Asymptotic Expansion of One-Factor Merton Models with Non-Gaussian and Serially Correlated Innovations

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Abstract

The one-factor Merton model in the context of CreditMetrics is specialized by a single factor common to all counterparties. We extend the structural credit risk model to a model that includes underlying single risk factor and issuer-specific process have non-Gaussian and serially correlated asset returns. By using a standard Edgeworth expansion, we arrive at the closed-form analytic expressions for the default rate distribution. We also provide estimators of the parameters of the asset value process. Our empirical results illustrate the non-negligible effects of the skewness and kurtosis of the distributions on the systematic risk of credit portfolio risk evaluations.

Key Words: asset correlation, credit risk, Edgeworth expansion, probability of default, single risk factor model.

1. Introduction

In recent years, the need for credit risk modeling has become essential for those responsible for granting bank loans or investing in financial products exposed to counterparty default risk. For regulators and internal risk managers, calculating capital adequacy requirements and allocating capital efficiency are important for understanding and tracking credit portfolio risk management. Since credit risk refers to the risk of incurring losses due to unexpected changes in the credit quality of a counterparty or issuer, credit risk modeling has become an important topic in the field of finance and banking.

In the finance literature and within the banking industry, focus is placed on modeling the risk inherent in the entire credit portfolio such as loans, pledges, and guarantees, namely the pool of defaultable instruments. The losses on the initial portfolio value due to the default of the underlying issuer depend on the default probability of each issuer and the losses derived from each default. The quantitative modeling of credit risk is used to evaluate the credit risk associated with the loan portfolio, which includes CreditMetrics by JP Morgan, CreditRisk+ by Credit Suisse Financial Products, and CreditPortfolioView by McKinsey. Moody's KMV, the most popular credit risk model, is used as the benchmark portfolio model because it is considered to be a reasonable internal model with which to assess regulatory capital related to credit risk (see, for details, Crouhy et al.[2]). To estimate the default probabilities, the historical default rates are required by CreditMetrics and CreditRisk+. For a comparison of the models and estimation methods, see Carey and Hrycay[1]. Koylouglu and Hickman[10] and Gordy[4] show that the evaluation of credit risk assessment for loans by the above-mentioned models is similar since the models are based on similar ideas and on the mathematical framework of the Vasicek or one-factor Merton representation of asset returns. The different databases and different approaches for parametrizing these credit risk models may often lead to substantial differences in practice.

In this paper, we focus on the method based on the Vasicek single factor model or the CreditMetrics framework. Such a one-factor Merton model provides a structural link between default events and the obligor's asset returns. By adapting the single asset model of

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Merton[11] to a credit portfolio, Vasicek[14] derives default probabilities conditional on a single systematic risk, which is caused by the fluctuation of returns in terms of macroeconomic factors. According to these models, the most important factor deriving the credit portfolio risk is the asset correlation of the underlying asset value processes.

Merton's model assumes a normal distribution for the logarithms of asset returns; however, this assumption has often proven to be invalid. Many empirical finance studies report that the returns of financial assets have a large sharper peak with greater density in the tails of the distribution and serially correlated heteroskedasticity. It is thus important to check whether the estimation of default probabilities satisfies normal distribution assumptions. If not, the suggested risk measurements by the model may tend to under- or overestimate the required risk tolerance level. Several techniques can be used to derive the distribution of the portfolio loss rate in non-normal situations. Gordy[5] provides a saddle point approximation to the default rate distribution in CreditRisk+ models. Hull et al.[8] consider a jump diffusion model to explain the volatility skew in the credit default swap Markets. Kawada and Shiohama[9] consider the asymptotic expansion of the asset value process with a non-Gaussian innovation to obtain the default probability in Merton's structural model. For the reduced-form approach in credit risk models, Miura et al.[12] consider the asymptotic expansion of the credit spread process with non-Gaussian and serially correlated errors.

The remainder of the paper is organized as follows. Section 2 provides the theoretical settings of the models and their assumptions. Section 3 presents the asymptotic credit risk evaluation. In Section 4, numerical examples are illustrated to highlight the effects of non-Gaussianity in asset returns. Section 5 contains a real data analysis using U.S. historical default frequency data to shed light on how non-Gaussian modeling affects the credit risk evaluation. All proofs are omitted to save space.

2. The Model and its Assumption

In Merton's model, the asset value of obligor i is assumed to follow a geometric Brownian motion under the Physical measure $\mathbb{P}$,

$$\mathrm{d}V_{i,t} = \mu V_{i,t}\mathrm{d}t + \sigma V_{i,t}\mathrm{d}W_{i,t},$$

where $V_{i,t}$ denotes the asset value at time t of obligor i, μ is the drift, and σ is the volatility parameters. To capture the dependencies between obligors, the Brownian motion $W_{i,t}$ is decomposed into two independent Brownian motions such that

$$\mathrm{d}W_{i,t} = \sqrt{\rho}\mathrm{d}X_{0,t} + \sqrt{1-\rho}\mathrm{d}X_{i,t},$$

where $X_{0,t}$ is a common systematic risk factor and $X_{i,t}$ is a obligor-specific risk factor. We also assume that two standard Brownian motions $X_{0,t}$ and $X_{i,t}$ are independent of each other. The parameter ρ is the asset correlation and $\sqrt{\rho}$ indicates the sensitivity of the asset value. Here, we assume a homogeneous portfolio consisting of N obligors that possess the same parameters μ and σ and the same correlation coefficient ρ between obligors i and j for $i = 1, \ldots, N$.

To implement the non-Gaussian and serially correlated structure in the one-factor Merton model, we consider the discrete time expression of the model that is known as the Euler scheme of the stochastic differential equations. The current time is 0 and we assume that the stochastic process $\{V_{i,j}\}$ is discretely sampled with interval Δ such that $V_{i,j}$ is sampled at times $0, \Delta, 2\Delta, \ldots, n\Delta (\equiv t)$ over $[0, t]$. Then, the ith obligor's asset value at time $j\Delta$ is defined as follows:

$$\begin{aligned}\ln V_{i,j\Delta} &= \ln V_{i,(j-1)\Delta} + \mu\Delta + \sigma\sqrt{\Delta}W_{i,(j-1)\Delta},\\ W_{i,(j-1)\Delta} &= \sqrt{\rho}\tilde{X}_{0,(j-1)\Delta} + \sqrt{1-\rho}\tilde{X}_{i,(j-1)\Delta}.\end{aligned} \tag{1}$$

In this paper, we consider that the process $\tilde{X}_{i,j}, i = 0, 1, \ldots, N$ is an independent stationary process with zero mean and unit variance, namely standardized process $\tilde{X}_{i,j} = X_{i,j}/(\mathrm{var}(X_{i,j}))^{1/2}$. Here, we assume that the process $\{X_{i,j}\}$ is the stationary process satisfying the following conditions.

Assumption 1 *The processes $\{X_{i,j}\}$ are fourth-order stationary in the sense that for each $i \in \{0, 1, \ldots, N\}$,*

1. $E[X_{i,j}] = 0$,
2. $\mathrm{cum}(X_{i,j}, X_{i,j+u}) = c_{X_i}(u)$,
3. *cum*$(X_{i,t}, X_{i,j+u_1}, X_{i,j+u_2}) = c_{X_i}(u_1, u_2)$,
4. *cum*$(X_{i,t}, X_{i,j+u_1}, X_{i,j+u_2}, X_{i,j+u_3}) = c_{X_i}(u_1, u_2, u_3)$.

Assumption 2 *The k-th order cumulants $c_{X_i}(u_1, \ldots, u_{k-1})$ of $\{X_t\}$, for $k = 2, 3, 4$ and $i \in \{0, 1, \ldots, N\}$ satisfy*

$$\sum_{u_1, \cdots, u_{k-1}=-\infty}^{\infty} |c_{X_i}(u_1, \ldots, u_{k-1})| < \infty.$$

Assumptions 1 and 2 are satisfied by a wide class of time series models containing the usual autoregressive moving average (ARMA) and generalized autoregressive conditional heteroskedasticity (GARCH) processes. For more details on this modeling, see Honda et al.[6], Miura et al.[12], and Kawada and Shiohama[9].

From (1), the obligor's asset value at time $t = n\Delta$ is expressed as

$$\ln V_{i,t} = \ln V_{i,0} + \mu n \Delta + \Delta^{1/2} \sigma \left(\sqrt{\rho} \sum_{j=1}^{n} \tilde{X}_{0,j} + \sqrt{1-\rho} \sum_{j=1}^{n} \tilde{X}_{i,j} \right).$$

Define $Y_{0,n} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \tilde{X}_{0,j}$ and $Y_{i,n} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \tilde{X}_{i,j}$. Let $W_{i,n} = \sqrt{\rho} Y_{0,n} + \sqrt{1-\rho} Y_{i,n}$, then we observe

$$\begin{aligned} V_{i,t} = V_{i,n\Delta} &= V_{i,0} \exp\left\{\mu t + \sigma \sqrt{t} W_{i,n}\right\} \\ &= V_{i,0} \exp\left\{\mu t + \sigma \sqrt{t} (\sqrt{\rho} Y_{0,n} + \sqrt{1-\rho} Y_{i,n})\right\}. \end{aligned}$$

The following lemma states the cumulants for $\{Y_{i,n}\},\ i \in \{0, 1, \ldots, N\}$. Recall that we have assumed that the random variables $Y_{i,n}$ and $Y_{j,n}$ are independent for all $i, j = 0, \ldots, N$ with $i \neq j$, all the joint cumulants of $Y_{i,n}$ and $Y_{j,n}$ equal zero.

Lemma 1 *Under Assumptions 1 and 2, the cumulants of $\{Y_{i,n}\}$ for $i \in \{0, 1, \ldots, N\}$ are evaluated as follows:*

1. $E[Y_{i,n}] = 0$,
2. $\mathrm{cum}(Y_{i,n}, Y_{i,n}) = 1 + o(n^{-1})$,
3. $\mathrm{cum}(Y_{i,n}, Y_{i,n}, Y_{i,n}) = n^{-1/2} C_{Y_i,3}^{(n)} + o(1)$,
4. $\mathrm{cum}(Y_{i,n}, Y_{i,n}, Y_{i,n}, Y_{i,n}) = n^{-1} C_{Y_i,4}^{(n)} + o(1)$,

where $C_{Y_i,3}^{(n)}$ and $C_{Y_i,4}^{(n)}$ are bounded for n.

The joint probability distribution function of $\boldsymbol{W}_{N,n} = (W_{1,n}, \ldots, W_{N,n})'$ can be obtained from the Edgeworth expansion below. To do this, we need to evaluate the joint cumulants of $\boldsymbol{W}_{N,n}$

Lemma 2 *Under Assumptions 1 and 2, the cumulants of $\{\boldsymbol{W}_{N,n}\}$ are evaluated as follows:*

1. $E[\boldsymbol{W}_{N,n}] = \mathbf{0}_N$,
2. $\mathrm{Var}(\boldsymbol{W}_{N,n}) = \boldsymbol{\Sigma}_{N,n} + o(1)$ *where the (i,j)th elements of $\boldsymbol{\Sigma}_{N,n}$ is ρ for $i,j = 1,\ldots,N$ and the diagonal elements are 1,*
3. *for $i,j,k \in \{1,\ldots,N\}$,*
$$\mathrm{cum}(W_{i,n}, W_{j,n}, W_{k,n}) = C^{(n)}_{W,ijk} + o(1),$$
where
$$C^{(n)}_{W,ijk} = \begin{cases} n^{-1/2}\rho^{3/2}C^{(n)}_{Y_0,3} & \text{for } i \neq j \neq k, \\ n^{-1/2}(\rho^{3/2}C^{(n)}_{Y_0,3} + (1-\rho)^{3/2}C^{(n)}_{Y_i,3}) & \text{for } i = j = k, \end{cases}$$
4. *for $i,j,k,\ell \in \{1,\ldots,N\}$,*
$$\mathrm{cum}(W_{i,n}, W_{j,n}, W_{k,n}, W_{\ell,n}) = C^{(n)}_{W,ijk\ell} + o(1),$$
where
$$C^{(n)}_{W,ijk\ell} = \begin{cases} n^{-1}\rho^{2}C^{(n)}_{Y_0,4} & \text{for } i \neq j \neq k \neq \ell, \\ n^{-1}(\rho^{2}C^{(n)}_{Y_0,4} + (1-\rho)^{2}C^{(n)}_{Y_i,4}) & \text{for } i = j = k = \ell, \\ n^{-1}(\rho^{2}C^{(n)}_{Y_0,4} + 2(1-\rho)) & \text{for } i = j, k = \ell. \end{cases}$$

To derive the Edgeworth expansion of $Y_{i,n}$ and $\boldsymbol{W}_n$, we need the following assumption.

Assumption 3 *The J-th order ($J \geq 5$) cumulants of $\{Y_{i,n}\}$ and $\{\boldsymbol{W}_n\}$ are of order $O(n^{-J/2+1})$.*

We then arrive at the following theorem. For more details and proofs, we refer Taniguchi and Kakizawa[13].

Theorem 1 *Under Assumptions 1-3, the third-order Edgeworth expansion of the density function of Y_i for $i \in \{0,1,\ldots,n\}$ is given by*

$$g_{Y_i}(y) = \phi(y)\left\{1 + \frac{C^{(n)}_{Y_i,3}}{6\sqrt{n}}H_3(y) + \frac{C^{(n)}_{Y_i,4}}{24n}H_4(y) + \frac{(C^{(n)}_{Y_i,3})^2}{72n}H_6(y)\right\} + o(n^{-1}), \quad (2)$$

where $\phi(\cdot)$ is the standard normal density function, while $H_k(\cdot)$ is the k-th order Hermite polynomial. The corresponding probability distribution function is of the form

$$G_{Y_i}(y) = \Phi(y) - \phi(y)\left\{\frac{C^{(n)}_{Y_i,3}}{6\sqrt{n}}H_2(y) + \frac{C^{(n)}_{Y_i,4}}{24n}H_3(y) + \frac{(C^{(n)}_{Y_i,3})^2}{72n}H_5(y)\right\} + o(n^{-1}).$$

The Edgeworth expansion for the distribution function for $\boldsymbol{W}_n$ is given by the following theorem. To do this, it is convenient notationally to adopt Einstein's summation convention, that is $a_r Z^r$ denotes the linear combination $a_1 Z^1 + \cdots + a_N Z^N$ and $a_{rs}Z^{rs}$ denotes $a_{11}Z^{11} + a_{12}Z^{12} + \cdots + a_{NN}Z^{NN}$, and so on.

Theorem 2 *Under Assumptions 1–3, the third-order Edgeworth expansion of the density function of $\boldsymbol{W}_{N,n}$ is given by*

$$g_{W_N}(\boldsymbol{y}) = \phi_{\Sigma_{N,n}}(\boldsymbol{y}) \left\{ 1 + \frac{1}{6\sqrt{n}} C^{(n)}_{W,ijk} H^{ijk}_{\Sigma_{N,n}}(\boldsymbol{y}) + \frac{1}{24n} C^{(n)}_{W,ijk\ell} H^{ijk\ell}_{\Sigma_{N,n}}(\boldsymbol{y}) + \frac{1}{72n} C^{(n)}_{W,ijk} C^{(n)}_{W,i'j'k'} H^{ijki'j'k'}_{\Sigma_{N,n}}(\boldsymbol{y}) \right\} + o(n^{-1}),$$

where $\phi_{\Sigma_{N.n}}(\boldsymbol{y})$ is the N-dimensional normal density function with mean $\mathbf{0}_N$ and covariance matrix $\boldsymbol{\Sigma}_{N,n}$, that is

$$\phi_{\Sigma_{N,n}}(\boldsymbol{y}) = \frac{1}{(2\pi)^{N/2} |\boldsymbol{\Sigma}_{N,n}|^{1/2}} \exp\left(-\frac{1}{2} \boldsymbol{y}' \boldsymbol{\Sigma}^{-1}_{N,n} \boldsymbol{y} \right),$$

and $H^{i_1,\cdots i_j}_{\Sigma_{N,n}}(\boldsymbol{y})$ are Hermite polynomials with $\phi_{\Sigma_{N,n}}(\boldsymbol{y})$ that is

$$H^{i_1\cdots i_j}_{\Sigma_{N,n}}(\boldsymbol{y}) = \frac{(-1)^j}{\phi_{\Sigma_{N,n}}(\boldsymbol{y})} \frac{\partial^j}{\partial y_{i_1} \cdots \partial y_{i_j}} \phi_{\Sigma_{N,n}}(\boldsymbol{y}).$$

To ensure the process $V_{i,t}$ has the expected return on assets as μ, define $m = \frac{\sigma^2}{2} + \frac{\sigma^3\sqrt{t}(\rho^{3/2}C^{(n)}_{Y_0,3}+(1-\rho)^{3/2}C^{(n)}_{Y_i,3})}{6\sqrt{n}} + \frac{\sigma^4 t(\rho^2 C^{(n)}_{Y_0,4}+(1-\rho)^2 C^{(n)}_{Y_i,4})}{24n}$. Note that the identity

$$\int_{-\infty}^{\infty} e^{\sigma\sqrt{t}z} H_k(z)\phi(z)dz = (\sigma\sqrt{t})^k e^{\sigma^2 t/2},$$

together with Theorem 2 above with $N = 1$, we observe that the expected value of the process $\tilde{V}_{i,t} = V_{i,0}\exp\{(\mu - m)t + \sqrt{t}\sigma_n W_{i,n}\}$ becomes

$$\begin{aligned} E_0[\tilde{V}_{i,t}] = V_{i,0} E_0[e^{(\mu-m)t+\sqrt{t}\sigma W_{i,n}}] = V_{i,0} e^{(\mu-m)t} \int_{-\infty}^{\infty} e^{\sigma\sqrt{t}y} g_{W_1}(y) dy \\ = \quad V_{i,0} e^{\mu t}. \end{aligned}$$

Hereafter, we consider the process $\tilde{V}_{i,t} = e^{(\mu-m)t+\sigma\sqrt{t}W_{i,n}}$.

3. Asymptotic Credit Risk Evaluation

We are interested in the default probabilities at time t of a portfolio within the same asset and liability structure. A credit portfolio is nothing but a collection of N transactions with certain counterparties. Since the portfolio would consist of N homogeneous obligors, the probability of default of obligor i and the initial asset value $V_{i,0} = V_0$ are identical for every obligor. We drop the i subscript for simplicity. For example, $V_{i,t}$ and $Y_{i,n}$ are denoted as V_t and $Y_{1,n}$, respectively for the sequel. To model the defaults of the loan in a portfolio, we consider the case with $\tilde{V}_t < D_t$, where D_t is the amount of debt interest at time t. Then, the default time t can be expressed in terms of the random variable W_n:

$$\tilde{V}_n < D_t \Leftrightarrow W_n < K_t \Leftrightarrow Y_{1,n} < C_t$$

where

$$K_t = \frac{\ln D_t/V_0 - (\mu - m)\sqrt{t}}{\sigma\sqrt{t}} \quad \text{and} \quad C_t = \frac{K_t - \sqrt{\rho}Y_{0,n}}{\sqrt{1-\rho}}. \tag{3}$$

Recall that we have assumed that all loans have the same cumulative probability distribution for the time to default, which are given in Theorem 2, and we denote this distribution as G_{W_1}. For notational convenience, we write the representative idiosyncratic random factor as $Y_{1,n}$.

A representative obligor defaults when its asset value $\tilde{V}_t$ falls below the default trigger K_t. Hence, the probability of default is stated in the following theorem.

Theorem 3 *Suppose that Assumptions 1–3 hold; then, the unconditional default probability is expressed as*

$$PD = P(W_{1,n} < K_t) = G_{W_1}(K_t) = p_t^{(1)} + p_t^{(2)} + p_t^{(3)} + p_t^{(4)} + o(n^{-1}),$$

where

$$\begin{aligned}
p_t^{(1)} &= \Phi(K_t),\\
p_t^{(2)} &= -\frac{1}{6\sqrt{n}}\left(\rho^{3/2}C_{Y_0,3}^{(n)} + (1-\rho)^{3/2}C_{Y_1,3}^{(n)}\right)H_2(K_t)\phi(K_t),\\
p_t^{(3)} &= \frac{1}{24n}\left(\rho^2 C_{Y_0,4}^{(n)} + (1-\rho)^2 C_{Y_1,4}^{(n)}\right)H_3(K_t)\phi(K_t),\\
p_t^{(4)} &= \frac{1}{72n}\left(\rho^3(C_{Y_0,3}^{(n)})^2 + 2(\rho(1-\rho))^{3/2}C_{Y_0,3}^{(n)}C_{Y_1,3}^{(n)} + (1-\rho)^3(C_{Y_1,4}^{(n)})^2\right)H_5(K_t)\phi(K_t),
\end{aligned}$$

where K_t is defined by (3).

According to this theorem, the probability of default at time t depends on the value of the factor Y_0. This variable can be thought of as an index of the macroeconomic conditions. If Y_0 is large, the macroeconomic conditions are good, and hence each $W_{1,n}$ tends to be large so that the default probability reduces, and vice versa when Y_0 is small. To explore this effect of the common factor Y_0, we consider the probability of default conditional on Y_0.

Corollary 1 *Let $C_t(Y_0) = \frac{K_t - \sqrt{\rho}Y_{0,n}}{\sqrt{1-\rho}}$. Then, the default rate that is the conditional default probability given $Y_{0,n} = y_0$ is expressed as*

$$\begin{aligned}
&DR = P(Y_{1,n} < C_t(Y_0)|Y_{0,n} = y_0) = G_{Y_1}(C_t(Y_0))\\
= \quad & p_t(Y_0) = \ p_t^{(1)}(Y_0) + p_t^{(2)}(Y_0) + p_t^{(3)}(Y_0) + p_t^{(4)}(Y_0) + o(n^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
p_t^{(1)}(Y_0) &= \Phi(C_t(Y_0)), \quad p_t^{(2)}(Y_0) = -\frac{1}{6\sqrt{n}}C_{Y_1,3}^{(n)}(C_t(Y_0)^2 - 1)\phi(C_t(Y_0)),\\
p_t^{(3)}(Y_0) &= \frac{1}{24n}C_{Y_1,4}^{(n)}(C_t(Y_0)^3 + 3C_t(Y_0))\phi(C_t(Y_0)),\\
p_{i,t}^{(4)}(Y_0) &= \frac{1}{72n}(C_{Y_1,3}^{(n)})^2(-C_t(Y_0)^5 + 10C_t(Y_0)^3 - 15C_t(Y_0))\phi(C_t(Y_0)).
\end{aligned}$$

We refer to this probability as the default rate hereafter. Obviously, the default rate is a function that depends on the common risk factor $Y_{0,n}$, asset correlation ρ, and default trigger K_t.

The default trigger K_t can be viewed as the following Cornish–Fisher expansion.

Corollary 2 *Since $PD = G_{W_1}(K_t)$, the default trigger K_t can be viewed as the following Cornish–Fisher expansion:*

$$\begin{aligned}
& K_t = G_{W_1}^{-1}(PD) \\
= \quad & \Phi^{-1}(PD) + \frac{\rho^{3/2}C_{Y_0,3}^{(n)} + (1-\rho)^{3/2}C_{Y_1,3}^{(n)}}{6\sqrt{n}}((\Phi^{-1}(PD))^2 - 1) \\
+ \quad & \frac{\rho^2 C_{Y_0,4}^{(n)} + (1-\rho)^2 C_{Y_1,4}^{(n)}}{24n}((\Phi^{-1}(PD))^3 - 3\Phi^{-1}(PD)) \\
- \quad & \frac{\rho^3 (C_{Y_0,3}^{(n)})^2 + (1-\rho)^3 (C_{Y_1,3}^{(n)})^2 + 2(\rho(1-\rho))^{3/2} C_{Y_0,3}^{(n)} C_{Y_1,3}^{(n)}}{36n}(2(\Phi^{-1}(PD))^3 - 5\Phi^{-1}(PD)) \\
+ \quad & o(1).
\end{aligned}$$

In a one-factor portfolio model with uniform asset correlation ρ and loss statistics $(L_1, \ldots, L_N)$, where L_i is a Bernoulli random variable with probability $p_t(Y_0)$, which is defined by Corollary 1, the joint default probability of two obligors is expressed by using the bivariate Edgeworth distribution function as

$$\begin{aligned}
& P(L_i = 1, L_j = 1) = G_{W_2}(\boldsymbol{K}_t) = \Phi_{\Sigma_{2,N}}((G_{W_1}^{-1}(PD), G_{W_1}^{-1}(PD)); \rho) \\
- \quad & \int_{y_1 < C_t \cap y_2 < C_t} \phi_{\Sigma_{2,N}}(\boldsymbol{y}) \left\{ \frac{C_{W,ijk}^{(n)} H_{\Sigma_{2,n}}^{ijk}(\boldsymbol{y})}{6\sqrt{n}} + \frac{C_{W,ijk\ell}^{(n)} H_{\Sigma_{2,n}}^{ijk\ell}(\boldsymbol{y})}{24n} \right. \\
& \left. + \frac{C_{W,ijk}^{(n)} C_{W,i'j'k'}^{(n)} H_{\Sigma_{2,n}}^{ijki'j'k'}(\boldsymbol{y})}{72n} \right\} d\boldsymbol{y} + o(n^{-1}),
\end{aligned}$$

where $\boldsymbol{K}_t = (G_{W_1}^{-1}(PD), G_{W_1}^{-1}(PD))$ and $\Phi_{\Sigma_{N,2}}(\cdot; \rho)$ denotes the cumulative bivariate normal distribution function with correlation ρ.

Recall that we have assumed that the ith conditional default rate $p_i(Y_0)$ is identical for all obligors and that the portfolio is large and well diversified. We can see that the portfolio loss given by Y_0 converges to the conditional default probability such that $L^{(N)} = \sum_{i=1}^N w_i L_i \to p_t(Y_0)$ as $N \to \infty$, almost surely, where w_i is the weight of loan i satisfying $\sum_{i=1}^N w_i = 1$ and $w_i > 0$.

The default rate distribution is the cumulative distribution function of the limit loss variable $p(Y_0)$. Then, the approximate distribution of the default rate in a large homogeneous portfolio becomes for every $p \in (0, 1)$,

$$\begin{aligned}
& P(p_t(Y_0) \le p) = P\left(G_{Y_1}\left(\frac{G_{W_1}^{-1}(PD) - \sqrt{\rho}(Y_0)}{\sqrt{1-\rho}} \right) \le p \right) \\
= \quad & P\left(Y_0 \le \frac{\sqrt{1-\rho}G_{Y_1}^{-1}(p) - G_{W_1}^{-1}(PD)}{\sqrt{\rho}} \right) = G_{Y_0}\left(\frac{\sqrt{1-\rho}G_{Y_1}^{-1}(p) - G_{W_1}^{-1}(PD)}{\sqrt{\rho}} \right).
\end{aligned}$$

Let

$$\tilde{C}_t(p) = \frac{\sqrt{1-\rho}G_{Y_1}^{-1}(p) - G_{W_1}^{-1}(PD)}{\sqrt{\rho}}.$$

By using the result of Theorem 1, the above cumulative probability distribution has the following expression:

$$P(p_t(Y_0) \le p) = p_t(p) = p_t^{(1)}(p) + p_t^{(2)}(p) + p_t^{(3)}(p) + p_t^{(4)}(p) + o(1),$$

where

$$p_t^{(1)}(p) = \Phi\left(\frac{\sqrt{1-\rho}G_{Y_1}^{-1}(p) - G_{W_1}^{-1}(PD)}{\sqrt{\rho}}\right) = \Phi(\tilde{C}_t(p)),$$

$$p_t^{(2)}(p) = -\frac{C_{Y_0,3}^{(n)}}{6\sqrt{n}}(\tilde{C}_t(p)^2 - 1)\phi(\tilde{C}_t(p)),$$

$$p_t^{(3)}(p) = \frac{1}{24n}C_{Y_0,4}^{(n)}(\tilde{C}_t(p)^3 + 3\tilde{C}_t(p))\phi(\tilde{C}_i(p)),$$

$$p_{i,t}^{(4)}(p) = \frac{1}{72n}(\tilde{C}_{Y_0,3}^{(n)})^2(-\tilde{C}_t(p)^5 + 10\tilde{C}_t(p)^3 - 15\tilde{C}_t(p))\phi(\tilde{C}_t(p)).$$

From the following expression

$$\frac{dG_{Y_1}^{-1}}{dp} = \frac{1}{g_{Y_1}(G_{Y_1}^{-1}(p))} \approx \phi^{-1}(G_{Y_1}^{-1}(p)) \exp\left\{-\frac{C_{Y_1,3}^{(n)}}{6\sqrt{n}}H_3(G_{Y_1}^{-1}(p)) - \frac{C_{Y_1,4}^{(n)}}{24n}H_4(G_{Y_1}^{-1}(p))\right\},$$

we arrive at the corresponding probability density function by calculating the derivative of $P(p_t(Y_0) < p)$ with respect to p, which is stated in the following theorem.

Theorem 4 *Under Assumptions 1–3, the default rate of a loan portfolio containing N obligors at time t converges for $N \to \infty$ to the conditional default probability $p_t(Y_0)$, given $Y_0 = y_0$. The probability density function of the default rate frequency is given by*

$$\begin{aligned} & f(p) = \frac{d}{dp}P(p_t(Y_0) < p) \\ = \; & \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \exp\left(-\frac{G_{W_1}^{-1}(PD)^2 - 2\sqrt{1-\rho}G_{Y_1}^{-1}(p)G_{W_1}^{-1}(PD) + (1-2\rho)G_{Y_1}^{-1}(p)^2}{2\rho}\right) \\ & \times \left[1 + \frac{C_{Y_0,3}^{(n)}}{6\sqrt{n}}H_3(\tilde{C}_t(p)) + \frac{C_{Y_0,4}^{(n)}}{24n}H_4(\tilde{C}_t(p)) + \frac{(C_{Y_0,3}^{(n)})^2}{72n}H_6(\tilde{C}_t(p))\right] \\ & \times \exp\left\{-\frac{C_{Y_1,3}^{(n)}}{6\sqrt{n}}H_3(G_{Y_1}^{-1}(p)) - \frac{C_{Y_1,4}^{(n)}}{24n}H_4(G_{Y_1}^{-1}(p))\right\} + o(1). \end{aligned}$$

The various risk measures such as expected loss, standard deviation, value-at-risk, economic capital, and expected shortfall for a large credit portfolio can be quantified by using Theorem 4. For the management of a loan portfolio, the decomposition of such risk measures to individual entities or segments is crucial. The above result enables us to calculate the appropriate level of capital requirements to cover any unexpected loss due to the uncertainty of the portfolio credit risks. By definition, the expected and unexpected losses are the mean and standard deviation of the portfolio loss distribution. Under our non-Gaussian and serially correlated factor modeling, these values are calculated as shown in the following lemma.

Theorem 5 *Under Assumptions 1–3, the mean and variance of a portfolio loss L are given* *by*

$$
\begin{aligned}
&E(L) = E(p_t(Y_0)) = p_t^{(1)} + p_t^{(2)} + p_t^{(3)} + p_t^{(4)} =: \bar{p}, \\
&\mathrm{Var}(L) = \mathrm{Var}(p_t(Y_0)) = \Phi_{\Sigma_{N,2}}(\boldsymbol{K_t}) \\
- &\int_{y_1<C_t \cap y_2<C_t} \phi_{\Sigma_{2,n}}(\boldsymbol{y}) \left\{ \frac{C_{W,ijk}^{(n)} H_{\Sigma_{2,n}}^{ijk}(\boldsymbol{y})}{6\sqrt{n}} + \frac{C_{W,ijk\ell}^{(n)} H_{\Sigma_{2,n}}^{ijk\ell}(\boldsymbol{y})}{24n} \right. \\
&\left. + \frac{C_{W,ijk}^{(n)} C_{W,i'j'k'}^{(n)} H_{\Sigma_{2,n}}^{ijki'j'k'}(\boldsymbol{y})}{72n} \right\} d\boldsymbol{y} - \bar{p}^2,
\end{aligned}
$$

where $p_t^{(m)}, m = 1, \ldots, 4$ *are given in Corollary 1 and* $\boldsymbol{K}_t = (G_{W_1}^{-1}(PD), G_{W_1}^{-1}(PD))^\top$.

From the results of Theorem 2, the higher order moments of the portfolio loss can be obtained in a similar manner.

4. Numerical Example

For the numerical example, the parameters for the asset correlation and default probability of the obligors are set at $\rho = 0.05$ and $PD = 0.05$, respectively. The time step of the asset returns is fixed at $n = 12$ or $\Delta = 1/12$, which indicates that the monthly asset returns are observable. We consider the four distributional assumptions in asset returns for the systematic and idiosyncratic risk factors, which includes the normal case ($C_{Y_i,3} = C_{Y_i,4} = 0$), left-skewed case ($C_{Y_i,3} = -0.6$ and $C_{Y_i,4} = 1.5$), right-skewed case ($C_{Y_i,3} = 0.6$ and $C_{Y_i,4} = 1.5$), and symmetric kurtic case ($C_{Y_i,3} = 0$ and $C_{Y_i,4} = 2.5$) for $i = 0, 1$. To avoid model complexity, we omit the case with both systematic and idiosyncratic risk factors that have a non-Gaussian distribution.

Figure 1 shows the default rate distribution for the different distributional assumptions in the systematic and idiosyncratic factors. As can be seen in this figure, the effects of the non-Gaussian assumption on the magnitude of the changes in the shape of the distributions in the idiosyncratic factors are greater than those in the systematic risk factors. This can be found in the form of the density function of the default rate distribution in Theorem 4. Hence, we can conclude that the non-Gaussian effects in idiosyncratic factors need special attention when being treated and evaluating credit risk measurements.

To highlight the differences in measuring various loan portfolio risks, Table 1 compares the statistics of the default rate distributions that incorporate either systematic or idiosyncratic factors and have a non-Gaussian distribution. As can be seen, the quantification of all the risk measures becomes large when the underlying idiosyncratic factor's asset returns exhibit a left-skewed distribution compared with the normal case. However, the non-Gaussian effects on the risk measures are highly complex, as shown in Theorem 4, and thus it is difficult to interpret how the parameters relate to the risk valuation.

5. Data Analysis

The data source for the default frequencies are obtained from Moody's for all rated companies between 1970 and 2010. The same data set was investigated in Chapter 11 of Hull[7].

To model the defaults of the loan portfolio with non-Gaussian assumptions using the proposed portfolio loan distribution given in Chapter 3, we use the maximum likelihood method to estimate the model parameters. As in Duellmann et al.[3], both the maximum likelihood and method-of-moment method can be possible for parameter estimation.

Here we use maximum likelihood estimation because of its tractability. The parameter estimation is based on observed default rates in periods, $1, \ldots, T$. The probability density of the default frequency $DF_t, t = 1, \ldots, T$ is given by Theorem 4. Then the log-likelihood function is given by

$$\begin{aligned} & LL(PD, \rho, C^{(n)}_{Y_0,3}, C^{(n)}_{Y_0,4}, C^{(n)}_{Y_1,3}, C^{(n)}_{Y_1,4}) \\ = & \sum_{t=1}^{T} \ln f(DF_t; PD, \rho, C^{(n)}_{Y_0,3}, C^{(n)}_{Y_0,4}, C^{(n)}_{Y_1,3}, C^{(n)}_{Y_1,4}) \end{aligned}$$

where $f(DF_t)$ is given in Theorem 4. The maximum likelihood estimates can be determined by numerical optimization.

Table 2 presents the estimated parameters, and the corresponding default rate distributions are shown in Figure 2. As for the modeling with the systematic risk factors that have non-Gaussian returns, the maximum likelihood estimation fails to provide reliable values for the parameters in $\hat{C}_{Y_0,3}$, since it lies on the boundary of the parameter space. Hence, we only compare the Gaussian modeling with the non-Gaussian modeling for the idiosyncratic factors. While the obtained log-likelihood for the non-Gaussian models is slightly higher than that of the Gaussian models, the likelihood ratio tests or AIC values suggest that the Gaussian models should be selected for parsimonious modeling. However, the estimated credit risk measurements obtained by using these models are quite different from each other, as illustrated in Table 2.

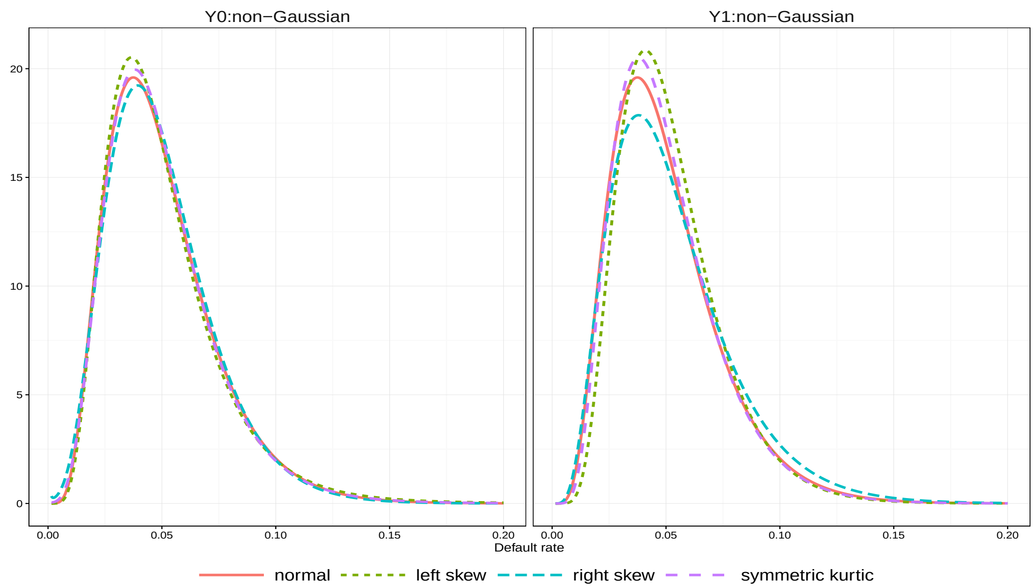

Figure 1: Default rate distribution for the different combinations of skewness and kurtosis in the one factor Merton models: .

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Table 1: Required economic capitals with other credit risk measurements for Gaussian and non-Gaussian modeling.

	Expected Loss	Unexpected Loss	Credit VaR 99%	Economic Capital
Non-Gaussian modeling in idiosyncratic risk factor.				
Normal	0.0500	0.0238	0.1243	0.0743
Normal	0.0500	0.0238	0.1243	0.0743
Left-skew	0.0516	0.0221	0.1242	0.0726
Right-skew	0.0527	0.0265	0.1331	0.0804
Symmetric kurtic	0.0503	0.0228	0.1147	0.0644
Non-Gaussian modeling in systematic risk factor.				
Normal	0.0500	0.0238	0.1243	0.0743
left-skew	0.0508	0.0265	0.1290	0.0782
right-skew	0.0501	0.0232	0.1145	0.0645
symmetric kurtic	0.0501	0.0243	0.1256	0.0755

Table 2: Estimated parameters for Gaussian and non-Gaussian modeling.

	$\widehat{PD}$	$\hat{\rho}$	$\hat{C}_{Y_0,3}$	$\hat{C}_{Y_0,4}$	$\hat{C}_{Y_1,3}$	$\hat{C}_{Y_1,4}$	Log-Likeli.
Gaussian	0.0134	0.1102					137.7313
Y_0: non-Gaussian	0.0143	0.1244	-1.9997	0.0030			138.7655
Y_1: non-Gaussian	0.0111	0.1273			0.8329	6.3493	138.5032

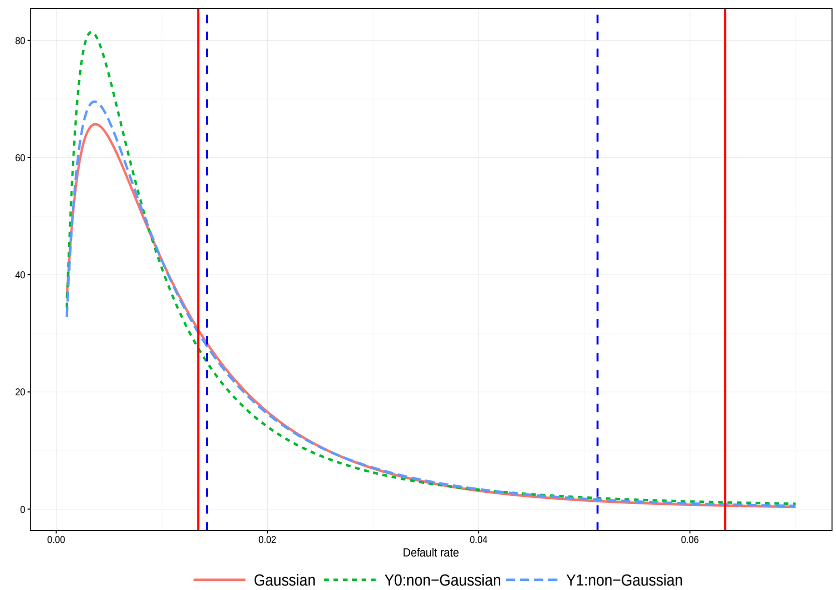

Figure 2: Estimated default rate distribution, using historical default data in the United States from 1970 to 2010.

Table 3: Credit risk valuation for different models.

	Expected Loss	Unexpected Loss	Credit VaR 99%	Economic Capital
Normal	0.0134	0.0131	0.0633	0.0499
Y_0: non-Gaussian	0.0189	0.0344	0.0469	0.0280
Y_1: non-Gaussian	0.0143	0.0146	0.0513	0.0370

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