Exchange Traded Funds with Variable Leverage

Valmira Hoxhaj and Ravindra Khattree

Department of Mathematics and Statistics, Oakland University
Rochester, MI, 48309-4401
(hoxhaj.valmira@gmail.com; khattree@oakland.edu)

Abstract

In this article we introduce the concept of an leveraged exchange traded fund where leverage is a random variable, rather than a constant. This allows one to have more control on hedging demand ratio and it reduces the volatility of the fund. Simulations are used to study the performance of such funds and to compare them with the corresponding constant leveraged funds.

1 Introduction

Exchange traded funds (ETF) are defined as shares of an investment that represent an interest in a portfolio of securities which tracks an underlying benchmark or index. In general, ETF's shares are traded daily on securities exchange. Prices are established by the market as they are bought or sold. ETFs typically have higher daily liquidity and lower fees than the mutual fund shares, making them an attractive alternative for individual investors. ETFs trade like stocks and unlike mutual funds, their Net Asset Values vary all through the day. Leveraged exchange traded funds (LETF) are a recent and very successful financial innovation and an extension of the ETFs. Specifically a leveraged exchange-traded fund is a publicly traded ETF whose goal is to generate daily returns that are a multiple of the daily returns of some benchmark index. This multiple may either be positive, usually $\{2, 3\}$ (for a long leveraged ETF) or negative (usually $\{-1, -2, -3\}$) for an inverse (or short) ETF. This additional return is made possible by investing the borrowed money. The borrowed money to attain the leverage may not necessarily be a loan and a manager may employ the use of other financial derivatives like options, futures and swaps.

The long term performance and related risk issues of leveraged ETFs have been studied recently by Trainor and Baryla (2008), Lu, Wang and Zhang (2009), and Jarrow (2010) and Little (2010). As indicated by above authors and adequately elaborated by Trainor and Baryla (2008), the performance of the leveraged ETF suffers from two effects. The first one is what Trainor and Baryla call the "constant leverage trap" which refers to a magnified compounding problem. Specifically the achievement of the daily constant leverage does not translate to the same multiple of leverage in long term since compounding of investment occurs in a multiplicative fashion, rather than in an additive fashion. More importantly, the constant leverage requirement results in daily rebalancing which negatively affects the performance of the leveraged fund. We will elaborate on it extensively in subsequent discussion through examples.

The second effect is the lack of normality and as Trainor and Baryla amply emphasize, the compounded returns tend to follow lognormal distribution rather than normal, even when the daily returns are normally distributed. The lognormal distribution is positively skewed and hence its density is more concentrated towards left, resulting in a situation where "median compounded return" may be considerably less than "mean compounded return". They also indicate that even on a day to day basis, there is significant volatility associated with the leverage multiple *actually* achieved. Further the requirement of daily rebalancing and positive skewness of the distribution of returns induces considerable volatility in the performance.

The above practical issues are the motivation for this work. How good the daily constant leverage requirement is if one is not able to attain that? That is especially a legitimate concern in terms of increased volatility. The work here attempts to modify that requirement and as shown here in the process, reduces the volatility. The modification that we suggest is practical for implementation and in some way provides more transparency about the daily action taken by the assets manager.

2 Conceptual Framework of Leveraged Returns

Suppose we have an index with a return $r_{t,t+1}$ during time $(t, t+1)$. The time t here represents a day. Index is tracked for days $\{0, 1, 2, \cdots, N\}$. Without any loss of generality, depending on the context, we can also consider any other time frequency (hour, week, month, quarter, year, etc). Let S_t be the net asset price of the index on day t. Then the return $r_{t,t+1}$ from t to $t+1$ is $r_{t,t+1} = \frac{S_{t+1}}{S_t} - 1$ and its compounded return over N days is

$$\begin{aligned} C_N &= \prod_{t=1}^{N} (1 + r_{t,t+1}) - 1 \\ &= (1 + r_{1,2})(1 + r_{2,3}) \cdots (1 + r_{N-1,N}) - 1. \end{aligned} \tag{1}$$

To understand how a leveraged fund works, we adopt the calculations given by Cheng and Madhavan (2009). Also see Avellanda and Zhang (2009) and Zhang (2010). We will use the following notations.

$r_{t,t+1}$: Return on the underlying index from day t to $t+1$.

A_t : NAV of leveraged ETF on day t.

L_t : Notional amount of total return swaps required (i.e. the amount to be invested by the leveraged fund) before day $t+1$.

x : The (constant) leverage factor (usually $x = -3, -2, -1, 2, 3$).

E_{t+1} : Exposure of the total return swaps on day $t+1$.

With above notations, we observe that for $t = 0, 1, 2, \cdots$, the notional amount is the $'x'$ multiple of NAV, that is,

$$L_t = xA_t.$$

Thus on day $(t+1)$, after a return of $r_{t,t+1}$ during day t, the exposure will be,

$$E_{t+1} = L_t(1 + r_{t,t+1}) = xA_t(1 + r_{t,t+1}).$$

Accordingly the NAV of leveraged fund will change to

$$A_{t+1} = A_t(1 + xr_{t,t+1}),$$

since the return for the leveraged fund is $xr_{t,t+1}$. Thus, the notional amount of total return swap after day $t+1$ and before day $t+2$ must be

$$L_{t+1} = L_t(1 + xr_{t,t+1}) = xA_t(1 + xr_{t,t+1}),$$

This will call for rebalancing in order to adjust the exposure. The amount of rebalancing on day $(t+1)$ is

$$\begin{aligned} \triangle_{t+1} &= L_{t+1} - E_{t+1} \\ &= xA_t(1 + xr_{t,t+1}) - xA_t(1 + r_{t,t+1}) \\ &= A_t x(x-1) r_{t,t+1}. \end{aligned} \tag{2}$$

The last expression in (2) shows that the amount of rebalancing is a quadratic function of the leverage multiple x and for the usual choices of $x = 2, 3 - 1, -2, -3$, it is positive. Thus this shows that the rebalancing has the same direction as the index return, in that if the index return is negative (price goes down), leveraged fund must sell while if the return is positive, the leveraged fund must buy. However since

$\triangle_{t+1}$ is a quadratic function of leverage multiple, on different days with different returns and for different leverage values, the rebalancing amount may also be different. For example, for $x = 3$ as well as for $x = -2$, $x^2 - x = 6$. Thus all other things given the same, the rebalancing amounts are equal for the leverage $3\times$ and inverse $2\times$, even though they could behave very differently in terms of their returns.

It is clear that when the market is in upward (downward) trend the leveraged (inverse) fund will result in higher profit over long run, albeit not by the same multiple as stated for daily return. However in a neutral market, both funds, over longer period, will result in loss. See Carver (2009) for a detailed discussion of this issue. We suggest the construction of a new leveraged fund, which results in better volatility properties. Throughout our analysis we will assume that the daily return of our leveraged or inverse ETFs are $R_{t,t+1} = x \cdot r_{t,t+1}$ and thus the compounded return over N days will be

$$\begin{aligned} C_N^\star &= \prod_{t=1}^{N}(1 + xr_{t,t+1}) - 1 \\ &= (1 + xr_{1,2})(1 + xr_{2,3}) \cdots (1 + xr_{N-1,N}) - 1. \end{aligned} \tag{3}$$

We will ignore the cost component in the entire analysis since it is present in all situations and is not likely to greatly depend on the particular approach.

3 Variable Leveraged Fund

The amount of rebalancing in (2) is a function of the index's return $r_{t,t+1}$, and is a quadratic function of the leverage multiple, x. It is positive for $x > 1$ and for $x < 0$. Rebalancing must be done daily and it is in the same direction as the direction of underlying index (buy when the index goes up; sell when index goes down). Notice also that $\triangle_{t+1}$ in (2) is a random variable since it depends on $r_{t,t+1}$. Although LETFs target daily multiple-x returns of underlying index, namely, $xr_{t,t+1}$, in reality the actual return may be somewhat different for a variety of reasons. In other words, the promised leverage multiple is not always attained even on daily basis. Further the amount $\triangle_{t+1}$ or the percent of rebalancing is also random. Market participants and fund holders may prefer an approach with less volatility. Specifically, uncertainty about not knowing how much of the rebalancing will be done on a particular day may make an investor more uneasy and anxious.

To provide some definiteness about $\triangle_{t+1}$, we suggest to keep the ratio $c = \frac{\triangle_{t+1}}{A_t}$, (called daily hedging demand ratio) constant, on daily basis. Equation (2) shows that this requirement will understandably affect the leverage multiple. It follows that $\frac{\triangle_{t+1}}{A_t} = \frac{A_t x(x-1) r_{t,t+1}}{A_t} = x(x-1)r_{t,t+1}$. We will choose to keep the absolute value of hedging demand ratio fixed at c and allow the leverage multiple x to vary. Say, it is x_t

at time t. Clearly x_t values now form a time series. The above change affects how now fund functions. Specifically, now we have,

$$E_{t+1} = L_t(1 + r_{t,t+1}) = x_t A_t(1 + r_{t,t+1}),$$

and the exposure of total return swaps on day $t + 1$ will be

$$L_{t+1} = x_{t+1}A_{t+1} = x_{t+1}A_t(1 + x_t r_{t,t+1}).$$

Therefore,

$$\begin{aligned}\triangle_{t+1} &= L_{t+1} - E_{t+1} \\ &= x_{t+1}A_t(1 + x_t r_{t,t+1}) - x_t A_t(1 + r_{t,t+1}).\end{aligned}$$

Since $\frac{\triangle_{t+1}}{A_t} = \pm c$, we must have

$$\pm c = x_{t+1}(1 + x_t r_{t,t+1}) - x_t(1 + r_{t,t+1}), \tag{4}$$

where $\pm$ is used because the appropriate sign must be retained with c. Thus if $r_{t,t+1} \geq 0$, we use $+c$, and if $r_{t,t+1} < 0$ we use $-c$ in the above formula. If follows from (4) that

$$x_{t+1} = \frac{\pm c + x_t(1 + r_{t,t+1})}{1 + x_t r_{t,t+1}} = \frac{sgn(r_{t,t+1}) \cdot c + x_t(1 + r_{t,t+1})}{1 + x_t r_{t,t+1}} \tag{5}$$

where

$$sgn(r_{t,t+1}) = \begin{cases} 1 & \text{if } r_{t,t+1} \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Thus (5) gives the leverage multiple x_{t+1} for day $t + 1$, which is clearly a function of x_t and $r_{t,t+1}$. The compounded return in this case, over N days is given by

$$\begin{aligned}C_N^{\star\star} &= \prod_{t=1}^{N}(1 + x_t r_{t,t+1}) - 1 \\ &= (1 + x_1 r_{1,2})(1 + x_2 r_{2,3}) \cdots (1 + x_{N-1} r_{N-1,N}) - 1.\end{aligned} \tag{6}$$

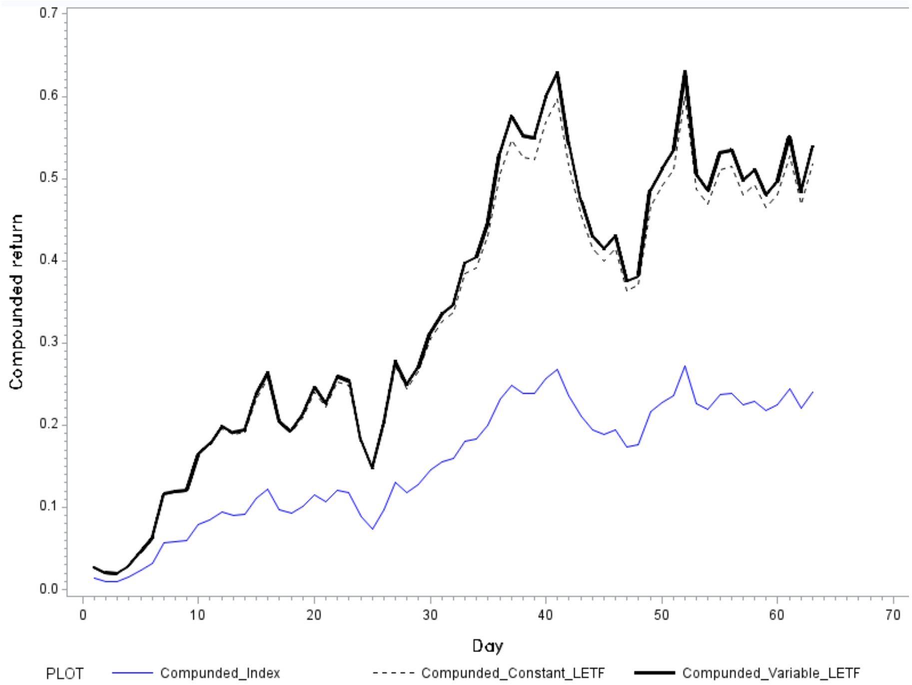
4 Implementation of Variable Leverage

We will implement the above strategy on simulated returns and assess the effects of various quantities on the short term performance. This analysis will give us insight into how to set up the strategy and what kind of performance one could expect. Needless to say that distributional assumptions must play a crucial role and thus in any given context must be chosen carefully. For our work, we will consider the target leverage multiple to be '2' although similar studies can be taken upon for other leverage multiples including the negative ones. We will also assume that percent daily returns $r_{t,t+1}$ are normally distributed with mean $\mu_r = 0.005$ and standard deviation $\sigma_r = 0.015$. The absolute value of daily hedging demand ratio $c = \frac{\triangle_{t+1}}{A_t}$ is being held fixed and we will choose $c = 0.01, 0.02, 0.03, 0.04$ and 0.05. The compounded returns C, C^* and C^{**} as defined in (1), (3) and (6) respectively are calculated over one week (five days), and 3 months (63 days). Typical relative performances for 63 days of index fund, constant leveraged fund and our variable leveraged fund are shown in Exhibit 1. The hedging demand for variable leveraged fund plays an important role and must be chosen judiciously so as to ensure that x_t has a distribution which is around $x = 2$ (or any other multiple if so desired). To investigate the statistical properties of the fund so constructed, we resort to simulation. To do so we generate the above scenario for $Nsim = 5000$ times. We present the summary statistics (mean and standard derivation over all simulations) of daily performances of the index fund, 2× constant leveraged fund and the 2× variable leveraged fund (with $c = 0.01$, 0.02, 0.03, 0.04 and 0.05) for first 5 days in Exhibit 2 and Exhibit 3 respectively.

Exhibit 2 and Exhibit 3 clearly show the dependence of the distribution of x_t on the choice of the daily hedging demand. Clearly the choices of $c = 0.01$ or 0.02 are inadequate to yield x_t values around 2. The situation is somewhat better with $c = 0.03$, even though this hedging demand also results in the under-performance of the leveraged fund. For $c = 0.04$ and 0.05 performance of x_t gets better. More precisely, for a choice of $c = 0.04$ the values of x_t for the variable leverage fund are very close to the target value of 2, whereas $c = 0.05$ also yields x_t values around 2 but slightly higher values of x_t than other cases. Hence the best choice for hedging demand may be $c = 0.04$. Exhibit 2 does show that the compounded returns of variable leveraged fund over five days for this choice and of fixed leveraged fund are very comparable. What is more important to note is that at any level of the hedging demand c, the 2× variable leveraged exchange fund's standard deviation is smaller compared to the standard deviation of 2× constant leverage exchange fund, over 5 days (See Exhibit 3). This is a big advantage that this alternative strategy provides.

Means and standard deviations (over $Nsim = 1000$) for the longer period (=63 days) for the index fund, 2× constant leverage fund and corresponding variable leverage fund (with $c = 0.04$) are plotted in Exhibit 4 and Exhibit 5. It is seen that means of constant leveraged fund and variable leveraged funds are

Exhibit 1: A Typical Daily Performance of Compounded Returns (Over $Nsim = 1$) of Index Fund, 2× Constant Leveraged Fund and 2× Variable Leveraged Fund Over Three Months (63 days) with Hedging Demand Ratio $c = 4\%$.



practically indistinguishable while while variable leverage fund seems to result in smaller standard deviation.

One important point must be made. As seen in (5), x_{t+1} depends on c, $r_{t,t+1}$ as well as on x_t. Consequently $\{x_t\}$ forms a time series of dependent random variables which may not be stationary. Monotonically increasing/decreasing values of x_t in Exhibit 2 certainly seem to suggest that in our simulations. Thus, adjustments must be made periodically to bring the values of x_t back to near target. This is especially so since, the series $\{r_{t,t+1}\}$ although independent, may drift from its mean and that will have strong influence on the choice of c as well as on the series $\{x_t\}$.

We also believe that in practical context, the return distribution will shift either in parameters or in terms of underlying distributional assumptions and/or shape. Such changes are important and should be taken into consideration to make adjustment in the management of the fund. This will in turn change the choice of c, so that the variable multiple is close to the chosen target multiple. Thus from time to time statistical tests for the distributional assumptions and/or for change point problems must be performed.

Exhibit 2: Mean (Over $Nsim = 5000$) Compounded Returns of Index Fund, 2× Constant Leveraged Fund and 2× Variable Leveraged Fund Over One Week for Various Values of Hedging Demand Ratio c. The table also shows that for $c = 4\%$, x_t tracks the target very closely.

	Index Fund	2×Constant Leveraged Fund	Hedging Demand Ratio (c) for 2× Variable Leveraged Fund									
			$c = 1\%$		$c = 2\%$		$c = 3\%$		$c = 4\%$		$c = 5\%$	
Day	Mean of C_t	Mean of $C_t^{\star}$	Mean of x_t	Mean of $C_t^{\star\star}$	Mean of x_t	Mean of $C_t^{\star\star}$	Mean of x_t	Mean of $C_t^{\star\star}$	Mean of x_t	Mean of $C_t^{\star\star}$	Mean of x_t	Mean of $C_t^{\star\star}$
1	0.005	0.010	2.000	0.010	2.000	0.010	2.000	0.010	2.000	0.010	2.000	0.010
2	0.010	0.021	1.993	0.020	1.996	0.020	1.998	0.020	2.000	0.020	2.003	0.021
3	0.015	0.032	1.986	0.030	1.991	0.031	1.996	0.031	2.001	0.031	2.005	0.031
4	0.020	0.041	1.981	0.039	1.988	0.040	1.995	0.040	2.002	0.041	2.009	0.041
5	0.025	0.052	1.974	0.050	1.983	0.050	1.993	0.051	2.002	0.051	2.011	0.052

Exhibit 3: Standard Deviation (Over $Nsim = 5000$) of Compounded Returns of Index Fund, 2× Constant Leveraged Fund and 2× Variable Leveraged Fund Over One Week for Various Values of Hedging Demand Ratio c. The table shows that standard deviations of variable leveraged options are smaller than the constant leveraged case.

	Index Fund	2× Constant Leveraged Fund	Hedging Demand Ratio (c) for 2× Variable Leveraged Fund									
			$c = 1\%$		$c = 2\%$		$c = 3\%$		$c = 4\%$		$c = 5\%$	
Day	STD of C_t	STD of $C_t^{\star}$	Mean of x_t	STD of $C_t^{\star\star}$	Mean of x_t	STD of $C_t^{\star\star}$	Mean of x_t	STD of $C_t^{\star\star}$	Mean of x_t	STD of $C_t^{\star\star}$	Mean of x_t	STD of $C_t^{\star\star}$
1	0.015	0.030	2.000	0.029	2.000	0.029	2.000	0.029	2.000	0.029	2.000	0.029
2	0.021	0.061	1.993	0.042	1.996	0.042	1.998	0.042	2.000	0.042	2.003	0.042
3	0.026	0.069	1.987	0.053	1.991	0.053	1.996	0.053	2.001	0.053	2.005	0.053
4	0.030	0.076	1.981	0.061	1.988	0.061	1.995	0.062	2.002	0.062	2.009	0.062
5	0.034	0.083	1.974	0.069	1.983	0.069	1.993	0.069	2.002	0.070	2.011	0.070

Exhibit 4: Mean (Over $Nsim = 1000$) Compounded Returns of Index Fund, $2\times$ Constant Leveraged Fund and $2\times$ Variable Leveraged Fund Over Three Months (63 days) with Hedging Demand Ratio $c = 4\%$.

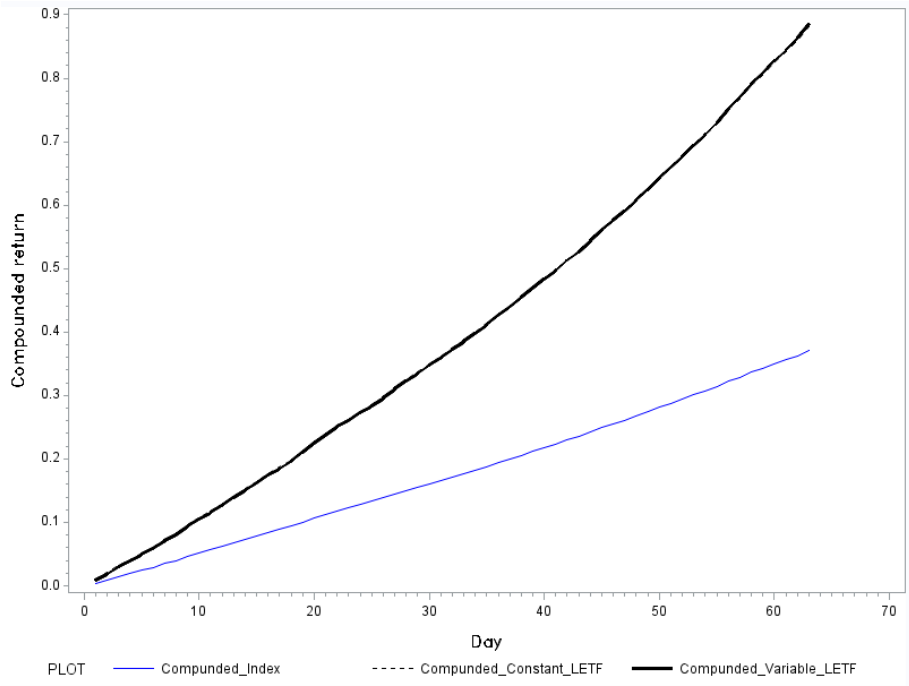
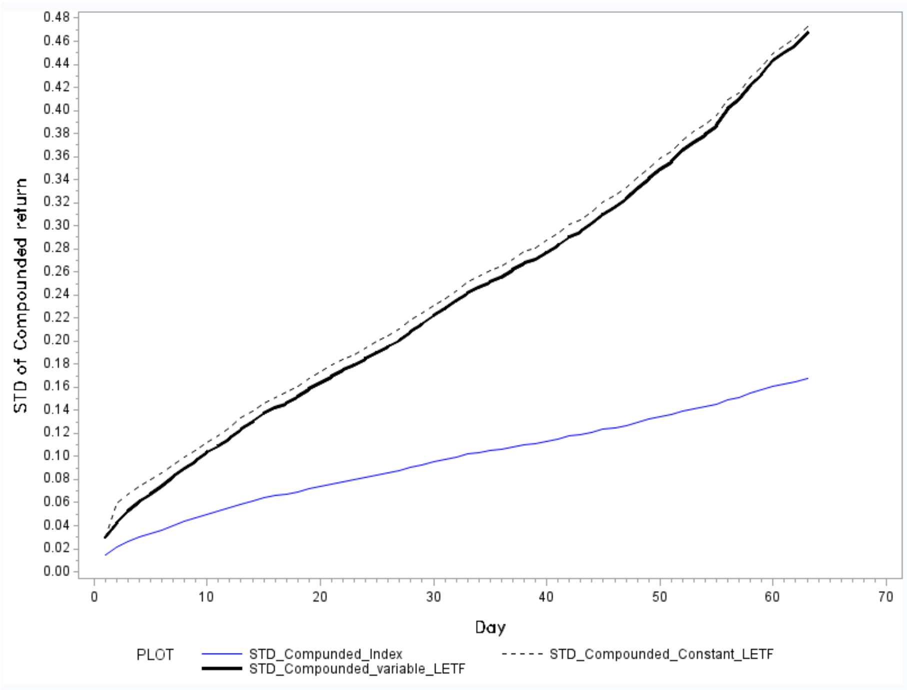


Exhibit 5: Standard Deviation (Over $Nsim = 1000$) of Compounded Returns of Index Fund, $2\times$ Constant Leveraged Fund and $2\times$ Variable Leveraged Fund Over Three Months (63 days) with Hedging Demand Ratio $c = 4\%$.



5 Comparison of Volatility

By design, the leveraged funds are more volatile than the corresponding index fund and there is no way to construct a leverage fund which at the same time is less volatile than the underlying index fund. Thus the natural question to ask is in longer terms how volatile is our $2\times$ variable leverage fund compared to the standard $2\times$ constant leverage fund? It must be noted that apart from index fund's variability, volatility is also a function of the daily hedging demand ratio. We believe that the changing hedging demand which is a natural component in Cheng and Madhavan's (2009) approach, adds considerably to volatility. Exhibit 6 presents the mean volatility of various funds and presents a comparative picture after a week, a month and 3 months, for leverage multiples $2, 3$ and for $c = 0.01$ to 0.05 over 1000 simulations. Note that any choice of c taken here is not necessarily the optimal one and will depend on the particular situation.

Clearly for both the $2\times$ and $3\times$ variable leveraged funds, compounded returns of the funds get more volatile as hedging demand c increases. This is intuitively obvious since higher level of buying and selling on daily basis will result in more fluctuations in the daily performance and consequently in the compounded performance. What is striking is that both $2\times$ and $3\times$ variable leveraged funds are less volatile than considering $2\times$ and $3\times$ constant leveraged funds for almost every value of the hedging demand c taken under consideration. This is an important and very desirable feature of variable leveraged funds. The only case when our $2\times$ variable leveraged fund is very slightly more volatile than the $2\times$ constant leveraged fund occurs for the time period of 63 days and for hedging demand $c = 0.05$.

Exhibit 6: Mean Volatility (Over $Nsim = 1000$) of Compounded Returns for Index Fund, $\{2\times, 3\times\}$ Constant Leveraged Fund and $\{2\times, 3\times\}$ Variable Leveraged Fund Over Different Time Periods for Various Values of Hedging Demand Ratio c. The table shows that for variable leveraged funds, the volatility depends on hedging demand Ratio c, yet it is always smaller than that for corresponding constant leveraged fund.

				Hedging Demand Ratio (c) for Variable Leverage				
Target Leverage multiple	Time period	Index Fund	Constant Leverage	$c = 1\%$	$c = 2\%$	$c = 3\%$	$c = 4\%$	$c = 5\%$
2	One Week(5 days)	0.025	0.063	0.050	0.050	0.050	0.050	0.050
	One Month(21 days)	0.050	0.116	0.102	0.104	0.105	0.107	0.109
	3 Months(63 days)	0.097	0.248	0.204	0.216	0.227	0.239	0.251
3	One Week(5 days)	0.025	0.096	0.075	0.075	0.075	0.075	0.075
	One Month(21 days)	0.050	0.185	0.153	0.155	0.156	0.158	0.159
	3 Months(63 days)	0.097	0.461	0.301	0.313	0.326	0.338	0.351

6 Concluding Remarks

We have introduced the idea of variable leverage with an intent to reduce the variability of the compounded returns, to make the fund less volatile and to provide more definiteness of the action taken by the fund, which an investor may prefer. The usefulness of the strategy has been illustrated by implementing it on simulated data under the assumption of normality of returns.

We find that with appropriate choice of hedging demand and daily rebalancing, the leveraged fund can be made to return, on an average, the same multiple of the market even though no promise can be made that the leverage will be constant on daily basis. In strict sense, that promise, however is never true even for the existing constant leveraged funds. What is important is that we are able to reduce the volatility and also provide a more assuring and mathematically well defined approach to rebalancing, thereby reducing the anxiety that an investor may face when investing in the fund.

Few more remarks are in line for future research as well as about care in interpreting our results. First, our simulations assumed normality of returns for convenience. Further mean and standard deviation were assumed as certain specific quantities. No generality is lost if the values of parameters were changed. Also, nothing prevents one to assume any other distribution for returns and adopt the simulation as indicated above. What the return distribution really is, depends on the particular index and thus in any specific situation, the return distribution must be carefully examined and tested using past data before making any such assumption.

Secondly, in this work the calculations were done for various choices of hedging demand ratio c. What is the correct value of c can be established only through intense calculations. We have chosen certain specific values of c in our work for leveraged as well as inverse funds, and in real situation this c must be carefully chosen by more detailed calculations so that the variable leverage x_t is as close as possible to the target. It is practically impossible to have a closed form mathematical formulation to find the optimal choice of c and one must necessarily rely on simulations. Further research is needed which will clearly depend on specific context.

What is the behavior of time series $\{x_t\}$ of variable leverages? Clearly for a highly nonstationary series the values of variable leverage x_t will soon drift from the target, which is highly undesirable. It is therefore a difficult question to answer. Under what assumptions would this series be stationary? At the outset, we do empirically observe that series can perhaps be nonstationary. Is there a range of c values, for which this series, under certain distributional assumptions, is stationary? These issues are not clear and require detailed empirical investigation.

Finally, it must be remembered that the return distributions themselves do change over time, and since the variable leverage does depend on return distribution, associated with our problem of constructing and managing a variable leverage fund is the problem of change point detection which should be investigated empirically from time to time using the data. A change in the assumptions of return distribution may effect the optimum choice of c so as to have the variable leverage multiple close to the target value. In that sense, one may view our variable leveraged fund as one which, in part, is actively managed.

Acknowledgement

This work and corresponding talk is based on first author's PhD dissertation at Oakland University. Travel support from Department of Mathematics and Statistics is acknowledged.

References

[1] Avellanda, M., Zhang, S.(2009). Path-dependence of leveraged ETF returns. *SIAM J. Financial Math* 1(1), 586-603.

[2] Carver, A.B.(2009). Do levereged and inverse ETFs converge to zero? *A Guide to Exchange-Traded Funds* 144-149.

[3] Cheng, M., Madhavan, A.(2009). The Dynamics of leveraged and inverse exchange- traded funds. *Journal of Investment Management* 7(4), 43-62.

[4] Jarrow, R.A.(2010). Understanding the risk of leveraged ETFs. *Finance Research Letters*, 7(3), 135-139.

[5] Little, P.K.(2010). Inversed and leveraged ETFs: Not your father's ETF. *Journal of Index Investing*, 1(1), 83-89.

[6] Lu, L., Wang, J., Zhang, G. (2009). Long term performance of leveraged ETFs. *Available at SSRN* 1344133.

[7] Trainor Jr, W., Baryla Jr, E. A. (2008). Leveraged ETFs: A risky double that doesn't muliply by two. *Journal of Financial Planning*, 21(5), 48-55.

[8] Zhang, J.(2010). Path-dependence properties of leveraged exchange-traded funds: Compunding volatility and option pricing. *PhD Thesis, Department of Mathematics, New York University.*