A New Distribution-free Phase II Shewhart Control Chart for Location

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Abstract
Most of the currently available distribution-free Phase II control charts for monitoring process location assume that the underlying process distributions have the same shape. However, in many applications in practice this assumption may not hold or may be hard to verify. In this paper, we examine the impact of the violation of the underlying assumptions on the performance of the Phase II Shewhart-type charts based on the Mann-Whitney statistic. As a more practical alternative, we consider a new distribution-free Shewhart-type chart based on the Fligner-Policello (1981) statistic that can be used to monitor location with fewer assumptions on the shapes of the underlying distributions. Performance of the proposed chart is studied in a simulation study. In the end, we find that traditional Mann-Whitney Shewhart-type chart is less robust than our new chart and it cannot sustain a relatively high performance under the violation of the same shape assumption.

Key Words: Nonparametric, Shewhart control chart, Shape, Location, Scale

1. Introduction

Control charts are well-known tools used to detect changes in a process. Before constructing and applying control charts, users need to make some assumptions, such as normality, symmetry, and so on. However, these assumptions may not all be reasonable and the consequence may be poor chart performance. Among the assumptions made about the underlying process the normality is a key one. In many situations normality may not hold, even approximately, and this can negatively impact the performance of the normal theory (parametric) control charts. Distribution-free or Nonparametric control charts may be useful in these situations which are applicable under much less restrictive assumptions about the distributions. There is now a rather substantial body of literature on nonparametric charts. Among these, the Shewhart chart based on the Mann-Whitney

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statistic, called the Shewhart Mann-Whitney chart (MW chart) is a Phase II distribution-free control chart proposed and studied by (Chakraborti & van de Wiel, 2008) to detect changes in location. This chart is used when a reference sample is available from a Phase I retrospective analysis and Phase II test samples are monitored sequentially in a prospective manner. In general, there are two assumptions to be satisfied for a correct application of this chart. One, the process is in-control when the reference sample and the test sample come from the same continuous distribution with the same location (median). Second, the Phase I and II distributions have the same shape and scale when the process is in-control. In other words, under the in-control stage, the two distributions, from which the reference sample and the test samples are drawn, are identical. Under these assumptions, the MW test is known to be distribution-free (see for example, Gibbons and Chakraborti, 2010) and so is the MW chart, so that the false alarm rate, the in-control average run-length, etc. can be calculated, without any specific knowledge about the underlying distribution. This is the key advantage of a distribution-free chart. Chakraborti & van de Wiel, 2008) show that the MW chart's performance is only slightly worse than the classical Shewhart $\bar{X}$ chart when the underlying distributions are in fact normal. For other symmetric, heavier-tailed non-normal and skewed distributions, the MW chart performs better than the $\bar{X}$ chart. Thus, in general, it is recommended to use the MW chart in practice when there may be even a slight doubt about normality. Li, et al., (2010) considered the CUSUM and EWMA adaptations of the MW chart, so these distribution-free charts can be used, if smaller shifts in the location are of interest.

Though the MW charts are good distribution-free charts, the assumption of the same shapes of the distributions (identical distributions) which is required for the charts to be distribution-free may be deemed to be somewhat restrictive in some situations. In fact, when processes start drifting, their shapes may simultaneously change along with the location. Thus, we may need a more flexible control chart for this problem.

Fligner & Policello (1981) considered a general method for modifying "many of the standard nonparametric rank tests for the two-sample location problem." After modification, the procedures can be used to test equality of two population medians with fewer assumptions on the shapes of the populations." "The major asset of our procedures is that the modified statistics still retain all the desirable properties of the original test statistics. Specifically, the modified procedures are still exactly distribution-free when the populations are identical." "In addition when the populations have equal medians but different shapes, the modified procedures are asymptotically distribution-free under some mild shape assumptions to be discussed later." Another important advantage for small sample sizes is that the modified test statistics tend to take a greater number of values, leading to more natural (achievable) □ levels at which to perform the test." In particular they consider a modification of the Mann-Whitney test without the shape assumption which we adapt in SPC and propose a control chart that can be used under more general conditions. To this end, let the cdf`s of the reference and the test distribution by F and G, respectively. For the MW test we need $F = G$, that is, the two distributions are identical when the process is in-control. This condition implies $\int_{-\infty}^{\infty} F\, dF = 0.5$. However, as noted earlier, it is desirable to relax the condition that $F = G$ when the process is in-control. Fligner and Policello (1981) show that one way to do this is to assume that when the locations are equal $\int_{-\infty}^{\infty} F\, dG = 0.5$ and a sufficient condition for this to hold is that F and

G be symmetric. This yields an asymptotically distribution-free test for all symmetric distributions when the medians are equal. Here we follow their modification of the MW test and call it the FP replacement. Thus, in this paper, we try to adapt the MW Shewhart chart of Chakraborti and van de Wiel (2008) by incorporating the FP modification idea so that the proposed chart is more broadly applicable. We compare the performance of the resulting Shewhart chart with the parametric $\bar{X}$ chart and the original MW Shewhart chart in a simulation study.

2. Methodology

In this section, we set up 3 types of Phase II control charts in order to compare to their performance under the change of shape. In general, we can define a Phase II Shewhart-type control chart, with a charting statistic CS, to declare that there is no signal at the h^{th} test (Phase II) subgroup when:

$$LCL = E(CS|IC) - k\sqrt{Var(CS|IC)} \leq CS^h \leq UCL = E(CS|IC) + k\sqrt{Var(CS|IC)}$$

where CS^h is charting statistic for the h^{th} Phase II sample, $h = 1,2,\ldots$. The LCL and the UCL are the lower and the upper control limit, respectively, and k is the charting constant.

2.1 The Shewhart MW chart

Suppose we have a reference (Phase I) sample of size m, denoted by $X = (X_1, \ldots, X_m)$, from an in-control process and that $Y = (Y_1, \ldots, Y_n)$ is an arbitrary test sample of size n. The superscript h is used to denote the h^{th} test sample, $Y^h = \left(Y_1^h, \ldots, Y_n^h\right)$, $h = 1,2, \ldots$, when necessary for notational clarity; otherwise, the superscript is suppressed. Let F and G denote the cumulative distribution functions of the $X's$ and the $Y's$, respectively. The process is in-control (IC) when $F = G$. Chakraborti & van de Wiel (2008) assume that the test samples are independent of each other and they are all independent of the reference sample. The MW test is based on the total number of (X, Y) pairs where the Y observation is larger than the X. This is the statistic

$$U = \sum_{i=1}^{m} \sum_{j=1}^{n} I(X_i < Y_j) = \sum_{j=1}^{n} \{I\left(X_1 < Y_j\right) + \cdots + I(X_m < Y_j)\}$$

where $I\left(X_1 < Y_j\right)$ is the indicator function for the event $\{X_1 < Y_j\}$. According to (Gibbons & Chakraborti, 2010), $\mathrm{E}(U|IC) = \frac{mn}{2}$ and $\mathrm{Var}(U|IC) = \frac{mn(m+n+1)}{12}$. In terms of the standardized statistic $U^* = \frac{U^h - \frac{mn}{2}}{\sqrt{\frac{mn(m+n+1)}{12}}}$, the control chart issues no signal if

$$-k \leq U^h \leq k$$

Otherwise, the chart signals a change in location.

2.2 The Shewhart MW chart with FP modification

Following Fligner and Policello (1981), recall that the process is in-control when F and G have the same location (say the median) and that F and G are both assumed to be symmetric. Let $X_{(1)} \leq \cdots \leq X_{(m)}$ and $Y_{(1)} \leq \cdots \leq Y_{(n)}$ denote that ordered X and Y sample, respectively and let $F_m(x)$ and $G_n(y)$ denote the empirical distribution functions of the X and Y sample, respectively. The number of Y's less than or equal to $X_{(i)}$, which is equal to $nG_n(X_{(i)})$ is denoted by P_i. Similarly, the number of X's less than or equal to $Y_{(i)}$ is equal to $mF_m(Y_{(j)})$ and is denoted S_j. The well-known Mann-Whitney statistic can be written as the sum of the $P_i = \sum_{i=1}^{m} P_i$. The "modified" Mann-Whitney test is based on the following statistic (Fligner and Policello, 1981):

$$\widehat{U} = \frac{\sqrt{n}\left(\frac{U}{mn} - \frac{1}{2}\right)}{\hat{\sigma}}$$

where $U = \sum_{i=1}^{m} P_i$, so the exact modified MW statistic with FP replacement:

$$\widehat{U} = \frac{\sum P_i - \sum S_j}{2\left(\frac{m-1}{m}\sum\left(S_j - \bar{S}\right)^2 + \frac{n-1}{n}\sum(P_i - \bar{P})^2 + \bar{P}\bar{S}\right)^{1/2}}$$

where $\bar{S} = \sum_{j=1}^{n} S_j / n$, $\bar{P} = \sum_{i=1}^{m} P_i / m$. When m and n are big enough, $(m-1)$ and $(n-1)$ can be replaced by m and n respectively, so the simplified modified MW statistic with FP replacement:

$$\widehat{U} = \frac{\sum P_i - \sum S_j}{2(\sum\left(S_j - \bar{S}\right)^2 + \sum(P_i - \bar{P})^2 + \bar{P}\bar{S})^{1/2}} \quad (1)$$

We use $\widehat{U}$ as the charting statistic at every Phase II sample. It has been shown that $E\left(\widehat{U}|IC\right) = 0$ and $Var\left(\widehat{U}|IC\right) = 1$ aympotically. The distribution of $\widehat{U}$ is known to be symmetric about 0 when the process is in-control and under the assumption that both distributions are symmetric. Also, as shown in Fligner and Policello (1981), in the in-control case, $\widehat{U}$ asymptotically follows the standard normal distribution.

Thus, in terms of the modified statistic, the chart declares no signal at the h^{th}Phase II sample if

$$-k \leq \widehat{U}^h \leq k$$

otherwise, the chart indicates a change in the location. Note that the proposed MW chart with FP chart for location is asymptotically distribution-free when the two populations are symmetric and their locations are the same.

2.3 The 2-sample t chart

For the same setting, for a reference sample of size m denoted by $X = (X_1, \dots, X_m)$ from an in-control process and that $Y = (Y_1, \dots, Y_n)$ is an arbitrary test sample of size n. The superscript h is used to denote the h^{th} test sample, $Y^h = \left(Y_1^h, \dots, Y_n^h\right), h = 1,2, \dots$, when necessary for notation clarity; otherwise, the superscript is suppressed. Resembling the traditional $\bar{X}$ chart (Montgomery, 2013), generally the $\bar{X}$ chart is no signals if

$$\bar{X} - k\frac{\sqrt{S^2(X)}}{\sqrt{m}} \leq \bar{Y}^h \leq \bar{X} + k\frac{\sqrt{S^2(X)}}{\sqrt{m}}$$

where $S^2(X) = \frac{1}{m}\sum_{i=1}^{m}(X_i - \bar{X})^2$ and $\bar{X}$ is the mean of reference sample X, $\bar{Y}^h$ is the mean of the h^{th} test sample. Otherwise, the chart signals a change in location.

2.4 Definition of the in-control process

The definition of in-control for these 3 control charts is different. For the MW chart, the in-control definition is that the reference sample X and the test sample Y are independently following an identical symmetric distribution. For MW with FP chart, the in-control definition is that the reference sample X and the test sample Y are independently following a symmetric distribution with a same location. For 2-sample t chart, the reference sample X and the test sample Y are independently following an identical normal distribution. Thus, we know that the 2-sample t chart has the strictest definition. Also, MW with FP chart has the weakest definition.

2.5 Calculation of the ARL for the MW chart with FP replacement

Chakraborti & van de Wiel (2008) show the probability of signal for any test sample, given the reference sample $X = x$ and a specified charting constant k, is

$$p_G(x) = P_G(\hat{U} < -k) + P_G(\hat{U} > k)$$

let N denote the run length random variable for the chart. Given the reference sample $X = x$, and that any two arbitrary test samples Y^h and Y^l, $(h \neq l)$ are independent, which implies the independence of $\hat{U}^h$ and $\hat{U}^l$. Hence,

$$ARL = E(N) = E_F\big(E_G(N|X = x)\big) = E_F\left(\frac{1}{p_G(x)}\right)$$
$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{1}{p_G(x)} dF(x_1) \dots dF(x_m)$$

In MW chart, (Chakraborti & van de Wiel, 2008) uses the assumption $F = G$ to obtain the in-control average run length, but this assumption has been changed. Due to Monte Carlo estimation, both of in-control run length and out-of-control run length can be written as the following:

$$ARL \approx \frac{1}{S}\sum_{i=1}^{S}\frac{1}{p_G(x_i)} \qquad (2)$$

where $x_i = (x_{i1}, \dots, x_{im})$ is the i^{th} Monte Carlo sample, $i = 1, \dots, S$. For the in-control case, the two distributions are symmetric with same locations. For example, X is from standard normal distribution and Y is from the T distribution with 5 degrees of freedom.

2.6 Determination of charting constants

Implementation of the chart will be affected by finding the control limits which is a function of the charting constant k. In practice, the charting constants are decided for some specified in-control average run length such as 370 or 500, so the constant k depends on the specified nominal in-control average run length. Thus, we can obtain k by Monte Carlo estimation. Let the given in-control average run length be ARL_{given} equal to some specific value,

$$ARL_0(k) - ARL_0 = 0 \qquad (3)$$

where $ARL_0(k)$ is the in-control average run length obtained by Monte Carlo estimation with a specified k. We can obtain k by solving this equation, because k is the root of this equation.

2.7 Performance

We will compare these 3 kinds of charts including the $\bar{X}$ chart, the MW chart and the MW chart with FP replacement. The unconditional statistics will be

Chart	Charting statistics(CS)	E(CS)	Var(CS)
MW	U	*mn/2*	*mn(m+n+1)/12*
FP	Modified U	0	1
T	$\bar{Y}$	$\bar{X}$	S^2*(X)/m*

where $\bar{Y}$ is the mean of test sample. $\bar{X}$ is the mean of reference sample. *Var(X)* is the variance of reference sample.

2.7.1 Simulation settings

Because the major improvement of MW chart with FP replacement is that we only have assumptions on the symmetry of population, we can set processes in control if their shapes are symmetric and have same median. Also, because we compare performance among charts at different shifts and scales, we define W to be a random variable with a specific distribution, θ is the shift parameter and τ is the scale parameter. Let W^* be the transformed random variable, so the relation between the transformed sample from W^* and the original sample from W will be the following:

$$W^* = \frac{W + \theta}{\tau}$$

so we can control the shift and dispersion of different distributions by just changing θ and τ instead of changing interior parameters of distributions. For example, if we need to compare the performance between the standard normal distribution and the transformed T distribution with 5 degrees of freedom and 1 standard deviation, we can generate the reference sample from the standard normal distribution and generate the test sample from the transformed T distribution with 5 degrees of freedom, $\theta = 0$ and $\tau = 1.2909$.

Then, we generate samples W based on the following settings and then use given θ's and τ's to transform to the true simulated sample W^* for the further comparison.

Distribution	Mean	Standard Deviation
Normal	0	1
t	0	1.290994

Laplace	0	1.414214

where T is the T distribution with 5 degrees of freedom. Laplace is the Laplace distribution with location 0 and scale 1.

2.7.2 Control limits

Obviously, the most restrictive definition of the in-control process is from the $\bar{X}$ chart, so, if processes meet the in-control requirement of the $\bar{X}$ chart, they also meet the requirement of other charts. To make the comparison fair to all charts and simplify our simulation, we only generate in-control processes from the standard normal distribution. Let $ARL_0 = a$ and processes are in control, we can obtain k by solving the following equation developed by the equations (2) and (3),

$$\frac{1}{S}\sum_{i=1}^{S}\frac{1}{p_G(x_i)} - a = 0$$

so k root of this equation. Here, we use bisection method to search the appropriate k. After 100000 simulations, we have the following charting constants:

Table 1: The k values for the MW chart for different m, n and ARL_0

		ARL_0= 370	ARL_0= 500
m	n	k	k
20	5	2.5094	2.5109
	10	2.4000	2.4611
50	5	2.6642	2.7228
	10	2.6976	2.7766
100	5	2.7269	2.7637
	10	2.8176	2.8750

Table 2: The k values for the MW chart with FP replacement for different m, n and ARL_0

		ARL_0= 370	ARL_0= 500
m	n	k	k
20	5	4.5625	4.8469
	10	3.0783	3.1835
50	5	6.2156	6.5703

	10	3.8651	4.0453
100	5	7.3582	7.8953
	10	4.2895	4.5246

Table 3: The k values for the 2-sample t chart for different m, n and ARL_0

		ARL_0= 370	ARL_0= 500
m	n	k	k
20	5	5.5850	5.7278
	10	4.0366	4.1315
50	5	9.3030	9.5432
	10	6.6705	6.8471
100	5	13.3272	13.7110
	10	9.5294	9.7964

2.7.3 Simulation scheme

Step 1: Obtain k.

Step 2: Individually generate the reference sample and the test sample at the different settings including in-control and out-of-control settings.

Step 3: Calculate a run length.

Step 4: keep repeating step 2 and step 3.

Step 5: When the amount of run lengths reach our maximum simulation, stop repeating and calculate statistics of run lengths such as, mean, standard deviation and so on.

2.7.4 Result

As we said before, for the $\bar{X}$ chart, it is in control when both of samples are identical and normal. In other word, they have Shift(θ) = 0 and Scale(τ) = 1. For the MW chart, it is in control when both of samples are identical and symmetric with Shift(θ) = 0 and Scale(τ) = 1. For the MW chart with FP replacement, it is in control when both of samples are symmetric with Shift(θ) = 0. The following tables and plots will show the performance of 3 charts.

Table 4: The performance based on that the reference sample is from the standard normal distribution and the test sample is from the normal distribution shown above with shift θ and scale τ at $ARL_0 = 500$.

	Scale = 0.5			Scale = 0.75			Scale = 1		
Shift	MW	FP	T	MW	FP	T	MW	FP	T
0.00	46.34	197.00	8.10	131.72	282.77	47.59	501.24	497.02	499.54
0.05	44.74	189.83	7.81	125.09	264.20	45.30	493.17	476.46	475.76
0.10	39.30	155.61	7.17	105.67	231.11	39.54	417.70	419.67	402.53
0.25	18.40	70.35	4.39	45.45	105.13	18.49	179.59	196.88	159.22
0.50	5.65	17.48	1.99	10.21	21.91	4.82	26.64	35.47	22.51

Table 5: The performance based on that the reference sample is from the standard normal distribution and the test sample is from the t distribution shown above with shift θ and scale τ at $ARL_0 = 500$.

	Scale = 0.5			Scale = 0.75			Scale = 1		
Shift	MW	FP	T	MW	FP	T	MW	FP	T
0.00	42.15	189.87	4.54	95.97	257.17	13.60	261.40	412.47	47.95
0.05	39.54	181.45	4.46	92.51	244.56	13.35	254.64	385.68	47.24
0.10	35.11	157.42	4.24	80.81	215.32	12.52	233.88	359.14	43.51
0.25	18.85	74.80	3.28	40.26	106.29	8.55	111.32	177.50	28.55
0.50	6.03	20.29	1.94	10.74	25.29	3.78	24.38	38.36	10.05

Table 6: The performance based on that the reference sample is from the standard normal distribution and the test sample is from the Laplace distribution shown above with shift θ and scale τ at $ARL_0 = 500$.

	Scale = 0.5			Scale = 0.75			Scale = 1		
Shift	MW	FP	T	MW	FP	T	MW	FP	T
0.00	45.45	195.14	3.77	100.24	272.55	9.80	243.55	454.85	30.46
0.05	43.54	183.72	3.73	96.22	264.29	9.68	236.41	435.41	29.74
0.10	37.61	156.66	3.62	83.98	230.50	9.22	210.57	384.30	27.83
0.25	19.20	66.55	2.96	42.28	103.67	6.86	107.84	184.61	19.54
0.50	5.84	18.00	1.92	10.84	22.75	3.49	23.94	36.57	8.09

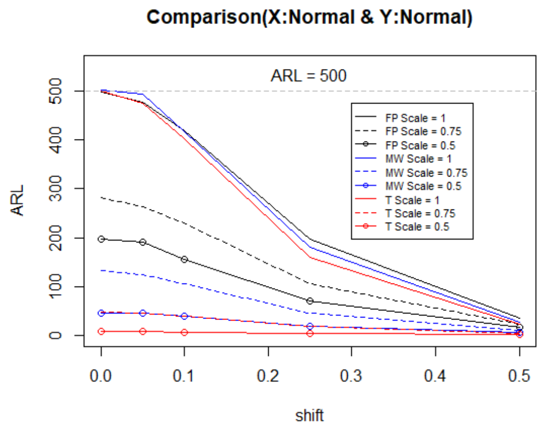

Figure 1: The comparison of 3 types of control charts at the setting that the reference sample is from the standard normal distribution and the test sample is from the normal distribution shown above with shift θ and scale τ at $ARL_0 = 500$.

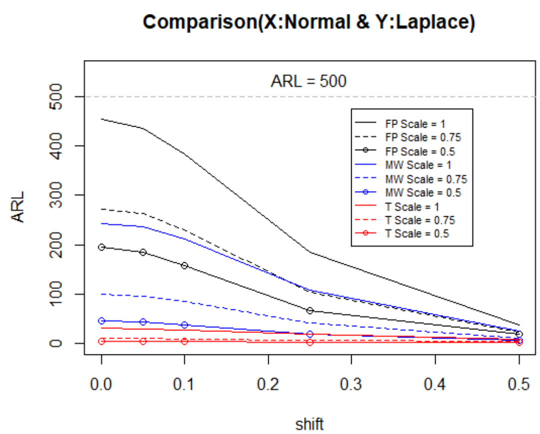

Figure 2: The comparison of 3 types of control charts at the setting that the reference sample is from the standard normal distribution and the test sample is from the t distribution shown above with shift θ and scale τ at $ARL_0 = 500$.

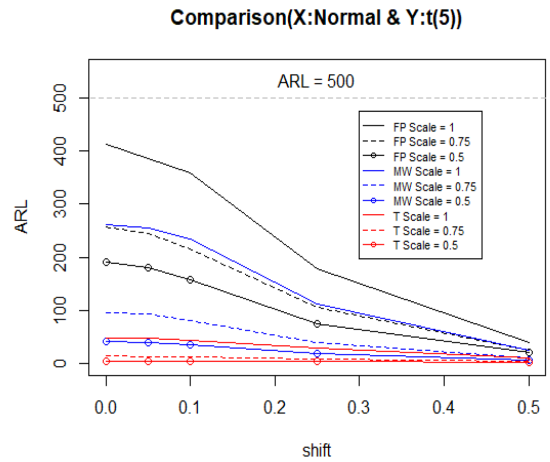

Figure 3: The comparison of 3 types of control charts at the setting that the reference sample is from the standard normal distribution and the test sample is from the Laplace distribution shown above with shift θ and scale τ at $ARL_0 = 500$.

2.7.5 Discussion of the results

According to the tables and plots shown above, the 2-sample t chart has better performance at the setting of the normal reference sample and the normal test sample with scale 1. However, if the reference sample and the test sample are not from a normal distribution or has scales, the performance of 2-sample chart drops tremendously. Also, at the in-control case of the 2-sample t chart, the MW chart and the MW chart with FP replacement have similar performance which is slightly worse than the 2-sample t chart, but at the our-of-control case of the 2-sample t chart, both are better than the 2-sample t chart. Furthermore, the MW chart with FP replacement can keep relatively high performance, comparing with others.

3. Conclusion

The proposed MW chart with FP replacement is a good alternative distribution-free chart because it can give similar performance as the basic MW chart but is more robust, as long as the underlying distributions are symmetric. Thus the MW chart with FP replacement can be applied under possibly different shapes of the two distributions (as long as each is symmetric).

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