Condition-Based Maintenance Policy under Gamma Degradation Process

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Abstract
Condition-based maintenance is an effective method to reduce unexpected failures as well as the operations and maintenance costs. This work discusses the condition-based maintenance policy with optimal inspection points under the gamma degradation process. A random effect parameter is used to account for population heterogeneities and its distribution is continuously updated at each inspection epoch. The observed degradation level along with the system age is utilized for making the optimal maintenance decision, and the structure of the optimal policy is examined along with the existence of the optimal inspection intervals.

Key Words: condition-based maintenance, gamma degradation process, optimal inspection interval, random effect model, stochastic order

1. Introduction

In the modern reliability studies, products are outfitted with numerous sensors in order to capture information on how, when, and under what environmental and operating conditions they are being used. This generates complex field reliability data with the emerging IoT technology, and feeds System Health Management for developing prognostic information systems. As a part of System Health Management, Condition-Based Maintenance utilizes modern sensor technology for periodic inspections of a system. The maintenance actions are based on the inspection of working conditions of the system unlike traditional methods. It has proven effective in reducing unexpected failures with lower operational costs, and it outperform the traditional age-based and block-based maintenance policies; see Jardine *et al.* (2006) and Wang (2002). In this work, we develop an optimal condition-based maintenance/replacement policy under the gamma degradation process with random effects to account for population heterogeneities. The problem is formulated using the Markov Decision Process where the replacement decision is based on the observed degradation level and the age of a unit. Aiming to minimize the total discounted operational costs, we investigate the structural properties of the optimal policy and determine the optimal inspection interval.

The rest of the paper is organized as follows. Section 2 describes the gamma degradation process with random effects to model population heterogeneities. The properties of the stochastic model considered in this work are laid out in Section 3. Then, the procedure of condition-based maintenance is explained in Section 4, and the optimization problem is formulated using the Markov Decision Process. The structure of the optimal maintenance policy is investigated in Section 5 along with the existence of the optimal inspection interval. Section 6 presents the results of a numerical study to understand the behavior of the optimal maintenance policy and the optimal inspection interval in relation to other model parameters. Finally, Section 7 draws a brief conclusion.

2. Gamma Process with Random Effects

Statistical degradation processes have been studied and used to construct prognostic models in characterizing the physical deterioration of a component and/or a system. The

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choice of a stochastic model affects the prediction of the Remaining Useful Life as well as influences the maintenance strategy and economic performance. Such stochastic processes include Markov chain models based on discrete degradation states (Block-Mercier (2002) and Chen *et al.* (2003)), continuous degradation state models with independent degradation increments – Wiener process (Elwany *et al.* (2011) and Guo *et al.* (2013)), inverse Gaussian process (Wang (2010) and Chen *et al.* (2015)), and gamma process (Lawless and Crowder (2004)). In particular, the gamma process is characterized by a monotonic degradation evolution $\{Y_t,\ t \geq 0\}$, which is appropriate for modeling crack growth and wears. In the gamma process, the increments $\langle Y_{t_2} - Y_{t_1}\rangle$ and $\langle Y_{s_2} - Y_{s_1}\rangle$ are independent for $0 \leq s_1 < s_2 \leq t_1 < t_2$, and they are gamma distributed. That is, $\langle Y_t - Y_s\rangle \sim Gamma(\alpha[\Lambda(t) - \Lambda(s)],\ \beta)$, where $\Lambda(t)$ is a monotonically increasing time transformation function with $\Lambda(0) = 0$.

Moreover, due to variations in raw materials, diverse usage, field conditions, environmental differences, etc., the degradation characteristics can differ among units of the same population; see Hong (2013). Hence, different units often exhibit different degradation patterns. It is then necessary to account for commonly observed unit-specific heterogeneities, and it can be achieved by introducing random effects or prior distributions for parameters. This distribution is then updated when more degradation observations become available. For the gamma degradation process considered here, different units can have different realizations of β and this random effect can be modeled by an inverse gamma distribution. That is, $\beta \sim Inverse\ Gamma(\gamma,\ \lambda)$ with the probability density function given by

$$f(\beta;\ \gamma, \lambda) = \frac{\lambda^\gamma}{\Gamma(\gamma)}\beta^{-\gamma-1}e^{-\lambda/\beta}, \quad \beta > 0.$$

In this work, the effect of random β on maintenance planning is also investigated.

3. Stochastic Model Properties

Under the model setup in Section 2, let $Y_j = Y_{t_j}$ be degradation levels observed at times t_j, $j = 1, 2, \ldots, n$, and let $\mathbf{Y}_n = (Y_1, Y_2, \ldots, Y_n)$ with $\Lambda_j = \Lambda(t_j)$. Then, it can be shown that $\langle \beta|\mathbf{Y}_n\rangle \sim Inverse\ Gamma(\alpha\Lambda_n + \gamma,\ Y_n + \lambda)$. That is, the updated posterior distribution of β depends only on the most recent degradation measure Y_n and Λ_n. Further, it can be shown that $\left\langle \frac{Y_{n+1} - Y_n}{Y_{n+1} + \lambda}\middle|\mathbf{Y}_n\right\rangle \sim Beta(\alpha[\Lambda_{n+1} - \Lambda_n],\ \alpha\Lambda_n + \gamma)$ by computing $f(y_{n+1}|\mathbf{y}_n) = \int_0^\infty f(y_{n+1} - y_n|\beta)\ f(\beta|\mathbf{y}_n)\ d\beta$. Hence, $\langle Y_{n+1}|\mathbf{Y}_n\rangle$ attains the Markov property since its distribution depends only on Y_n.

Before we formulate and understand the structure of the optimal maintenance policy, we first need to study the age-dependent degradation behaviors with heterogeneity, using the properties of stochastic orders and likelihood ratio orders; see Shaked and Shanthikumar (2007). Following are the essential lemmas to understand the structural properties of the optimal maintenance/replacement policy.

Lemma 1. *$\langle Y_{t+\Delta}|Y_t\rangle$ is stochastically non-decreasing in Y_t. That is, $\langle Y_{t+\Delta}|Y_t = y_1\rangle \prec \langle Y_{t+\Delta}|Y_t = y_2\rangle$ given $y_1 < y_2$.*

Lemma 2. *$\langle Y_{t+\Delta}|Y_t\rangle$ is stochastically non-increasing in t. That is, $\langle Y_{t_1+\Delta_1}|Y_{t_1} = y\rangle \prec \langle Y_{t_2+\Delta_2}|Y_{t_2} = y\rangle$ given $t_1 > t_2$ and $\Lambda(t_1 + \Delta_1) - \Lambda(t_1) \leq \Lambda(t_2 + \Delta_2) - \Lambda(t_2)$.*

4. Procedure of Condition-Based Maintenance

Here we illustrate the procedure of condition-based maintenance with $\Lambda(t) = t$. A system is inspected periodically with δ denoting an inspection interval. It is assumed that failure of the system is not self-announcing and only revealed through inspections. Also, any maintenance action is assumed instantaneous. Let c_i be the cost of inspection, c_f the cost of corrective replacement, c_p the cost of preventive maintenance ($c_p < c_f$), and c_d the downtime cost per unit time while $\exp(-rt)$ denotes the conventional discounting factor for the purpose of calculating the present values. First, a system is initialized with the state of $t = 0$ and $Y(t) = 0$. Then, at each inspection point, the system is inspected at cost c_i and $Y(t)$ is observed. With the degradation threshold D, if $Y(t) > D$, it is considered that (soft) failure of the system has occurred. Then, the Corrective Maintenance (CM) action is performed at cost c_f in order to restore the state to the *as-good-as-new* state. That is, the system is restored to the initial state. On the other hand, if $Y(t) \leq D$, failure has not occurred yet. Then, there are two options. Either we perform the Preventive Maintenance (PM) action at cost c_p in order to restore the state to the *as-good-as-new* state, or the maintenance decision is deferred to the next inspection epoch by setting $t = t + \delta$. In the latter case, failure may occur at unknown failure time $T_k \in (t,\ t + \delta)$. If so, the downtime cost incurs due to loss of efficiency or quality by operating in the failure state. The question is "How do we decide between these two options?" This question can be answered by formulating the problem using the Markov Decision Process.

Here, $(\tau_k,\ Y_{\tau_k})$ forms a discrete-time continuous-state Markov chain with age-dependent transition probability. At each state, one action is taken from CM, PM, or NULL (decision deferment). Under this framework, the cost of maintenance is captured by the Value function defined by

$$V_\delta(u, v) = \begin{cases} \min\left\{e^{-r\delta} U_\delta(u, v) + W_\delta(u, v),\ c_p + V_\delta(0, 0)\right\}, & v \leq D; \\ c_f + V_\delta(0, 0), & v > D \end{cases} \tag{1}$$

for equi-spaced inspection time $u = 0, \delta, 2\delta, \ldots$ and degradation level $v > 0$. Eq.(1) is the minimum total discounted cost, starting from the state (u, v) on the infinite horizon. It satisfies the Bellman equation; see Puterman (2009). In eq.(1), $U_\delta(u, v) = E\big[V_\delta(u + \delta, Y_{u+\delta}) \mid \tau_k = u, Y_{\tau_k} = v\big]$ is the expected value with one period transition from the current state (u, v) while $W_\delta(u, v) = E\big[\rho(T_k) \mid \tau_k = u, Y_{\tau_k} = v\big]$ is the expected downtime cost based on the current state (u, v). The discounted downtime cost is given by

$$\rho(T_k) = \begin{cases} \displaystyle\int_{T_k}^{\tau_k+\delta} c_d e^{-r(t-\tau_k)} dt = \frac{c_d}{r}\Big(e^{-r(T_k-\tau_k)} - e^{-r\delta}\Big), & \tau_k < T_k < \tau_k + \delta; \\ 0, & \text{otherwise} \end{cases}$$

Then, the expected total maintenance cost is represented by $V_\delta(0, 0)$. On the other hand, the total discounted inspection cost is expressed by

$$S(\delta) = \sum_{k=0}^{\infty} c_i e^{-rk\delta} = c_i\Big(1 - e^{-r\delta}\Big)^{-1} \tag{2}$$

The aim of the optimization is then to find the optimal inspection interval δ^* and to find the corresponding maintenance policy for minimizing the total operational costs $V_\delta(0, 0) + S(\delta)$.

5. Structure of the Optimal Maintenance Policy

The following lemmas are critical to understand the behavior of the Value function $V_\delta(u, v)$.

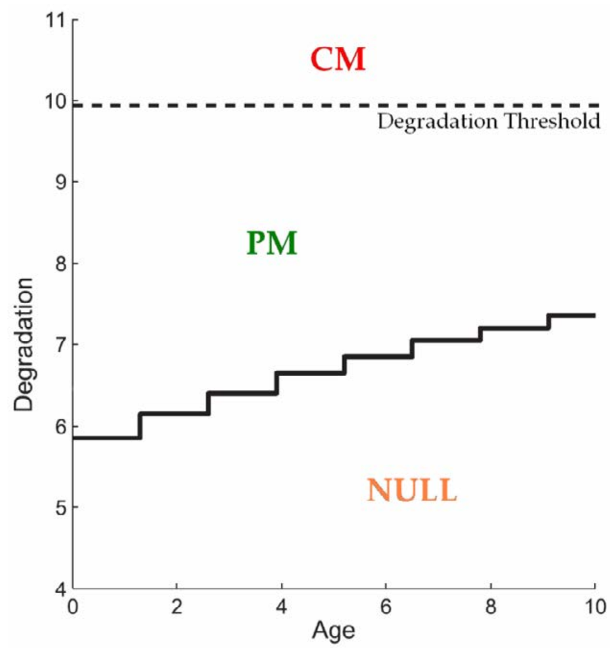

Figure 1: Schematic illustration of the optimal maintenance policy that minimizes the total operational costs, which is a monotone control limit policy

Lemma 3. *$U_\delta(u, v)$ is non-decreasing in v and non-increasing in u.*

Lemma 4. *$W_\delta(u, v)$ is non-decreasing in v and non-increasing in u.*

Then, by induction using Lemmas 3 and 4, it can be shown that $V_\delta(u, v)$ also has the monotone property; Puterman (2009). This monotonicity substantially reduces the computational burden of finding the optimal maintenance policy.

Lemma 5. *The optimal value function $V_\delta(u, v)$ is non-decreasing in v and non-increasing in u.*

Theorem 1. *Given δ, the optimal maintenance policy that minimizes $V(0, 0)$ is a monotone control limit policy. That is, there is a non-decreasing sequence $\{\xi_k\}$ such that the optimal action at state $(k\delta, y)$ is PM if $y > \xi_k$, and NULL otherwise.*

Without heterogeneity, the optimal maintenance policy is still a monotone control limit policy with respect to β. With larger β, the optimal maintenance policy has a lower (constant) control limit in order to balance the costs, and the corresponding optimal inspection interval becomes smaller ($i.e.$, more conservative).

Corollary 1. *With no heterogeneity in the population, all units share the same degradation rate and the optimal maintenance policy is a constant control limit policy $\xi_k = \xi_0$, $\forall k$.*

The result of Theorem 1 can be extended to an arbitrary monotone increasing concave function $\Lambda(t)$.

Theorem 2. *If $\Lambda(t)$ is a concave shaped function, then the optimal maintenance policy that minimizes $V(0, 0)$ is still a monotone control limit policy.*

It can be shown that as the inspection interval δ increases, the expected total maintenance cost $V_\delta(0, 0)$ increases monotonically to a constant as the downtime cost dominates.

On the other hand, as δ increases, the total discounted inspection cost $S(\delta)$ decreases monotonically to 0. Hence, the total operational costs $V_\delta(0,0) + S(\delta)$ is convex with respect to δ with $\lim_{\delta\to\infty} V_\delta(0,0) + S(\delta) = E\big[c_d e^{-rT}/r \mid Y_0 = 0\big]$, and the existence of the optimal inspection interval δ^*, which minimizes the total costs $V_\delta(0,0) + S(\delta)$, is guaranteed. From the numerical study, δ^* was found to be quite robust as the total operational costs $V_\delta(0,0) + S(\delta)$ is pretty flat near the optimal inspection point δ^*.

6. Numerical Results

An extensive numerical study was conducted to examine the structure of the optimal maintenance policy and the optimal inspection interval in relation to other model parameters. For an illustration, we let $\alpha = 1.3$, $\gamma = 23.5$, $\lambda = 19.0$ with $D = 10$, $r = 0.01$, and $\Lambda(t) = t$. The cost parameters were chosen to be $c_i = 0.05$, $c_f = 10$, $c_p = 3$, and $c_d = 1.0$. The numerical study verified that the optimal maintenance policy is the monotone control limit policy in age of a unit. It was also observed that as the inspection interval δ decreases, the optimal maintenance policy has higher control limits ($i.e.$, less conservative). As the inspection interval δ increases, the optimal maintenance policy has lower control limits ($i.e.$, more conservative). The optimal policy was also found to have varying degrees of sensitivity to the cost parameters. As c_f increases, the optimal inspection interval δ^* becomes shorter and the optimal maintenance policy has lower control limits ($i.e.$, more conservative). As c_p increases, δ^* becomes shorter as well but the optimal maintenance policy has higher control limits. As c_d increases, δ^* becomes shorter but the impact on the control limits was minimal due to relatively high D. On the other hand, as c_i increases, δ^* becomes longer and the optimal maintenance policy has lower control limits.

7. Conclusion

In this work, the optimal condition-based maintenance/replacement policy was studied under the gamma degradation process with random effects to accommodate population heterogeneities. Using the Markov Decision Process, the maintenance decision at each inspection point is based on the observed degradation level and the age of a unit. It was found that the optimal maintenance policy that minimizes the total discounted operational costs is a monotone control limit policy. Since the total operational cost forms a convex function of the inspection interval, the optimal inspection interval to minimize the total costs can be determined numerically as well.

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