Uniform- and Triangular-Based Third-Order Power Method Distributions Using a Doubling Technique

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Abstract
Power method (PM) polynomials have been used for simulating non-normal distributions in a variety of settings such as toxicology research, price risk, business-cycle features, microarray analysis, computer adaptive testing, and structural equation modeling. A majority of the applications associated with the PM polynomials are based on the method of matching conventional moments (e.g., skew and kurtosis). However, estimators of skew and kurtosis can be (a) substantially biased, (b) highly dispersed, or (c) influenced by outliers. To address this limitation, two families of third-order PM distributions are developed through the method of L-moments (Hosking, 1990) using a doubling technique (Morgenthaler & Tukey, 2000) and contrasted with the method of moments in the contexts of estimation of parameters. The methodology is based on simulating uniform- and triangular-based third-order PM distributions with specified values of L-skew and L-kurtosis. Monte Carlo simulation results indicate that the estimators based on method of L-moments are superior to their conventional moment-based counterparts.

1. Introduction

The third-order power method (PM) polynomial is defined as (Headrick, 2010)

$$p(V) = c_0 + c_1 V + c_2 V^2 + c_3 V^3, \tag{1}$$

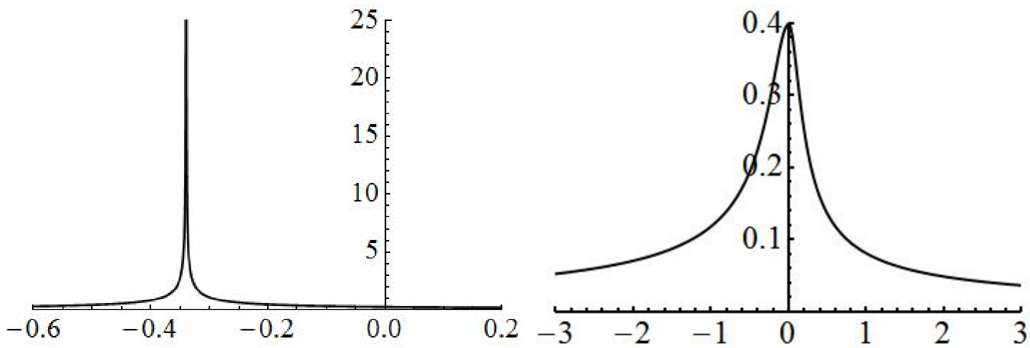
where V is a random variable with probability density function (pdf) and cumulative distribution function (cdf) denoted as $\phi(v)$ and $\Phi(v)$. If the random variable V in (1) is drawn from a standard normal distribution, then the expression in (1) is the Fleishman's (1978) third-order PM polynomial. The Fleishman's PM polynomial in (1) has been used in a variety of contexts for the purpose of simulating non-normal distributions with specified values of skew and kurtosis. Some examples include: asset pricing theory (Affleck-Graves & MacDonald, 1989), business-cycle features (Hess & Iwata, 1997), microarray analysis (Powell, Anderson, Cheng, & Alvord, 2002), price risk (Mahul, 2003), multivariate analysis (Steyn, 1993), analysis of variance (ANOVA) (Berkovits, Hancock, & Nevitt, 2000; Lix & Fouladi, 2007; Keselman, Wilcox, Algina, Othman, & Fradette, 2008), analysis of covariance (ANCOVA) (Harwell & Serlin, 1988; Headrick & Sawilowsky, 2000), regression analysis (Headrick & Rotou, 2001), item response theory (Stone, 2003), nonparametric statistics (Beasley & Zumbo, 2003), toxicology research (Hothorn & Lehmacher, 2007), and structural equation modeling (Henson, Reise, & Kim, 2007).

If the random variable V in (1) is drawn from a standard logistic and standard uniform distributions, respectively, then the corresponding expressions in (1) are referred to as logistic-based and uniform-based PM polynomials (Hodis & Headrick, 2007; Hodis, 2008; Headrick, 2010). A triangular-based PM polynomial has been developed by considering the random variable V from a standard triangular distribution and this triangular-based PM polynomial is contrasted with the normal-, logistic-, and uniform-based PM polynomials (see Hodis, Headrick, & Sheng, 2012). For the PM polynomial in (1) to produce a valid pdf, it is required that the expression in (1) be a strictly increasing monotone function. This requirement implies that an inverse function (p^{-1}) exists. As such, the parametric forms of cdf and pdf associated with (1) can be expressed as (Headrick & Kowalchuk, 2007; Headrick, 2010)

$$F\big(p(V)\big) = \big(p(V), \Phi(V)\big) \tag{2}$$

$$f\big(p(V)\big) = (p(V), \phi(V)/p'(V)) \tag{3}$$

One of the limitations associated with the PM polynomials is that the non-normal distributions with values of skew and (or) kurtosis that lie in the upper right region of the skew-kurtosis boundary graph (e.g., Headrick, 2010, p. 20) can be excessively leptokurtic and thus may not be representative of real world data (Pant & Headrick, 2012). For example, Figure 1 (Panel A) shows a pdf of uniform-based PM polynomial with skew (γ_3) of 1.2 and kurtosis (γ_4) of 1.2. This example illustrates the limitation that the PM can have in terms of excessive peakedness.



$c_0 = -0.332659, c_1 = 0.116965,$

$c_2 = 0.635331,$ and $c_3 = 1.177987$

$C_{\mathcal{L}} = 2.558204,$

$C_{\mathcal{R}} = 7.098050$

A B

Figure 1. Probability density function (pdf) of a traditional third order uniform-based (Panel A) and a double-uniform (Panel B) PM distributions based on matching the conventional skew of 1.2 and kurtosis of 1.2. The values of coefficients $c_{i=0,1,2,3}$ for the distribution in Panel A were determined by solving the system of equations (2.26)—(2.29) from Headrick (2010, p. 16), whereas the values of $C_{\mathcal{L}}$ and $C_{\mathcal{R}}$ for the distribution in Panel B were determined by solving (9)—(10).

Another limitation associated with the PM distributions is that the conventional-moment-based estimators of γ_3 and γ_4 have unfavorable attributes insofar as they can be substantially biased, highly dispersed, or can be influenced by outliers (Hosking, 1990, 1992; Hosking & Wallis, 1997), therefore, may not be good representatives of their corresponding parameters. Table 1 gives the parameters and sample estimates of skew (γ_3) and kurtosis (γ_4) for the distribution in Fig. 1 (Panel B). Inspection of Table 1 indicates that the bootstrap estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skew and kurtosis (γ_3 and γ_4) are substantially attenuated below their corresponding parameter values with greater bias and variance as the order of the estimate increases. Specifically, for the sample size of $n = 25$, the values of the estimates are 96.92%, and 135.33% of their corresponding parameters, respectively. The estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skew and kurtosis (γ_3 and γ_4) in Table 1 were calculated based on Fisher's k-statistics formulae (see, e.g., Kendall & Stuart, 1977, pp. 299-300), currently used by most commercial software packages such as SAS, SPSS, Minitab, etc., for computing the values of skew and kurtosis (where $\gamma_{3,4} = 0$ for the standard normal distribution).

Table 1: Conventional moment-based parameter values of skew (γ_3) and kurtosis (γ_4) and L-moment-based parameter values of L-skew (τ_3) and L-kurtosis (τ_4) with their corresponding estimates for the pdf in Fig. 1 (Panel B). Each bootstrapped estimate (Estimate), associated 95% bootstrap confidence interval (95% C.I.), and the standard error (SE) were based on resampling 25,000 statistics. Each statistic was based on a sample size of $n = 25$.

Skew: $\gamma_3 = 1.2$			Kurtosis: $\gamma_4 = 1.2$		
Estimate: $\hat{\gamma}_3$	95% C.I.	SE	Estimate: $\hat{\gamma}_4$	95% C.I.	SE
1.163	1.1568, 1.1691	0.0031	1.624	1.5999, 1.6477	0.0121

L-skew (τ_3) $= 0.2409$			L-kurtosis (τ_4) $= 0.2342$		
Estimate: $\hat{\tau}_3$	95% C.I.	SE	Estimate: $\hat{\xi}_4$	95% C.I.	SE
0.2374	0.2362, 0.2385	0.0006	0.2466	0.2455, 0.2479	0.0006

In order to address above limitations, Pant and Headrick (2012) have characterized double-normal- and double-logistic-PM distributions using L-moment-based procedure and contrasted this procedure with the conventional-moment-based procedure. Additionally, to address the latter limitation, Headrick (2011) has characterized the PM distributions through the method of L-moments. The method of L-moments (Hosking, 1990) is an attractive alternative to conventional moment-based method as it can be used in fitting theoretical and empirical distributions, estimating parameters, and testing of hypothesis (Hosking, 1990, 1992; Hosking & Wallis, 1997; Headrick, 2011). In the context of PM distributions, some of the advantages that L-moment based estimators (of L-skew and L-kurtosis) have over conventional moments are that they (a) exist whenever the mean of the distribution exists, (b) are nearly unbiased for all sample sizes and distributions, and (c) are more robust in the presence of outliers (Hosking, 1990, 1992; Hosking & Wallis, 1997; Headrick, 2011; Pant & Headrick, 2013).

For example, for the double-uniform-PM pdf in Fig. 1 (Panel B), the L-moment-based estimates ($\hat{\tau}_3$ and $\hat{\tau}_4$) of L-skew and L-kurtosis (τ_3 and τ_4) in Table 1 are relatively closer to their respective parameter values with much smaller variance compared to their conventional moment-based counterparts. Inspection of Table 1 shows that for the sample size of $n = 25$, the values of the estimates are on average 98.55% and 105.29% of their corresponding parameters.

In the context of the limitations described above, the main purpose of this study is to develop a double-uniform-PM and a double-triangular-PM distributions using a doubling technique (Morgenthaler & Tukey, 2000), characterize these distributions through the method of L-moments, and contrast the estimates of L-moments with their conventional-moment-based counterparts.

The two families of double-uniform-PM and double-triangular-PM distributions can be derived by using a doubling technique (Morgenthaler & Tukey, 2000; Pant & Headrick, 2012) and special cases of PM polynomials in (1) as

$$p(U) = \begin{cases} U + C_{\mathcal{L}}\, U^3, & \text{for } U \leq 0 \\ U + C_{\mathcal{R}}\, U^3, & \text{for } U \geq 0 \end{cases} \tag{4}$$

$$p(T) = \begin{cases} T + C_{\mathcal{L}}\, T^3, & \text{for } T \leq 0 \\ T + C_{\mathcal{R}}\, T^3, & \text{for } T \geq 0 \end{cases} \tag{5}$$

where the random variables U and T in (4) and (5) are drawn respectively from symmetric uniform- and triangular distributions: $U \sim Unif(-\sqrt{\pi/2}, \sqrt{\pi/2})$ and $T \sim Tri(-\sqrt{2\pi}, \sqrt{2\pi})$. These specific uniform and triangular distributions are used so that the maximum height of the pdf associated with the double-uniform-PM and double-triangular-PM distributions in (4) and (5), respectively, is $1/\sqrt{2\pi}$, which is also the maximum height of standard normal pdf (see Pant & Headrick, 2012).

The remainder of the paper is organized as follows. In Section 2, the systems of equations for the conventional-moment-based skew (γ_3) and kurtosis (γ_4) associated with these new families of double-uniform-PM and double-triangular-PM distributions are derived. Also provided in Section 2 is a methodology for solving the systems of equations for the shape parameters ($C_{\mathcal{L}}$ and $C_{\mathcal{R}}$) associated with these families of distributions. In Section 3, a brief introduction to L-moments is given. Section 3 also provides the derivation of the systems of equations for the L-moment-based L-skew (τ_3) and L-kurtosis (τ_4) for the two families of PM distributions. Also provided in Section 3 is an L-moment-based methodology for solving the systems of equations for the shape parameters ($C_{\mathcal{L}}$ and $C_{\mathcal{R}}$) associated with the two families of distributions. In Section 4, a comparison between conventional-moment- and L-moment-based double-uniform- and double-triangular-PM distributions is presented in the context of estimation of parameters. The simulation results are provided for the comparison of estimates. In Section 5, the simulation results are discussed.

2. Conventional-Moment-Based System

2.1. Conventional-Moment-Based System for Double-Uniform-PM Distributions

The conventional moments $\left(\mu_{r=1,\ldots,4}\right)$ associated with (4) can be obtained from

$$\mu_r = \int_{-\sqrt{\pi/2}}^{0} (u + C_{\mathcal{L}} u^3)^r\, \phi(u) du + \int_{0}^{\sqrt{\pi/2}} (u + C_{\mathcal{R}} u^3)^r\, \phi(u) du. \tag{6}$$

The mean (μ), variance (σ^2), skew (γ_3), and kurtosis (γ_4) of double-uniform-PM distributions can be given as (Kendall & Stuart, 1977):

$$\mu = \frac{(C_{\mathcal{R}} - C_{\mathcal{L}})\pi^{3/2}}{16\sqrt{2}} \tag{7}$$

$$\sigma^2 = \frac{\pi}{6} + \frac{1}{20}(C_{\mathcal{L}} + C_{\mathcal{R}})\pi^2 + \frac{\left(25C_{\mathcal{L}}{}^2 + 14C_{\mathcal{L}}C_{\mathcal{R}} + 25C_{\mathcal{R}}{}^2\right)\pi^3}{3584} \tag{8}$$

$$\begin{aligned}\gamma_3 = &-[6\sqrt{105}(C_{\mathcal{L}} - C_{\mathcal{R}})\pi\left(4480 + 9\pi\left(224(C_{\mathcal{L}} + C_{\mathcal{R}}) + 3\pi\left(9C_{\mathcal{L}}^2 + \right.\right.\right.\\ &\left.\left.\left.+ 14C_{\mathcal{L}}\,C_{\mathcal{R}} + 9C_{\mathcal{R}}^2\right)\right)\right)]/[8960 + 3\pi\left(896(C_{\mathcal{L}} + C_{\mathcal{R}}) + 5\pi\left(25C_{\mathcal{L}}^2 + \right.\right.\\ &\left.\left.+ 14C_{\mathcal{L}}\,C_{\mathcal{R}} + 25C_{\mathcal{R}}^2\right)\right)]^{3/2}\end{aligned} \tag{9}$$

$$\begin{aligned}\gamma_4 = \Big[6\Big\{&-2296053760 - 984023040(C_{\mathcal{L}} + C_{\mathcal{R}})\pi\\ &+ 512512\left(67C_{\mathcal{L}}{}^2 - 966C_{\mathcal{L}}C_{\mathcal{R}} + 67C_{\mathcal{R}}{}^2\right)\pi^2\\ &+ 174720(C_{\mathcal{L}} + C_{\mathcal{R}})\left(491C_{\mathcal{L}}{}^2 - 854C_{\mathcal{L}}C_{\mathcal{R}} + 491C_{\mathcal{R}}{}^2\right)\pi^3\\ &+ 165\left(65773C_{\mathcal{L}}{}^4 + 26572C_{\mathcal{L}}{}^3C_{\mathcal{R}} - 82290C_{\mathcal{L}}{}^2C_{\mathcal{R}}{}^2\right.\\ &\left.+ 26572C_{\mathcal{L}}C_{\mathcal{R}}{}^3 + 65773C_{\mathcal{R}}{}^4\right)\pi^4\Big\}\Big]\\ &/\Big[143(8960 + 3\pi(896(C_{\mathcal{L}} + C_{\mathcal{R}}) + 5\pi(25C_{\mathcal{L}}{}^2 + 14C_{\mathcal{L}}C_{\mathcal{R}}\\ &+ 25C_{\mathcal{R}}{}^2)))^2\Big]\end{aligned} \tag{10}$$

2.2. Conventional-Moment-Based System for Double-Triangular-PM Distributions

The conventional moments $\left(\mu_{r=1,\dots,4}\right)$ associated with (5) can be obtained from

$$\mu_r = \int_{-\sqrt{2\pi}}^{0}(t + C_{\mathcal{L}}t^3)^r\,\phi(t)dt + \int_{0}^{\sqrt{2\pi}}(t + C_{\mathcal{R}}t^3)^r\,\phi(t)dt. \tag{11}$$

The mean (μ), variance (σ^2), skew (γ_3), and kurtosis (γ_4) of double-triangular-PM distributions can be given as (Kendall & Stuart, 1977):

$$\mu = \frac{(C_{\mathcal{R}} - C_{\mathcal{L}})\pi^{3/2}}{5\sqrt{2}} \tag{12}$$

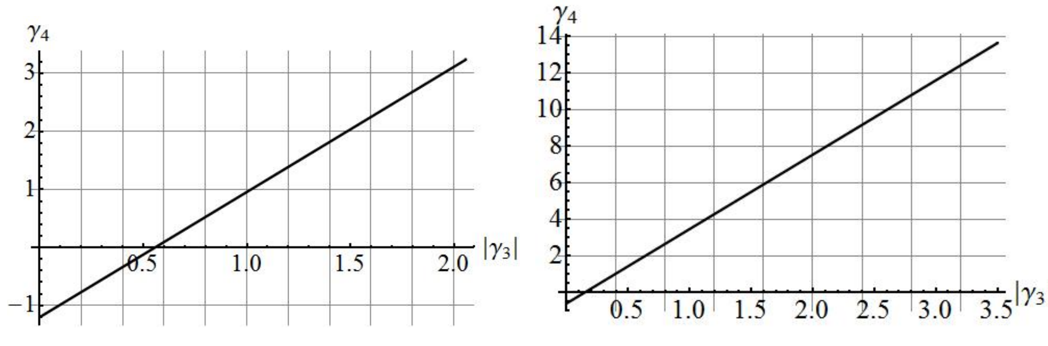
$$\sigma^2 = \frac{\pi(350 + 280\pi(C_{\mathcal{L}} + C_{\mathcal{R}}) + 3\pi^2(43C_{\mathcal{L}}{}^2 + 14C_{\mathcal{L}}C_{\mathcal{R}} + 43C_{\mathcal{R}}{}^2))}{1050} \tag{13}$$

$$\gamma_3 = -[2\sqrt{21}(C_{\mathcal{L}} - C_{\mathcal{R}})\pi\Big(10725 + 2\pi\Big(7315(C_{\mathcal{L}} + C_{\mathcal{R}}) + 81\pi(38{C_{\mathcal{L}}}^2 + 49C_{\mathcal{L}}C_{\mathcal{R}} + 38{C_{\mathcal{R}}}^2)\Big)\Big)] / [11\{\Big(350 + 280\pi(C_{\mathcal{L}} + C_{\mathcal{R}}) + 3\pi^2(43{C_{\mathcal{L}}}^2 + 14C_{\mathcal{L}}C_{\mathcal{R}} + 43{C_{\mathcal{R}}}^2)\Big)\}^{3/2}] \tag{14}$$

$$\gamma_4 = [6\{-1751750 + 1001000\pi(C_{\mathcal{L}} + C_{\mathcal{R}}) + 200200\pi^2(55{C_{\mathcal{L}}}^2 - 17C_{\mathcal{L}}C_{\mathcal{R}} + 55{C_{\mathcal{R}}}^2) + 3640\pi^3(C_{\mathcal{L}} + C_{\mathcal{R}})(3887{C_{\mathcal{L}}}^2 - 4074C_{\mathcal{L}}C_{\mathcal{R}} + 3887{C_{\mathcal{R}}}^2) + 3\pi^4(1803829{C_{\mathcal{L}}}^4 + 502684{C_{\mathcal{L}}}^3C_{\mathcal{R}} - 598026{C_{\mathcal{L}}}^2{C_{\mathcal{R}}}^2 + 502684C_{\mathcal{L}}{C_{\mathcal{R}}}^3 + 1803829{C_{\mathcal{R}}}^4)\}] / [143(350 + 280\pi(C_{\mathcal{L}} + C_{\mathcal{R}}) + 3\pi^2(43{C_{\mathcal{L}}}^2 + 14C_{\mathcal{L}}C_{\mathcal{R}} + 43{C_{\mathcal{R}}}^2))^2] \tag{15}$$

The conventional-moment-based procedure for simulating the double-uniform- and double-triangular-PM distributions involves a moment-matching approach in which specified values of skew (γ_3) and kurtosis (γ_4) are substituted on the left-hand sides of (9)—(10) and (14)—(15), respectively, and then these systems are simultaneously solved for the shape parameters ($C_{\mathcal{L}}$ and $C_{\mathcal{R}}$). The solved values of $C_{\mathcal{L}}$ and $C_{\mathcal{R}}$ can be substituted into (7)—(8) and (12)—(13), respectively, to determine the values of mean and variance associated with the double-uniform- and double-triangular-PM distributions. The solved values of $C_{\mathcal{L}}$ and $C_{\mathcal{R}}$ can be substituted into (3) to plot the pdfs associated with the corresponding distribution. For example, the pdf of double-uniform-PM distribution in Fig. 1 (Panel B) was plotted by first substituting the solved values of $C_{\mathcal{L}} = 2.558204$ and $C_{\mathcal{R}} = 7.098050$ into (4) for generating the double-uniform-PM distribution with $\gamma_3 = \gamma_4 = 1.2$, and subsequently substituting it into (3) for the parametric form of pdf.

The boundary graphs plotted in $|\gamma_3| - \gamma_4$ plane in Figure 2 (Panel A and Panel B) can be used for finding possible combinations of skew (γ_3) and kurtosis (γ_4) associated with conventional-moment-based double-uniform- and double-triangular-PM distributions. Fig. 2 (Panel A) shows the boundary graph for possible combinations of skew (γ_3) and kurtosis (γ_4) associated with a valid double-uniform-PM distribution, where the values of $|\gamma_3|$ range between 0 and 2.0573 and those of γ_4 range between -1.2 to 3.2381. Fig. 2 (Panel B) shows the boundary graph for possible combinations of skew (γ_3) and kurtosis (γ_4) associated with a valid double-triangular-PM distribution, where the values of $|\gamma_3|$ range between 0 and 3.5007 and those of γ_4 range between -0.6 to 13.6443.



A B

Figure 2. Boundary graphs of the regions for possible combinations of (absolute value) skew ($|\gamma_3|$) and kurtosis (γ_4) for the double-uniform- (Panel A) and the double-triangular- (Panel B) PM distributions.

3. *L*-Moment-Based Methodology

3.1. General Definition

L-moments can be expressed as a linear combination of probability weighted moments (PWMs). Let X be a random variable with the pdf $f(x)$, cdf $F(x)$, and the quantile function $F^{-1}(x)$. Then, the PWMs associated with X can be defined as (Hosking & Wallis, 1997)

$$\beta_r = \int F^{-1}(x)\big(F(x)\big)^r f(x)dx \tag{16}$$

Then, the first four L-moments based on the first four PWMs $\left(\beta_{r=0,1,2,3}\right)$ from (16) are expressed in their simplified forms as (Hosking & Wallis, 1997, pp. 20-22)

$$\lambda_1 = \beta_0 \tag{17}$$

$$\lambda_2 = 2\beta_1 - \beta_0 \tag{18}$$

$$\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0 \tag{19}$$

$$\lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0 \tag{20}$$

The notations λ_1 and λ_2 denote the location and scale parameters. In the literature of L-moments, λ_1 is called the L-location (which is equal to the arithmetic mean) and λ_2 (> 0) is called the L-scale, which is one-half of Gini's coefficient of mean difference (Kendall & Stuart, 1977, pp. 47-48). Dimensionless L-moment ratios (i.e., L-skew and L-kurtosis) are defined as the ratios of higher-order L-moments (i.e., λ_3 and λ_4) to λ_2. Thus, $\tau_3 = \lambda_3/\lambda_2$ and $\tau_4 = \lambda_4/\lambda_2$ are, respectively, the indices of L-skew and L-kurtosis. In general, these indices of L-skew and L-kurtosis are bounded such that $|\tau_3| < 1$ and $|\tau_4| < 1$, and as in conventional-moment theory, a symmetric distribution has L-skew $(\tau_3) = 0$ (Headrick, 2011).

Empirical L-moments for a sample (n) of real data, are computed as a linear combination of the sample order statistics $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$. The unbiased sample estimates of the PWMs are given as (Hosking, 1990; Headrick, 2011):

$$\hat{\beta}_r = \frac{1}{n}\sum_{i=r+1}^{n} \frac{(i-1)(i-2)\ldots(i-r)}{(n-1)(n-2)\ldots(n-r)} X_{i:n} \tag{21}$$

where $r = 0, 1, 2, 3$. Here, $\hat{\beta}_0$ is the sample mean. The first four sample L-moments $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4)$ are obtained by substituting $\hat{\beta}_r$ instead of β_r in equations (17)–(20). The symbols used for the sample L-moment ratios (i.e., L-skew and L-kurtosis) are $\hat{\tau}_3$ and $\hat{\tau}_4$, where $\hat{\tau}_3 = \hat{\lambda}_3/\hat{\lambda}_2$ and $\hat{\tau}_4 = \hat{\lambda}_4/\hat{\lambda}_2$.

3.2. *L*-Moment-Based System for Double-Uniform-PM Distributions

The L-moment-based system of equations for the double-uniform-PM distributions can be derived by first defining the PWMs based on (16) in terms of $p(U)$ in (4) and the standard uniform pdf $\phi(u) = 1/\sqrt{2\pi}$ and cdf $\Phi(u) = \left(u + \sqrt{\pi/2}\right)/\sqrt{2\pi}$ as

$$\beta_r = \int_{-\sqrt{\frac{\pi}{2}}}^{0} (u + C_{\mathcal{L}}\, u^3)\{\Phi(u)\}^r \phi(u) du + \int_{0}^{\sqrt{\frac{\pi}{2}}} (u + C_{\mathcal{R}}\, u^3)\{\Phi(u)\}^r \phi(u) du. \tag{22}$$

Integrating (22) for $\beta_{r=0,1,2,3}$ and substituting into (17)—(20) yields the first four L-moments; which are eventually substituted into the formulae for L-skew (τ_3) and L-kurtosis (τ_4) to obtain the following system of equations:

$$\lambda_1 = \frac{(C_{\mathcal{R}} - C_{\mathcal{L}})\pi^{3/2}}{16\sqrt{2}} \tag{23}$$

$$\lambda_2 = \frac{\sqrt{\pi}\left(20 + 3\pi(C_{\mathcal{L}} + C_{\mathcal{R}})\right)}{60\sqrt{2}} \tag{24}$$

$$\tau_3 = \frac{15(C_{\mathcal{R}} - C_{\mathcal{L}})\pi}{24(C_{\mathcal{L}} + C_{\mathcal{R}})\pi + 160} \tag{25}$$

$$\tau_4 = \frac{6(C_{\mathcal{L}} + C_{\mathcal{R}})\pi}{21(C_{\mathcal{L}} + C_{\mathcal{R}})\pi + 140} \tag{26}$$

The solutions for $C_{\mathcal{L}}$ and $C_{\mathcal{R}}$ for a valid double-uniform-PM distribution can also be determined by evaluating the following expressions for specified values of τ_3 and τ_4:

$$C_{\mathcal{L}} = \frac{2(16\tau_3 - 35\tau_4)}{3\pi(7\tau_4 - 2)} \tag{27}$$

$$C_{\mathcal{R}} = \frac{2(16\tau_3 + 35\tau_4)}{3\pi(2 - 7\tau_4)} \tag{28}$$

3.3. *L*-Moment-Based System for Double-Triangular-PM Distributions

The L-moment-based system of equations for the double-triangular-PM distributions can be derived by first defining the PWMs based on (16) in terms of $p(T)$ in (5) and then by integrating the following integral:

$$\beta_r = \int_{-\sqrt{2\pi}}^{0} (t + C_{\mathcal{L}} t^3)\{\Phi(t)\}^r \phi(t) dt + \int_{0}^{\sqrt{2\pi}} (t + C_{\mathcal{R}} t^3)\{\Phi(t)\}^r \phi(t) dt. \quad (29)$$

where $\phi(t)$ and $\Phi(t)$ are the standard triangular pdf and cdf, defined as: $\phi(t) = \begin{cases} (\sqrt{2\pi} + t)/2\pi, & \text{for } t \leq 0 \\ (\sqrt{2\pi} - t)/2\pi, & \text{for } t > 0 \end{cases}$ and $\Phi(t) = \begin{cases} (\sqrt{2\pi} + t)^2/4\pi, & \text{for } t \leq 0 \\ 1 - (\sqrt{2\pi} - t)^2/4\pi, & \text{for } t > 0 \end{cases}$.

Integrating (29) for $\beta_{r=0,1,2,3}$ and substituting into (17)—(20) yields the first four L-moments; which are eventually substituted into the formulae for L-skew (τ_3) and L-kurtosis (τ_4) to obtain the following system of equations:

$$\lambda_1 = \frac{(C_{\mathcal{R}} - C_{\mathcal{L}})\pi^{3/2}}{5\sqrt{2}} \quad (30)$$

$$\lambda_2 = \frac{\sqrt{\pi}\left(49 + 18\pi(C_{\mathcal{L}} + C_{\mathcal{R}})\right)}{105\sqrt{2}} \quad (31)$$

$$\tau_3 = \frac{53\pi(C_{\mathcal{R}} - C_{\mathcal{L}})}{72\pi(C_{\mathcal{L}} + C_{\mathcal{R}}) + 196} \quad (32)$$

$$\tau_4 = \frac{1116\pi(C_{\mathcal{L}} + C_{\mathcal{R}}) + 583}{2376\pi(C_{\mathcal{L}} + C_{\mathcal{R}}) + 6468} \quad (33)$$

The solutions for $C_{\mathcal{L}}$ and $C_{\mathcal{R}}$ for a valid double-triangular-PM distribution can be determined by evaluating the following expressions for specified values of τ_3 and τ_4:

$$C_{\mathcal{L}} = \frac{(30899 + 176760\tau_3 - 342804\tau_4)}{3816\pi(66\tau_4 - 31)} \quad (34)$$

$$C_{\mathcal{R}} = \frac{(30899 - 176760\tau_3 - 342804\tau_4)}{3816\pi(66\tau_4 - 31)} \quad (35)$$

For specified values of L-skew (τ_3) and L-kurtosis (τ_4) associated with the valid double-uniform- and double-triangular-PM distributions, the systems of equations (25)—(26) and (32)—(33) can be simultaneously solved for the values of shape parameters ($C_{\mathcal{L}}$ and $C_{\mathcal{R}}$). Alternatively, the specified values of L-skew (τ_3) and L-kurtosis (τ_4) associated with the valid double-uniform- and double-triangular-PM distributions can be directly substituted into (27)—(28) and (34)—(35), respectively, to obtain the values of $C_{\mathcal{L}}$ and $C_{\mathcal{R}}$. The solved values of $C_{\mathcal{L}}$ and $C_{\mathcal{R}}$ can be substituted into (4) and (5), respectively, for generating the double-uniform- and double-triangular-PM distributions. Further, the solved values of $C_{\mathcal{L}}$ and $C_{\mathcal{R}}$ can be substituted into (23)—(24) and (30)—(31) to determine the values of mean or L-location (λ_1) and L-scale (λ_2) associated with the double-uniform- and double-triangular-PM distributions, respectively.

The boundary graphs in Figure 3 (Panel A and Panel B) can be used for finding possible combinations of L-skew (τ_3) and L-kurtosis (τ_4) associated with the L-moment-based valid double-uniform- and double-triangular-PM distributions. Fig. 3 (Panel A) shows the boundary graph for possible combinations of L-skew (τ_3) and L-kurtosis (τ_4)

associated with a valid double-uniform-PM distribution, where the values of $|\tau_3|$ range between 0 and 0.625 and those of τ_4 range between 0 and 0.2857. Fig. 3 (Panel B) shows the boundary graph for possible combinations of L-skew (τ_3) and L-kurtosis (τ_4) associated with a valid double-triangular-PM distribution, where the values of $|\tau_3|$ range between 0 and 0.7361 and those of τ_4 range between 0.0901 to 0.4697.

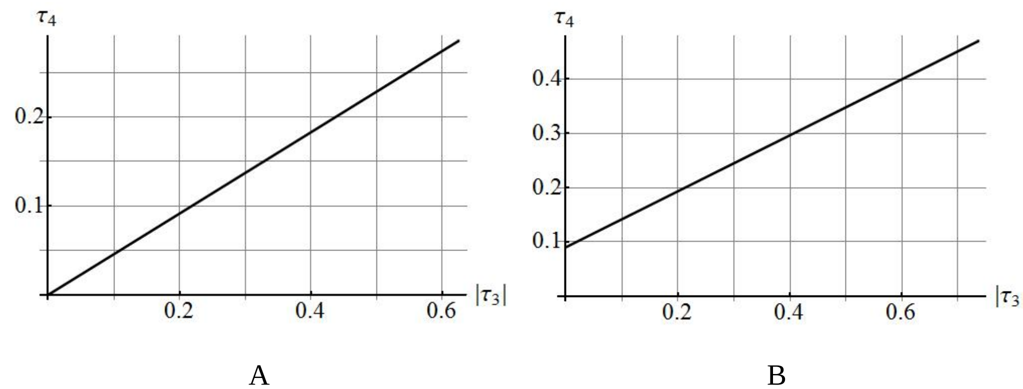


Figure 3: Boundary graphs of the regions for possible combinations of (absolute value) L-skew ($|\tau_3|$) and L-kurtosis (τ_4) for the double-uniform- (Panel A) and the double-triangular- (Panel B) PM distributions.

In the next section, examples are provided to demonstrate the aforementioned methodology and the advantages of L-moment-based procedure over the conventional-moment-based procedure in the contexts of estimation of parameters.

4. Comparison of *L*-Moments with Conventional Moments: Estimation

In the context of estimation of parameters, an example is provided in Figure 4 and Tables 2-3 to demonstrate the advantages of L-moment-based procedure over the conventional-moment-based procedure. Given in Fig. 4 (Panel B) are the pdfs of four distributions of which the first two (Distributions 1 and 2) are the double-uniform-PM and the last two (Distributions 3 and 4) are the double-triangular-PM distributions. The values of conventional-moment- and L-moment-based parameters of skew (γ_3) and L-skew (τ_3), kurtosis (γ_4) and L-kurtosis (τ_4) along with their solved values of shape parameters ($C_{\mathcal{L}}$ and $C_{\mathcal{R}}$) associated with these four distributions, are given in Fig. 4 (Panel A). The pdfs in Fig. 4 (Panel B) were plotted by first substituting the solved values of $C_{\mathcal{L}}$ and $C_{\mathcal{R}}$ into (4) and (5), respectively, to generate the double-uniform-PM and double-triangular-PM distributions and then substituting these into (3) to plot the parametric forms of pdfs associated with these four distributions.

The advantages of L-moment-based procedure over the conventional-moment-based procedure can be demonstrated in the context of estimation of parameters associated with the four distributions in Fig. 4 by considering the Monte Carlo simulation results associated with the indices for the percentage of relative bias (RB%) and standard error (SE) reported in Tables 2 and 3.

Specifically, a Fortran (Microsoft, 1994) algorithm was written to simulate 25,000 independent samples of size $n = 25$, and the L-moment-based estimates ($\hat{\tau}_3$ and $\hat{\tau}_4$) of L-skew and L-kurtosis (τ_3 and τ_4) and the conventional-moment-based estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skew and kurtosis (γ_3 and γ_4) were computed for each of the $(2 \times 25{,}000)$ samples based on the parameters and the values of shape parameters ($C_{\mathcal{L}}$ and $C_{\mathcal{R}}$) listed in Fig. 4 (Panel A). The estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of γ_3 and γ_4 were computed based on Fisher's k-

statistics formulae (Kendall & Stuart, 1977, pp. 47-48), whereas the estimates ($\hat{\tau}_3$ and $\hat{\tau}_4$) of τ_3 and τ_4 were computed by substituting sample estimates of PWMs from (21) into (17)−(20) for obtaining the sample estimates of L-moments and subsequently substituting these into the formulae for estimates $\hat{\tau}_3$ and $\hat{\tau}_4$. Bias-corrected accelerated bootstrapped average estimates (Estimate), associated 95% confidence intervals (95% C.I.), and standard errors (SE) were obtained for each type of estimates using 10,000 resamples via the commercial software package Spotfire S+ (TIBCO, 2008). Further, if a parameter was outside its associated 95% bootstrap C.I., then the percentage of relative bias (RB%) was computed for the estimate as

$$\text{RB\%} = 100 \times (\text{Estimate} - \text{Parameter})/\text{Parameter} \quad (36)$$

In order to demonstrate the advantages of L-moment-based procedure over the conventional-moment-based procedure, the results of simulation are discussed in the next section.

5. Discussion and Conclusion

This study introduced an L-moment based methodology for generating the double-uniform- and double-triangular-PM distributions, which may be useful to researchers in any discipline for simulating non-normal distributions in their studies. One of the advantages of the L-moment-based procedure over the conventional-moment-based procedure can be expressed in the context of estimation. Inspection of Tables 2 and 3 indicates that the estimates of L-moment-based L-skew (τ_3) and L-kurtosis (τ_4) are much less biased than the conventional-moment-based estimates of skew (γ_3) and kurtosis (γ_4) when samples are drawn from the distributions with more severe departures from normality. For example, for samples of size $n = 25$, the estimates of γ_3 and γ_4 for Distribution 4 in Fig. 4 were, on average, 81.06% and 65.67% of their associated parameters, whereas the estimates of τ_3 and τ_4 were 96.72% and 103.06% of their associated parameters. This advantage of L-moment-based estimates can also be expressed by comparing their relative standard errors (RSEs), where RSE $= \{(\text{SE}/\text{Estimate}) \times 100\}$. Comparing Tables 2 and 3, it is evident that the estimators of τ_3 and τ_4 are more efficient as their RSEs are considerably smaller than the RSEs associated with the conventional-moment-based estimators of γ_3 and γ_4. For example, in terms of Distribution 4 in Fig. 4, inspection of Tables 2 and 3 (for $n = 25$), indicates that RSE measures of: RSE $(\hat{\tau}_3) = 0.14\%$ and RSE $(\hat{\tau}_4) = 0.19\%$ are considerably smaller than the RSE measures of: RSE $(\hat{\gamma}_3) = 0.20\%$ and RSE $(\hat{\gamma}_4) = 0.41\%$. This demonstrates that the estimators of τ_3 and τ_4 have more precision because they have less variance around their bootstrapped estimates.

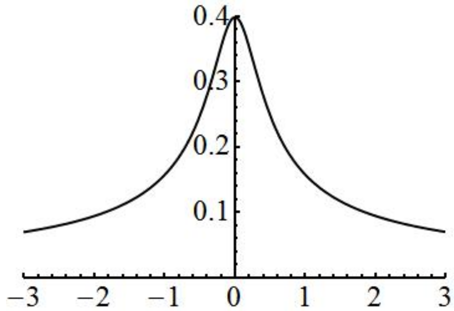
In summary, the proposed L-moment-based procedure is an attractive alternative to the conventional-moment-based procedure in the context of double-uniform- and double-triangular-PM distributions. In particular, the L-moment-based procedure has distinct advantages when distributions with large departures from normality are used. Finally, we note that Mathematica (Wolfram, 2012) source codes are available from the authors for implementing both the L-moment-based and conventional-moment-based procedures.

Distribution 1

$\tau_3 = 0, \tau_4 = 0.15$

$C_{\mathcal{L}} = C_{\mathcal{R}} = 1.1727$

$\gamma_3 = 0, \gamma_4 = -0.2398$

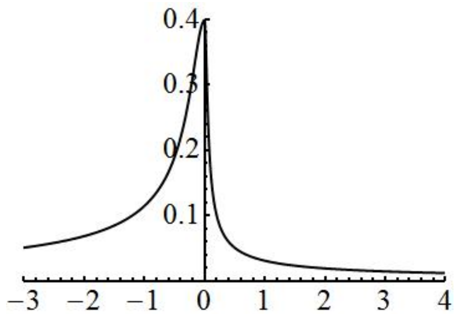


Distribution 2

$\tau_3 = 0.58, \tau_4 = 0.28$

$C_{\mathcal{L}} = 2.7587, C_{\mathcal{R}} = 101.2225$

$\gamma_3 = 2.0210, \gamma_4 = 3.1005$

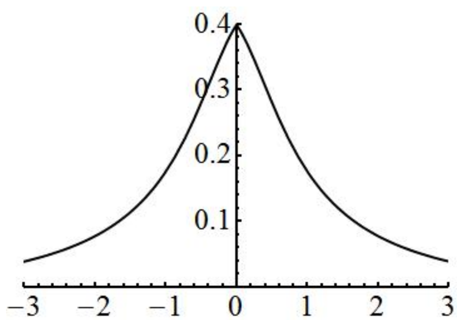


Distribution 3

$\tau_3 = 0, \tau_4 = 0.22$

$C_{\mathcal{L}} = C_{\mathcal{R}} = 0.2253$

$\gamma_3 = 0, \gamma_4 = 1.1257$

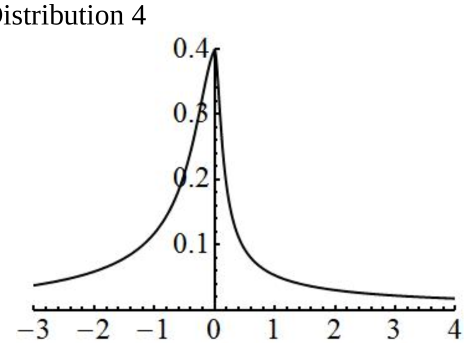


Distribution 4

$\tau_3 = 0.6, \tau_4 = 0.45$

$C_{\mathcal{L}} = 1.1105, C_{\mathcal{R}} = 14.7207$

$\gamma_3 = 3.3332, \gamma_4 = 12.5470$



A B

Figure 4: The parameters of skew (L-skew), kurtosis (L-kurtosis), and the solved values of shape parameters ($C_{\mathcal{L}}$ and $C_{\mathcal{R}}$) of the four distributions are shown in Panel A. The corresponding pdfs are shown in Panel B. Distributions 1 and 2 are the double-uniform-PM distributions, whereas Distributions 3 and 4 are double-triangular-PM distributions.

Table 2. The estimates of L-skew (τ_3) and L-kurtosis (τ_4) for the distributions in Fig. 4. Each estimate was based on a sample size of $n = 25$.

Dist.	Parameter	Estimate	95% C.I.	SE	RB%
1	$\tau_3 = 0$	0.0005	-0.0005, 0.0015	0.0005	-----
	$\tau_4 = 0.15$	0.1564	0.1555, 0.1573	0.0005	4.27
2	$\tau_3 = 0.58$	0.5968	0.5956, 0.5980	0.0006	2.90
	$\tau_4 = 0.28$	0.3145	0.3125, 0.3164	0.0010	12.32
3	$\tau_3 = 0$	0.0009	-0.0005, 0.0025	0.0008	-----
	$\tau_4 = 0.22$	0.2216	0.2207, 0.2226	0.0005	0.73
4	$\tau_3 = 0.6$	0.5803	0.5788, 0.5818	0.0008	-3.28
	$\tau_4 = 0.45$	0.4638	0.4620, 0.4655	0.0009	3.07

Table 3. The estimates of skew (γ_3) and kurtosis (γ_4) for the distributions in Fig. 4. Each estimate was based on a sample size of $n = 25$.

Dist.	Parameter	Estimate	95% C.I.	SE	RB%
1	$\gamma_3 = 0$	0.0029	-0.0017, 0.0074	0.0023	-----
	$\gamma_4 = -0.2398$	-0.0155	-0.0244, -0.0069	0.0045	-93.54
2	$\gamma_3 = 2.0210$	2.1030	2.0951, 2.1117	0.0042	4.06
	$\gamma_4 = 3.1005$	4.2020	4.1560, 4.2536	0.0249	35.53
3	$\gamma_3 = 0$	0.0072	-0.0015, 0.0156	0.0043	------
	$\gamma_4 = 1.1257$	1.1190	1.1023, 1.1352	0.0084	-0.60
4	$\gamma_3 = 3.3332$	2.7020	2.6911, 2.7120	0.0053	-18.94
	$\gamma_4 = 12.5470$	8.2400	8.1740, 8.3074	0.0340	-34.33

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