sum(x<u[1]);
o[2]=sum(x>=u[1] & x<u[2])
o[3]=sum(x>=u[2] & x<u[3])
o[4]=sum(x>=u[3] & x<u[4])
o[5]=sum(x>=u[4] & x<u[5])
o[6]=sum(x>=u[5] & x<u[6])
o[7]=sum(x>=u[6] & x<u[7])
o[8]=sum(x>=u[7])
print(o)
chi2.2=round(sum((o-e)^2/e), digits=3);
pval.2=pchisq(chi2.2, df=6,nc=0,lower.tail=F)
like2=((1/est2)^n)* exp((-n*est1/est2))
aic2=-(2*log(like2))+2*nu
bic2=-2*log(like2)+nu*log(n)
print(c(chi2.1, pval.1, chi2.2, pval.2,qchisq(0.05, df=6,nc=0,lower.tail=F)))
print(c(aic1, bic1))
print(c(aic2, bic2))
```

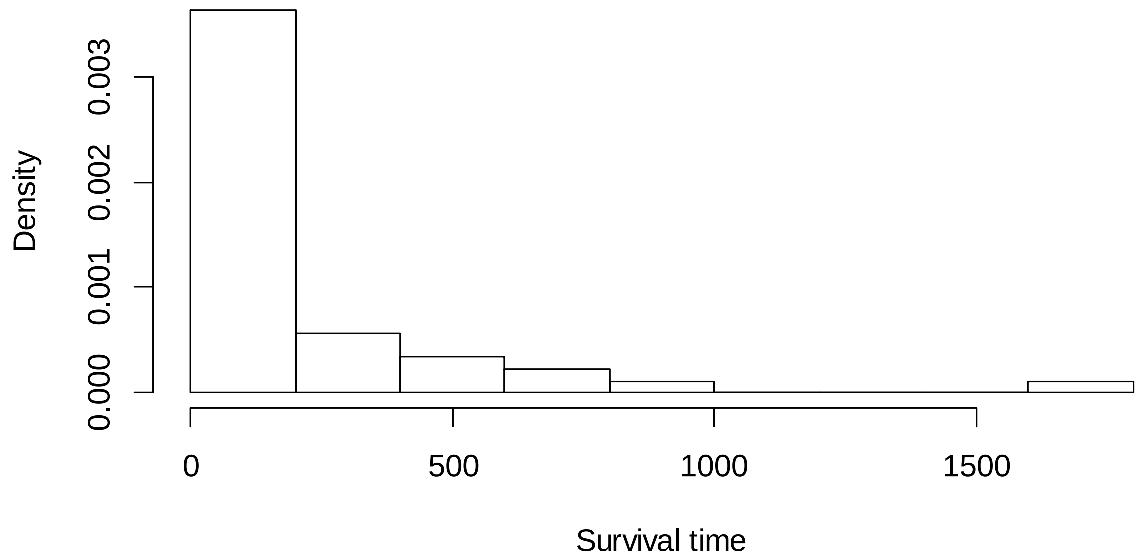
Example 2

The data set reported by Efron [14] and also appeared in [5] represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT).

12.2	23.6	23.7	25.9	32	37	41.35	47.38	55.46	58.4	63.5
68.46	78.3	74.5	81.4	84	92	94	110	112	119	127
130	133	140	146	155	159	173	179	194	195	209
249	281	319	339	432	469	519	633	725	817	1776

The density histogram in Figure 2 demonstrates that the shape of the distribution of the data is positive skewed.

Figure 2: Density of survival time of patients with head and neck cancer for data in Example 2



The mean survival time is 223.477. We test the null hypothesis that the data come from an exponential distribution with mean 223.

Table 3 below provides observed and expected frequencies evaluated using equations (2) and (3) using the MLE estimate $\hat{\theta} = \bar{x}$.

Table 3: Observed and expected frequencies of survival times of patients suffering from Head and Neck cancer treated with a combination of radiotherapy and chemotherapy (RT+CT) for Example 2 using the MLE $\hat{\theta} = \bar{x}$ of exponential parameter θ.

Interval	o_i	e_i	$(o_i - e_i)^2/e_i$
[0, 29.841)	4	5.5	0.409
[29.841, 64.290)	7	5.5	0.409
[64.290, 105.035)	7	5.5	0.409
[105.035, 154.902)	8	5.5	1.136
[154.902, 219.193)	7	5.5	0.409
[219.193, 309.805)	2	5.5	2.227
[309.805, 464.707)	3	5.5	1.136
[464.707, ∞)	6	5.5	0.045
Total	44	44	6.182

Then, the observed value of the Chi-square test statistic under the MLE estimate $\hat{\theta} = \bar{x}$ is

$$\chi^2_{\hat{\theta}} = \sum_{i=1}^{8} \frac{(o_i - e_i)^2}{e_i} = 6.182$$

and the p-value is 0. 40311 with $d.f. = (8 - 1 - 1) = 6$.

It also follows that $AIC = 566.0191$ and $BIC = 567.8033$.

For the new estimate $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ with $t = 0.00041$, the observed and expected frequencies evaluated using equations (2) and (3) appear in Table 4 below.

Table 4: Observed and expected frequencies of survival times of patients suffering from Head and Neck cancer treated with a combination of radiotherapy and chemotherapy (RT+CT) for Example 2 using the new estimate $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ of exponential parameter θ.

Interval	o_i	e_i	$(o_i - e_i)^2/e_i$
[0, 28.515)	4	5.5	0.409
[28.515, 61.433)	6	5.5	0.045
[61.433, 100.367)	8	5.5	1.136
[100.367, 148.018)	8	5.5	1.136
[148.018, 209.451)	7	5.5	0.409
[209.451, 296.036)	2	5.5	2.227
[296.036, 444.054)	3	5.5	1.136
[444.054, ∞)	6	5.5	0.045
Total	44	44	6.545

Then, the observed value of the Chi-square test statistic under the new estimate $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$\chi^2_{\tilde{\theta}} = \sum_{i=1}^{8} \frac{(o_i - e_i)^2}{e_i} = 6.545$$

and the p-value is 0.36498 with $d.f. = (8 - 1 - 1) = 6$.

It also follows that $AIC = 566.1115$ and $BIC = 567.8957$.

Given above analyses, at 5% level of significance there is a strong evidence that the data come from an exponential distribution with mean 223 using the both estimates $\hat{\theta}$ and $\tilde{\theta}$. However, the relative efficiency of the proposed estimate $\tilde{\theta}$ compared to $\hat{\theta}$ is 108.7%.

6. Relative Efficiency of the New Estimator

In this section, we investigate relative efficiency of the proposed estimator $\tilde{\theta}$ compared to $\hat{\theta}$ for given values of t, θ and n using R code.

We consider various values of the parameter θ fixed at 0.5, 2.5, 5, 10, 15, 20, 25, 30, 50, 100, arbitrarily and sample size at 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 and 100. For each combination of θ and n, we consider values of t between a and b with an increment of

0.0001, notationally expressed as $t \in [a, b, @\ 0.0001]$, where $a = 0.0001$ and values of b are evaluated using the search so as to satisfy (1) and are reported along with the relative efficiency for a given combination of θ and n in the Table 5.

Table 5: Relative efficiency of proposed estimate compared to the maximum likelihood estimate for varying sample size and t

θ	n	Range of t in the search	Range of relative efficiency (re)
0.50	5	$0.0001 \leq t \leq 2.1600$	$100.01 \leq re \leq 164.85$
	10	$0.0001 \leq t \leq 1.2320$	$100.00 \leq re \leq 135.59$
	15	$0.0001 \leq t \leq 0.8789$	$100.01 \leq re \leq 124.59$
	20	$0.0001 \leq t \leq 0.6862$	$100.01 \leq re \leq 118.79$
	25	$0.0001 \leq t \leq 0.5637$	$100.00 \leq re \leq 115.21$
	30	$0.0001 \leq t \leq 0.4786$	$100.00 \leq re \leq 112.78$
	35	$0.0001 \leq t \leq 0.4159$	$100.01 \leq re \leq 111.02$
	40	$0.0001 \leq t \leq 0.3679$	$100.00 \leq re \leq 109.68$
	45	$0.0001 \leq t \leq 0.3298$	$100.00 \leq re \leq 108.64$
	50	$0.0001 \leq t \leq 0.2989$	$100.00 \leq re \leq 107.80$
	100	$0.0002 \leq t \leq 0.1544$	$100.00 \leq re \leq 103.95$
2.5	5	$0.0001 \leq t \leq 0.4330$	$100.06 \leq re \leq 164.85$
	10	$0.0001 \leq t \leq 0.2464$	$100.00 \leq re \leq 135.59$
	15	$0.0001 \leq t \leq 0.1757$	$100.04 \leq re \leq 124.59$
	20	$0.0001 \leq t \leq 0.1372$	$100.02 \leq re \leq 118.79$
	25	$0.0001 \leq t \leq 0.1127$	$100.02 \leq re \leq 115.21$
	30	$0.0001 \leq t \leq 0.0957$	$100.01 \leq re \leq 112.78$
	35	$0.0001 \leq t \leq 0.0831$	$100.04 \leq re \leq 111.02$
	40	$0.0001 \leq t \leq 0.0735$	$100.04 \leq re \leq 109.68$
	45	$0.0001 \leq t \leq 0.0659$	$100.03 \leq re \leq 108.64$
	50	$0.0001 \leq t \leq 0.0597$	$100.04 \leq re \leq 107.80$
	100	$0.0001 \leq t \leq 0.0308$	$100.04 \leq re \leq 103.95$
5	5	$0.0001 \leq t \leq 0.2166$	$100.01 \leq re \leq 164.85$
	10	$0.0001 \leq t \leq 0.1232$	$100.00 \leq re \leq 135.59$
	15	$0.0001 \leq t \leq 0.0877$	$100.11 \leq re \leq 124.59$
	20	$0.0001 \leq t \leq 0.0686$	$100.02 \leq re \leq 118.79$
	25	$0.0001 \leq t \leq 0.0563$	$100.06 \leq re \leq 115.21$
	30	$0.0001 \leq t \leq 0.0478$	$100.05 \leq re \leq 112.78$
	35	$0.0001 \leq t \leq 0.0415$	$100.09 \leq re \leq 111.02$
	40	$0.0001 \leq t \leq 0.0367$	$100.08 \leq re \leq 109.68$
	45	$0.0001 \leq t \leq 0.0329$	$100.08 \leq re \leq 108.64$
	50	$0.0001 \leq t \leq 0.0298$	$100.09 \leq re \leq 107.80$
	100	$0.0001 \leq t \leq 0.0154$	$100.04 \leq re \leq 103.95$
10	5	$0.0001 \leq t \leq 0.1083$	$100.01 \leq re \leq 164.85$
	10	$0.0001 \leq t \leq 0.0616$	$100.00 \leq re \leq 135.59$
	15	$0.0001 \leq t \leq 0.0439$	$100.07 \leq re \leq 124.59$
	20	$0.0001 \leq t \leq 0.0343$	$100.02 \leq re \leq 118.79$
	25	$0.0001 \leq t \leq 0.0281$	$100.14 \leq re \leq 115.21$
	30	$0.0001 \leq t \leq 0.0239$	$100.05 \leq re \leq 112.78$
	35	$0.0001 \leq t \leq 0.0207$	$100.18 \leq re \leq 111.02$
	40	$0.0001 \leq t \leq 0.0183$	$100.17 \leq re \leq 109.68$
	45	$0.0001 \leq t \leq 0.0164$	$100.17 \leq re \leq 108.64$

	50 100	$0.0001 \leq t \leq 0.0149$ $0.0001 \leq t \leq 0.0077$	$100.09 \leq re \leq 107.80$ $100.04 \leq re \leq 103.95$
15	5 10 15 20 25 30 35 40 45 50 100	$0.0001 \leq t \leq 0.0722$ $0.0001 \leq t \leq 0.0410$ $0.0001 \leq t \leq 0.0292$ $0.0001 \leq t \leq 0.0228$ $0.0001 \leq t \leq 0.0187$ $0.0001 \leq t \leq 0.0159$ $0.0001 \leq t \leq 0.0138$ $0.0001 \leq t \leq 0.0122$ $0.0001 \leq t \leq 0.0109$ $0.0001 \leq t \leq 0.0099$ $0.0001 \leq t \leq 0.0051$	$100.01 \leq re \leq 164.85$ $100.13 \leq re \leq 135.59$ $100.22 \leq re \leq 124.59$ $100.18 \leq re \leq 118.79$ $100.23 \leq re \leq 115.21$ $100.14 \leq re \leq 112.78$ $100.18 \leq re \leq 111.02$ $100.17 \leq re \leq 109.68$ $100.26 \leq re \leq 108.64$ $100.18 \leq re \leq 107.79$ $100.14 \leq re \leq 103.95$
20	5 10 15 20 25 30 35 40 45 50 100	$0.0001 \leq t \leq 0.0541$ $0.0001 \leq t \leq 0.0308$ $0.0001 \leq t \leq 0.0219$ $0.0001 \leq t \leq 0.0171$ $0.0001 \leq t \leq 0.0140$ $0.0001 \leq t \leq 0.0119$ $0.0001 \leq t \leq 0.0103$ $0.0001 \leq t \leq 0.0091$ $0.0001 \leq t \leq 0.0082$ $0.0001 \leq t \leq 0.0074$ $0.0001 \leq t \leq 0.0038$	$100.10 \leq re \leq 164.85$ $100.00 \leq re \leq 135.59$ $100.22 \leq re \leq 124.59$ $100.18 \leq re \leq 118.79$ $100.31 \leq re \leq 115.21$ $100.23 \leq re \leq 112.78$ $100.35 \leq re \leq 111.02$ $100.35 \leq re \leq 109.68$ $100.17 \leq re \leq 108.64$ $100.27 \leq re \leq 107.79$ $100.23 \leq re \leq 103.95$
25	5 10 15 20 25 30 35 40 45 50 100	$0.0001 \leq t \leq 0.0433$ $0.0001 \leq t \leq 0.0246$ $0.0001 \leq t \leq 0.0175$ $0.0001 \leq t \leq 0.0137$ $0.0001 \leq t \leq 0.0112$ $0.0001 \leq t \leq 0.0095$ $0.0001 \leq t \leq 0.0083$ $0.0001 \leq t \leq 0.0073$ $0.0001 \leq t \leq 0.0065$ $0.0001 \leq t \leq 0.0059$ $0.0001 \leq t \leq 0.0030$	$100.06 \leq re \leq 164.85$ $100.13 \leq re \leq 135.59$ $100.30 \leq re \leq 124.59$ $100.10 \leq re \leq 118.79$ $100.31 \leq re \leq 115.21$ $100.31 \leq re \leq 112.78$ $100.09 \leq re \leq 111.02$ $100.26 \leq re \leq 109.68$ $100.44 \leq re \leq 108.64$ $100.36 \leq re \leq 107.80$ $100.42 \leq re \leq 103.95$
30	5 10 15 20 25 30 35 40 45 50 100	$0.0001 \leq t \leq 0.0361$ $0.0001 \leq t \leq 0.0205$ $0.0001 \leq t \leq 0.0146$ $0.0001 \leq t \leq 0.0114$ $0.0001 \leq t \leq 0.0093$ $0.0001 \leq t \leq 0.0079$ $0.0001 \leq t \leq 0.0069$ $0.0001 \leq t \leq 0.0061$ $0.0001 \leq t \leq 0.0054$ $0.0001 \leq t \leq 0.0049$ $0.0001 \leq t \leq 0.0025$	$100.01 \leq re \leq 164.85$ $100.13 \leq re \leq 135.59$ $100.22 \leq re \leq 124.59$ $100.18 \leq re \leq 118.79$ $100.48 \leq re \leq 115.21$ $100.40 \leq re \leq 112.78$ $100.18 \leq re \leq 111.02$ $100.17 \leq re \leq 109.68$ $100.53 \leq re \leq 108.64$ $100.45 \leq re \leq 107.79$ $100.42 \leq re \leq 103.95$
50	5 10 15 20	$0.0001 \leq t \leq 0.0216$ $0.0001 \leq t \leq 0.0123$ $0.0001 \leq t \leq 0.0087$	$100.27 \leq re \leq 164.85$ $100.13 \leq re \leq 135.58$ $100.67 \leq re \leq 124.59$

	25	$0.0001 \le t \le 0.0068$	$100.50 \le re \le 118.79$
	30	$0.0001 \le t \le 0.0056$	$100.31 \le re \le 115.21$
	35	$0.0001 \le t \le 0.0047$	$100.74 \le re \le 112.78$
	40	$0.0001 \le t \le 0.0041$	$100.53 \le re \le 111.02$
	45	$0.0001 \le t \le 0.0036$	$100.70 \le re \le 109.68$
	50	$0.0001 \le t \le 0.0032$	$100.88 \le re \le 108.64$
	100	$0.0001 \le t \le 0.0029$	$100.80 \le re \le 107.79$
		$0.0001 \le t \le 0.0015$	$100.42 \le re \le 103.94$
100	5	$0.0001 \le t \le 0.0108$	$100.27 \le re \le 164.84$
	10	$0.0001 \le t \le 0.0061$	$100.79 \le re \le 135.58$
	15	$0.0001 \le t \le 0.0043$	$101.43 \le re \le 124.59$
	20	$0.0001 \le t \le 0.0034$	$100.50 \le re \le 118.79$
	25	$0.0001 \le t \le 0.0028$	$100.31 \le re \le 115.20$
	30	$0.0001 \le t \le 0.0023$	$101.59 \le re \le 112.76$
	35	$0.0001 \le t \le 0.0020$	$101.39 \le re \le 111.02$
	40	$0.0001 \le t \le 0.0018$	$100.70 \le re \le 109.68$
	45	$0.0001 \le t \le 0.0016$	$100.88 \le re \le 108.64$
	50	$0.0001 \le t \le 0.0014$	$101.67 \le re \le 107.78$
	100	$0.0001 \le t \le 0.0007$	$101.28 \le re \le 103.94$

7. Results and Discussion

We write program in R to search for values of t and the relative efficiency of the proposed estimator of exponential parameter (mean) as compared to the MLE estimator. It appears that the values of t for the example data model remain positive for relative efficiency to be more than 100% for the proposed estimator compared to the MLE estimator. In the search of values of t, we restrict ourselves to positive values of t nearing to 0 for relative efficiency more than 100% for the proposed estimator. Theoretically, since the proposed estimate is unbiased as $t \to 0$, we wish to achieve efficiency as well as nearing unbiased estimate by choosing values of t nearing 0. For example, when $\theta = 0.5$ and the sample size $n = 5$, the relative efficiency of the proposed estimate ranges from 100.01 to 164.85 when t ranges from 0.0001 to 2.16 with an increment of 0.0001. This means that the by choosing a value of $t = 2.16$ in the estimator $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$, the relative efficiency of the estimator can be increased approximately 167% compared to the estimate $\hat{\theta} = \bar{x}$ when $\theta = 0.5$. However, when $\theta = 0.5$ and the sample size $n = 10$, the relative efficiency ranges from 100.00 to 135.59 when t ranges from 0.0001 to 1.232 with an increment of 0.0001. From the reported results, it appears that for a fixed parameter, lower sample size provides better efficiency for the proposed estimate, which makes sense because as sample size gets larger, the values of $MSE(\tilde{\theta})$ and $V(\hat{\theta})$ both get smaller so as to lead to the equally efficient estimates $\tilde{\theta}$ and θ. It also follows that relative efficiency of the proposed estimate is not sensitive to the values of the parameter θ, rather it is sensitive to the sample size and the values of t.

8. Concluding Remarks

We proposed a new estimate, $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$, $t \neq 0$, for estimating the unknown exponential parameter θ using mgf. Some properties of the new estimator such as Expected value, Bias, MSE, Variance and RE have been studied. As $t \to 0$, the new estimator is unbiased, and

MSE and Variance are identical to the variance of the MLE. By searching values of t nearing 0, we can have the higher relative efficiency of the proposed estimate $\tilde{\theta}$ compared to the ML estimate, $\hat{\theta} = \bar{x}$. The new estimator has been justified using two real-life examples, where the new estimate $\tilde{\theta}$ and the competitor estimate $\hat{\theta} = \bar{x}$ provide approximately similar fit, but the new estimate provide higher efficiency in the estimation of the parameter. In a broader search of relative efficiency, with varying values of the parameter θ, sample size n and t, it appears that the proposed estimator has much higher relative efficiency as compared to the MLE for smaller sample size. We write program in R to search for the range of t and range of relative efficiency (RE) of the proposed estimate as compared to MLE, which will provide a guide to implement the new method. Given facts of the study and success in real-life application of the proposed estimate, we could conclude that the proposed new estimate is more efficient than usual MLE for values of t nearing 0, and therefore, we recommend the new method of estimation for fitting exponential model to survival time data and the estimation of exponential parameter.

References

1. Davis, D. J (1952) An analysis of some failure data. Journal of American Statistical Association 47(258): 113-150.
2. Epstein, B and Sobel, M. (1953) Life testing. Journal of American Statistical Association 48(263): 486-502.
3. Epstein, B. (1958) The exponential distribution and its role in life testing. Ind Qual Control 15: 2-7.
4. Feigl, P. and Zelen, M. (1965) Estimation of exponential survival probabilities with concomitant information. Biometrics 21(4): 826-838.
5. Shanker, R., Fesshaye, H.and Selvaraj, S. (2015). On Modeling of Lifetimes Data Using Exponential and Lindley Distributions. Biom Biostat Int J 2(5): 00042. DOI: 10.15406/bbij.2015.02.00042.
6. Hogg, R.V., McKean, J.W. and Craig, A.T. (2013). Introduction to Mathematical Statistics. Seventh Edition, Prentice Hall.
7. Casella, G. and Berger, R. L. (2002). Statistical Inference, 2nd ed., Duxbury.
8. Rohatgi, V. K. (1984). Statistical Inference, John Wiley & Sons, NY.
9. D’Agostino, R. B. (1986). Goodness-of-Fit-Techniques, CRC Press.
10. Schwarz, G. E. (1978). Estimating the dimension of a model, Annals of Statistics, **6** (2): 461–464
11. Wit, E., Heuvel, E. V. D. and Romeijn, J. (2012). ‘All models are wrong...’: an introduction to model uncertainty. Statistica Neerlandica. **66** (3): 217–236.
12. Burnham, K. P. and Anderson, D. R. (2004), Multimodel inference: understanding AIC and BIC in Model Selection, Sociological Methods & Research, **33**: 261–304.
13. Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. Am J Hyg 72: 130-148.
14. Efron, B (1988) Logistic regression, survival analysis and the KaplanMeier curve. Journal of the American Statistical Association 83(402): 414-425.